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Duality and Symmetry of Waves and Particles

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§0. Introduction.

The aim of this note is to present the summary of mirror symmetry, which appears in the theory of partial differential equations. The original of this note is Sjöstrand [2]. The plan of this note is as follows. In §1, we explain the senses of several words : *duality of waves and particles, pseudo quantum physics,* etc. In §2, we give a preparation about geometry. We define Sjöstrand space in §3 and mother function of mirror symmetry in §4, respectively. In §5, we write some items which illustrates the outline of mirror symmetry.

The contents of this note are only supplement of Morioka [1].

Remark. [1] is the English version of the author's lecture note which has been written in Japanese.

§1. Pseudo quantum physics.

Duality of waves and particles means the fact that light has the characters of both waves and particles. Duality of waves and particles can be described by the pseudo quantum physics.

Definition 1.1. A combination : (*Symplectic manifold, Quantization* I) is called pseudo quantum physics.

Definition 1.2. *Quantization* I is a realization of functions on a symplectic manifold as integral operators.

Concerning the pseudo quantum physics, there is *A model* and *B model.* Their symplectic manifolds are as follows.

A model —- Cotangent bundle

B model —- Complex manifold with a weight function

At B model, the weight function is assumed to be strictly pluri-subharmonic. Concerning the definition of strictly pluri-subharmonic functions, see [1, §7-2].

The description of *duality of waves and particles* by the pseudo quantum physics is mathematically the theorem about the propagation of singularities for principal type partial differential equations, whose statement is given in [1, Theorem F in §2 (page 18)] .

Definition 1.3. *Quantization* II is a realization of symplectiomorphisms as integral operators.

Between A model and B model, Quantization I and Quantization II are compatible. This relation is called mirror symmetry.

We will explain the mathematical contents of mirror symmetry.

§2. Geometry.

Concerning the notations, isomorphisms between manifolds etc. without definitions in this section, see [1, §7] .

Let X be a complex manifold. We denote by $X^{\mathbf{R}}$ the underlying real analytic manifold of X. As set, $X = X^{\mathbf{R}}$ holds. We have $TX \cong TX^{\mathbf{R}}$ and $T^*X \cong T^*X^{\mathbf{R}}$. Assume that 2-form θ on T^*X is given. By $T^*X \cong T^*X^{\mathbf{R}}$, 2-forms $\mathrm{Re}\,\theta$ and $\mathrm{Im}\,\theta$ on $T^*X^{\mathbf{R}}$ are determined from θ. Let σ be the canonical 2-form of T^*X. Then, both $\mathrm{Re}\,\sigma$ and $\mathrm{Im}\,\sigma$ are symplectic forms of $T^*X^{\mathbf{R}}$. That is to say, both $(T^*X^{\mathbf{R}}, \mathrm{Re}\,\sigma)$ and $(T^*X^{\mathbf{R}}, \mathrm{Im}\,\sigma)$ are symplectic manifolds. Especially, $-\mathrm{Im}\,\sigma$ coincides with the canonical 2-form of $T^*X^{\mathbf{R}}$. As set, $(T^*X)^{\mathbf{R}} = T^*X$. Therefore, we obtain an identification $(T^*X)^{\mathbf{R}} \cong T^*X^{\mathbf{R}}$ from $T^*X \cong T^*X^{\mathbf{R}}$. We denote by same notations ($\mathrm{Re}\,\sigma$ and $\mathrm{Im}\,\sigma$) the symplectic forms of $(T^*X)^{\mathbf{R}}$ determined by the symplectic forms $\mathrm{Re}\,\sigma$ and $\mathrm{Im}\,\sigma$ of $T^*X^{\mathbf{R}}$ and $(T^*X)^{\mathbf{R}} \cong T^*X^{\mathbf{R}}$. Let Λ be a real analytic submanifold of T^*X.

