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ON ASYMPTOTIC MOMENTS OF L-FUNCTIONS

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ABSTRACT. We give a survey on recent results about the asymptotic moments of L-functions associated to primitive forms and of symmetric square of primitive forms.

1. INTRODUCTION TO MODULAR FORMS

For any positive integer N, let $\Gamma_0(N)$ denote the congruence modular group

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a,b,c,d) \in \mathbb{Z}^4,\ ad-bc=1,\ c \equiv 0 \mod N\right\}.$$

For any positive even integer k, a parabolic form of level k and weight N is a function f which is holomorphic on the upper half-plane $\mathcal{H} = \Im z > 0$ and which satisfies to the two conditions

i) for every $z \in \mathcal{H}$ and every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z);$$

ii) the function $z \mapsto (\Im z)^{k/2}|f(z)|$ is bounded on $\mathcal{H}$.

Let $S_k(N)$ denote the set of these forms. It is an hermitian space, when endowed with Petersson's scalar product

$$\langle f, g\rangle = \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Any form in $f \in S_k(N)$ has a Fourier expansion at infinity

$$f(z) = \sum_{n=1}^{+\infty} \widehat{f}(n)\exp(2i\pi nz).$$

For n a positive integer, one defines the n-th Hecke operator T_n by

$$\begin{array}{cccc} T_n & : & S(k,N) & \to & S(k,N) \\ & & \sum_{m=1}^{+\infty} \widehat{f}(m)e^{2i\pi mz} & \mapsto & \sum_{m=1}^{+\infty}\left[\sum_{\substack{d|(m,n)\\(d,N)=1}} d^{k-1}\widehat{f}\left(\frac{mn}{d^2}\right)\right]e^{2i\pi mz}. \end{array}$$

These operators commute and are selfadjoint when $(n,N)=1$. They also enjoy the following multiplicative property

$$T_mT_n = \sum_{\substack{d|(m,n)\\(d,N)=1}} d^{k-1}T_{mn/d^2}.$$

Let d, N' be two divisors of N such that $dN' \mid N$ and $N' < N$. For $f \in S(k, N')$, the function $z \mapsto f(dz)$ belongs to $S(k, N)$. The space spanned by such forms is called the space of old forms. Its orthogonal is the space of new forms. The space of new forms has a special orthogonal basis $\mathrm{H}_k^*(N)$, whose elements are called primitive forms of level k and weight N. Primitive forms are eigenvectors of the Hecke operators: $T_n f = \widehat{f}(n) f$, and they are normalized with the condition $\widehat{f}(1) = 1$.

The eigenvalues of the Hecke operators are usually written as

$$\widehat{f}(n) = \lambda_f(n) n^{(k-1)/2},$$

which is motivated by Deligne's estimate: $\lambda_f(p) \in [-2, 2]$ for p a prime. We then have $\lambda_f(1) = 1$ and

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right).$$

Let us define the harmonic factor

$$\omega(f) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle}.$$

It allows to state a trace formula [I-L-S]: when N is squarefree, and when $(m.N) = (n, N) = 1$, we have

$$\sum_{f \in \mathrm{H}_k^*(N)} \omega(f)\lambda_f(n)\lambda_f(n) = \frac{\varphi(N)}{N}\delta_{m,n} + \text{remainder term}, \tag{1}$$

where δ denotes here the Kronecker symbol.

2. Moments of L-functions

A. L-functions.

For each $f \in \mathrm{H}_k^*(N)$, define the L-function

$$L(f, s) = \sum_{n=1}^{+\infty} \frac{\lambda_f(n)}{n^s}.$$

It may be written as an eulerian product

$$L(f, s) = \prod_{p \in \mathcal{P}} L(f_p, s)$$

with

$$L(f_p, s) = \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\epsilon_N(p)}{p^{2s}}\right)^{-1} \quad \text{and} \quad \epsilon_N(p) = \begin{cases} 1 & \text{if } (p, N) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

B. Symmetric square of primitive forms.

For $f \in \mathrm{H}_k^*(N)$, let us write

$$L(f_p, s)^{-1} = 1 - \frac{\lambda_f(p)}{p^s} + \frac{\epsilon_N(p)}{p^{2s}} = \left(1 - \frac{\alpha_f(p)}{p^s}\right)\left(1 - \frac{\beta_f(p)}{p^s}\right)$$

and define

$$L(\mathrm{sym}_p^2 f, s)^{-1} = \left(1 - \frac{\alpha_f(p)^2}{p^s}\right)\left(1 - \frac{\alpha_f(p)\beta_f(p)}{p^s}\right)\left(1 - \frac{\beta_f(p)^2}{p^s}\right).$$

From now on, we assume that N is squarefree.

