Strang-Fix Theory for Approximation Order in Weighted $L^p$-spaces

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We consider the Strang-Fix theory for approximation order in the weighted $L^p$-spaces. Let $\varphi$ be an element of $C_c(\mathbb{R}^n)$. For a sequence $c$ on $\mathbb{Z}^n$, the semi-discrete convolution product $\varphi \ast' c$ is the function defined by

$$\varphi \ast' c = \sum_{\nu \in \mathbb{Z}^n} \varphi(\cdot - \nu)c(\nu).$$

The collection $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ of $C_c(\mathbb{R}^n)$ is said to satisfy the Strang-Fix condition of order $k$ if there exist finitely supported sequences $b_j (j = 1, \cdots, N)$ such that the function $\varphi = \sum_{j=1}^{N} \varphi_j \ast' b_j$ satisfies

$$\hat{\varphi}(0) \neq 0$$

and

$$(\partial^\alpha \hat{\varphi})(2\pi \nu) = 0 \quad (|\alpha| < k, \nu \in \mathbb{Z}^n \setminus \{0\}),$$

where $\hat{\varphi}$ denotes the Fourier transform of $\varphi$. For a positive integer $k$, $L_k^p(\mathbb{R}^n)$ denotes the Sobolev space. For $f \in L_k^p(\mathbb{R}^n)$, we define semi-norms by

$$|f|_{k,p} = \sum_{|\alpha|=k} ||\partial^\alpha f||_{L^p(\mathbb{R}^n)}.$$

For $h > 0$, $\sigma_h$ is the scaling operator defined by

$$\sigma_h f(x) = f(hx) \quad (x \in \mathbb{R}^n).$$

We say that $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ provides local $L^p$-approximation of order $k$ if there exist constants $C$ and $r$ such that for each $f \in L_k^p(\mathbb{R}^n)$ there exist sequences $c_j^h$ ($h > 0, j = 1, \cdots, N$) so that

(i) $||f - \sigma_{1/h}(\sum_{j=1}^{N} \varphi_j \ast' c_j^h)||_{L^p(\mathbb{R}^n)} \leq Ch^k |f|_{k,p},$
(ii) \( c_j^h(\nu) = 0 \) \((j = 1, \ldots, N)\) whenever \( \text{dist}(h\nu, \text{supp } f) > r. \)

Boor and Jia [1] proved that \( \Phi \) satisfies the Strang-Fix condition of order \( k \) if and only if \( \Phi \) provides local \( L^p \)-approximation of order \( k \).

We give the definition of \( A_p \) in \( \mathbb{R}^n \). A weight \( w \geq 0 \) is said to belong to \( A_p \) for \( 1 < p < \infty \) if

\[
A_p(w) = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'}dx \right)^{p-1} < \infty,
\]

where \( Q \) is a cube in \( \mathbb{R}^n \) and \( p' \) is a conjugate exponent of \( p \). \( A_p(w) \) is called the \( A_p \)-constant of \( w \).

The class \( A_{\infty} \) is the union of the classes of \( A_p \), \( 1 \leq p < \infty \). These classes were introduced by Muckenhoupt in [3]. Let \( 1 \leq p \leq \infty \) and \( w \in A_p \). Then the weighted \( L^p \)-space \( L^p(w) \) consists of all measurable functions on \( \mathbb{R}^n \) such that

\[
\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right)^{1/p} < \infty,
\]

with the usual modifications when \( p = \infty \). We define the weighted Sobolev spaces \( L_k^p(w) \), where \( 1 \leq p \leq \infty \), \( w \) is an \( A_p \)-weight and \( k \) is a positive integer. A function \( f \) belongs to \( L_k^p(w) \) if \( f \in L^p(w) \) and the partial derivatives \( \partial^\alpha f \), taken in the sense of distributions, belong to \( L^p(w) \), whenever \( 0 \leq |\alpha| \leq k \).

The norm in \( L_k^p(w) \) is given by

\[
\|f\| = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p(w)}.
\]

In the weighted case, we use the following notation

\[
|f|_{k,p,w} = \sum_{|\alpha| = k} \|\partial^\alpha f\|_{L^p(w)}.
\]

and say that \( \Phi = \{\varphi_1, \ldots, \varphi_N\} \) provides local \( L^p(w) \)-approximation of order \( k \) if there exist constants \( C \) and \( r \) such that for each \( f \in L_k^p(w) \) there exist sequences \( c_j^h \) \((h > 0, j = 1, \ldots, N)\) so that (ii) and the following condition (iii) are satisfied

(iii) \[ \|f - \sigma_1/h(\sum_{j=1}^N \varphi_j * c_j^h)\|_{L^p(w)} \leq Ch^k|f|_{k,p,w}. \]
Based on [2], using boundedness of the Hardy-Littlewood maximal operator on $L^p(w)$, we prove the following theorem.

**Theorem.** Let $\Phi = \{\varphi_1, \cdots, \varphi_N\}$ be a finite collection of $C_c(\mathbb{R}^n)$. Then the following statements are equivalent.

(i) $\Phi$ satisfies the Strang-Fix condition of order $k$.

(ii) For all $p \in [1, \infty]$ and $w \in A_p$, $\Phi$ provides local $L^p(w)$-approximation of order $k$.

(iii) For some $p \in [1, \infty]$ and $w \in A_p$, $\Phi$ provides local $L^p(w)$-approximation of order $k$.

Lastly, we introduce the main lemma to prove the above theorem.

**Lemma.** Let $1 \leq p < \infty$ and $w \in A_p$. Suppose that $\varphi$ is a function on $\mathbb{R}^n$ which is non-negative, radial, decreasing and integrable. Then there exists a constant $C$ such that

$$
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x+\alpha y)| \varphi(y) dy \right)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx
$$

for all $f \in L^p(w)$ and $\alpha \in \mathbb{R}$.

N. Tomita proved the above lemma when $1 < p < \infty$, using Calderón-Zygmund Operator. Then Professor E. Nakai provided the simple proof when $1 < p < \infty$, using Hardy-Littlewood maximal operator. Then Professor K. Yabuta proved the case $p = 1$.

**References**