Define the symmetric square of a primitive form f by the formula

$$L(\mathrm{sym}^2 f, s) = \prod_{p \in \mathcal{P}} L(\mathrm{sym}_p^2 f, s).$$

It has the expansion

$$L(\mathrm{sym}^2 f, s) = \zeta^{(N)}(2s) \sum_{n=1}^{+\infty} \frac{\lambda_f(n^2)}{n^s},$$

with

$$\zeta^{(N)}(s) = \sum_{\substack{r=1 \\ (r,N)=1}}^{+\infty} \frac{1}{r^s}.$$

One can compute explicitly its Dirichlet series: for $\Re s > 1$,

$$L(\mathrm{sym}^2 f, s) = \sum_{r=1}^{+\infty} \frac{\rho^+(r)}{r^s},$$

with

$$\rho^+(r) = \sum_{\substack{m^2 l = r \\ (m,N)=1}} \lambda_f(l^2).$$

C. Asymptotic moments.
For $\kappa \in]0,1]$, define

$$\mathcal{N} = \{N \in \mathbb{N} \colon \mu(N) \neq 0,\, p \mid N \Rightarrow p \geq N^\kappa\},$$

and, for $n \in \mathbb{Z}$, put

$$H_n(N) = \frac{1}{\#\mathrm{H}_k^*(N)} \sum_{f \in \mathrm{H}_k^*(N)} \omega_N(f) L(f,1)^n,$$

$$M_n(N) = \frac{1}{\#\mathrm{H}_k^*(N)} \sum_{f \in \mathrm{H}_k^*(N)} \omega_N(f) L(\mathrm{sym}^2 f,1)^n.$$

By using the trace formula (1) and density results, Royer [R1, R2] proved the existence of

$$H_n = \lim_{\substack{N \to \infty \\ N \in \mathcal{N}}} H_n(N) \quad \text{and} \quad M_n = \lim_{\substack{N \to \infty \\ N \in \mathcal{N}}} M_n(N).$$

He also gave explicit expressions for these asymptotic moments. Let n be a nonnegative integer. Define

$$\mathcal{F}_n(b_1, \dots, b_n) = \left\{ (d_1, \dots, d_{n-1}) \in \mathbb{N}^{n-1} \colon d_i \mid \left(\frac{b_1 \cdots b_i}{(d_1 \cdots d_{i-1})^2}, b_{i+1} \right) \right\}$$

and

$$h_n(r) = \sum_{\substack{b_1, \dots, b_n \\ b_1 \cdots b_n = r}} \sum_{\substack{(d_1, \dots, d_{n-1}) \in \mathcal{F}_n(b_1, \dots, b_n) \\ (d_1 \cdots d_{n-1})^2 = r}} 1.$$

Then we have

$$H_n = \sum_{r=1}^{+\infty} \frac{h_n(r)}{r}.$$

Similarly, put

$$\mathcal{E}_n(b_1,\ldots,b_n) = \left\{(d_1,\ldots,d_{n-1}) \in \mathbb{N}^{n-1} : d_i \mid \left(\frac{b_1\cdots b_i}{d_1\cdots d_{i-1}}, b_{i+1}\right)^2\right\}$$

and

$$m_n(r) = \sum_{\substack{b_1,\ldots,b_n\\ b_1\cdots b_n=r}} \sum_{\substack{(d_1,\ldots,d_{n-1})\in\mathcal{E}_n(b_1,\ldots,b_n)\\ d_1\cdots d_{n-1}=r}} 1,$$

so that

$$M_n = \zeta(2)^n \sum_{r=1}^{+\infty} \frac{m_n(r)}{r}.$$

The values of the negative asymptotic moments involve the Möbius function μ. Define

$$h_{-n}(r) = \sum_{\substack{((a_1,\ldots,a_n,b_1,\ldots,b_n)\in\mathbb{N}^{2n}\\ a_1\cdots a_n(b_1\cdots b_n)^2=r}} \prod_{i=1}^{n} \mu(a_i)\mu(a_ib_i)^2 \times \sum_{\substack{(d_1,\ldots,d_{n-1})\in\mathcal{F}_n(a_1,\ldots,a_n)\\ (d_1\cdots d_{n-1})^2=a_1\cdots a_n}} 1.$$

Then we have

$$H_{-n} = \sum_{r=1}^{+\infty} \frac{h_{-n}(r)}{r}.$$

Similarly put

$$m_{-n}(r) = \sum_{\substack{(a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,\ldots,c_n)\in\mathbb{N}^{3n}\\ a_1\cdots a_n(b_1\cdots b_n)^2(c_1\cdots c_n)^3=r}} \prod_{i=1}^{n} \mu(a_ib_ic_i)\mu(b_i) \times \sum_{\substack{(d_1,\ldots,d_{n-1})\in\mathcal{E}_n(a_1b_1,\ldots,a_nb_n)\\ d_1\cdots d_{n-1}=a_1b_1\cdots a_nb_n}} 1.$$

Then we get

$$M_{-n} = \zeta(2)^{-1} \sum_{r=1}^{+\infty} \frac{m_{-n}(r)}{r}.$$

3. Eulerian products for the asymptotic moments

The aim of the section is to provide nice explicit eulerian products for the asymptotics moments which are defined above. More precisely we would like each p-factor to be a fixed polynomial in some variable depending on p. This can be performed using generating functions of various classes of paths.

A. Combinatorial paths.

A Dyck path of semilength n is a path in the first quadrant, which begins at the origin, ends at $(2n,0)$ and consists of steps $(1,1)$ and $(1,-1)$. Let C_n denote the number of Dyck paths of semilength n. We have

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

the n-th Catalan number.

A Riordan path of length n is a path in $\mathbb{Z}^2$ which begins at the origin, ends at $(n,0)$, consists of steps $(1,1)$,$(1,-1)$ and $(1,0)$, and remains above the x-axis except in the case of a double step $(1,-1)-(1,1)$. Let R_{n+2} denote the number of Riordan path of length n.

These numbers may be expressed as integrals:

$$C_n = \frac{2}{\pi}\int_0^2 x^{2n}\sqrt{1-\frac{x^2}{4}}\mathrm{d}x \quad \text{and} \quad R_n = \frac{2}{\pi}\int_0^2 (x^2-1)^n\sqrt{1-\frac{x^2}{4}}\mathrm{d}x. \tag{2}$$

We shall need the following two polynomials:

$$s_n(x) = \sum_{j=0}^{n/2} \binom{n}{2j} C_j x^{2j} \quad \text{and} \quad \ell_n(x) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} R_m x^m .$$

From the integral formulas (2), we deduce integral expressions for these polynomials:

$$s_n(x) = \frac{1}{\pi}\int_{-2}^{2} (1+tx)^n \sqrt{1-\frac{t^2}{4}}\mathrm{d}t \tag{3}$$

and

$$\ell_n(x) = \frac{1}{\pi}\int_{-2}^{2} \left(1+(1-t^2)x\right)^n \sqrt{1-\frac{t^2}{4}}\mathrm{d}t. \tag{4}$$

B. Connections between the moments and the paths.

In [R2, H-R], Royer and I showed the following formulas, for every nonnegative integer n:

$$\begin{aligned}
H_{-n} &= \frac{\zeta(2)^n}{\zeta(4)^n} \prod_{p\in\mathcal{P}} s_n\left(\frac{p}{1+p^2}\right), \\
M_{-n} &= \frac{1}{\zeta(2)\zeta(3)^n} \prod_{p\in\mathcal{P}} \ell_n\left(\frac{p}{p^2+p+1}\right), \\
H_{n+3} &= \frac{\zeta(2)^{3n+3}}{\zeta(4)^n} \prod_{p\in\mathcal{P}} s_n\left(\frac{p}{1+p^2}\right), \\
M_{n+2} &= \frac{\zeta(2)^{3n+3}\zeta(3)^n}{\zeta(6)^n} \prod_{p\in\mathcal{P}} \ell_n\left(-\frac{p}{p^2-p+1}\right).
\end{aligned}$$

Royer [R2] proved the negative case by simplifying the formulas involving the Möbius function, and by interpreting the remaining sums with paths. The positive case was treated in [H-R]. The main idea is to build recursively the sums defining the asymptotic moments, and to consider each step of the construction as a weighted path step.

C. Applications.

From (3-4), we can deduce the asymptotic behaviour of the asymptotic moments when n goes to infinity. Let γ denote the Euler constant. When n goes to infinity, we get

$$\begin{aligned}
\log H_n &= 2n\log\log n + 2\gamma n + O\left(\frac{n}{\log n}\right), \\
\log H_{-n} &= 2n\log\log n + \log\left(\frac{\zeta(2)}{\zeta(4)}\right)n + O\left(\frac{n}{\log n}\right), \\
\log M_n &= 3n\log\log n + 3\gamma n + O\left(\frac{n}{\log n}\right), \\
\log M_{-n} &= n\log\log n + (\gamma - 2\log\zeta(2))n + O\left(\frac{n}{\log n}\right).
\end{aligned}$$

Explicit versions of these equalities provide estimates for the extremal values of $L(f,1)$ and $L(\mathrm{sym}^2 f, 1)$. More precisely, put

$$\mathcal{N}_0 = \{N \in \mathbb{N}^* : \mu(N) \neq 0 \quad \text{and} \quad P^-(N) \geq \log(3N)\}.$$

Then there exist K_1, K_2, K_3, K_4 only depending on k such that, for every $N \in \mathcal{N}_0$, there exist four forms $f_1, f_2, f_3, f_4 \in \mathrm{H}_k^*(N)$ satisfying to the conditions

$$\begin{aligned} L(f_1,1) &\geq K_1 (\log\log(3N))^2\,, \\ L(f_2,1) &\leq K_2 (\log\log(3N))^{-2}\,, \\ L(\mathrm{sym}^2 f_3,1) &\geq K_3 (\log\log(3N))^3\,, \\ L(\mathrm{sym}^2 f_4,1) &\leq K_4 (\log\log(3N))^{-1}\,. \end{aligned}$$

These results are obtained by Royer and Wu [R-W]. They also proved that, under the generalized Riemann hypothesis, these are the correct order of magnitude: only the constants may be improved.

4. Extensions

We know that

$$H_{-n} = \prod_{p\in\mathcal{P}} \frac{1}{\pi}\int_{-2}^{2}\left(1+\frac{x}{p}+\frac{1}{p^2}\right)^n \sqrt{1-\frac{x^2}{4}}\mathrm{d}x\,,$$

$$H_{n+3} = \zeta(2)^{2n+3}\prod_{p\in\mathcal{P}} \frac{1}{\pi}\int_{-2}^{2}\left(1+\frac{x}{p}+\frac{1}{p^2}\right)^n \sqrt{1-\frac{x^2}{4}}\mathrm{d}x\,.$$

The same method allows to replace $1/p$ by $1/p^s$ in the above formula, for $\Re s > 1$. The same kind of result holds for M_n, $n \in \mathbb{Z}$.

Cogdell and Michel [C-M] found these formulas in a completely different way, which enables them to deal with the case $n \in \mathbb{C}$. Under suitable hypothesis, they also found analogues of these formulas for higher symmetric powers: the term of order p in the eulerian product associated to the z-moment related to m-th symmetric power is given by

$$\int_0^{2\pi}\left(\prod_{\substack{-m\leq j\leq m\\ j\equiv m \bmod 2}}(1-te^{ij\theta})\right)^{-z}\frac{1}{\pi}\sin^2\theta\mathrm{d}\theta\,.$$

(One can get back to the original formulas for $m = 1, 2$ using the change of variable $x = 2\cos\theta$.)

The same kind of asymptotic behaviors hold when $z = n$ is real and goes to infinity. For n positive, they got

$$(m+1)n\log\log n + (m+1)\gamma n + O\left(\frac{n}{\log n}\right)\,.$$

For $z = -n$, n positive and odd, they found

$$(m+1)n\log\log n + (m+1)(\gamma - \log\zeta(2))n + O\left(\frac{n}{\log n}\right)\,.$$

For $z = -n$, n positive and even, they got

$$A_m n\log\log n + B_m n + O\left(\frac{n}{\log n}\right)\,,$$

where $\exp(A_m)$ is the minimum of the Chebyshev polynomial of the second kind on $[-2, 2]$.

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