Definition 2.1.
(i) Λ is said to be R - [isotropic, involutive, lagrangian, symplectic] in T^*X if Λ is [isotropic, involutive, Lagrange, symplectic] submanifold of symplectic manifold $((T^*X)^{\mathbf{R}}, \mathrm{Re}\,\sigma)$, respectively.
(ii) Λ is said to be I - [isotropic, involutive, lagrangian, symplectic] in T^*X if Λ is [isotropic, involutive, Lagrange, symplectic] submanifold of symplectic manifold $((T^*X)^{\mathbf{R}}, \mathrm{Im}\,\sigma)$, respectively.

Remark. Definition 2.1 is equivalent to [1, Definition 7.1.6 in §7]

Let X be a complex manifold and M be a real analytic manifold. Assume that X is complexification of M. Then, an imbedding ι from T^*M to T^*X is canonically determined. $\iota(T^*M)$ is I-lagrangian and R-symplectic in T^*X. Moreover, T^*X is complexification of $\iota(T^*M)$. Let ω be the canonical 2-form of T^*M and σ be the canonical 2-form of T^*X. Then we have $\iota^*(\mathrm{Re}\,\sigma) = \omega$ and $\iota^*(\mathrm{Im}\,\sigma) = 0$.

Let Y be a complex manifold and $\varphi \in C^\omega(Y, \mathbf{R})$. An imbedding $\tilde{j}_\varphi$ from Y to $T^*Y^{\mathbf{R}}$ is determined by φ as follows.

$$\tilde{j}_\varphi(z) = (z\,,\,(d\varphi)_z) \quad ; \quad z \in Y \tag{2.1}$$

Moreover, an imbedding j_φ from Y to T^*Y is cdetermined from $\tilde{j}_\varphi$ and $T^*Y \cong T^*Y^{\mathbf{R}}$. We define $\Lambda \subset T^*Y$ by $\Lambda = j_\varphi(Y)$. Then Λ is I-lagrangian in T^*Y. Let σ be the canonical 2-form of T^*Y. Then we have $(j_\varphi)^*(\mathrm{Re}\,\sigma) = -2i\bar{\partial}\partial\varphi$ and $(j_\varphi)^*(\mathrm{Im}\,\sigma) = 0$. Assume that φ is strictly pluri-subharmonic on Y. Then Λ is I-lagrangian and R-symplectic in T^*Y. Moreover, T^*Y is complexification of Λ.

Let Y be a complex manifold and $\varphi \in C^\omega(Y, \mathbf{R})$. Assume that φ is strictly pluri-subharmonic on Y. Let $\theta = -2i\bar{\partial}\partial\varphi$. Then, $(Y^{\mathbf{R}}, \theta)$ is symplectic manifold. Let σ be the canonical 2-form of T^*Y, π be the projection from T^*Y to Y and $\Lambda = j_\varphi(Y)$. Then $(\mathrm{Im}\,\sigma)\,|_\Lambda = 0$ holds. Moreover, $(\mathrm{Re}\,\sigma)\,|_\Lambda$ is symplectic form of

Λ. That is to say, $(\Lambda, (\mathrm{Re}\,\sigma)\,|_\Lambda)$ is symplectic manifold. π is symplectiomorphism from $(\Lambda, (\mathrm{Re}\,\sigma)\,|_\Lambda)$ to $(Y^{\mathbf{R}}, \theta)$.

Finally we write a review of this section.

Let Y and X be complex manifolds and M be a real analytic manifold. Assume that X is complexification of M. Let $\varphi \in C^\omega(Y, \mathbf{R})$ and assume that φ is strictly pluri-subharmonic on Y. Let $\Lambda = j_\varphi(Y)$. Then we have the following claims.

- $\iota(T^*M)$ is I-lagrangian and R-symplectic in T^*X.
- T^*X is complexification of $\iota(T^*M)$.
- Λ is I-lagrangian and R-symplectic in T^*Y.
- T^*Y is complexification of Λ.

Remark. $\iota(T^*M)$ is not section of T^*X. On the other hand, Λ is section of T^*Y.

Remark. Mother function of mirror symmetry generates an isomorphism which transforms $\iota(T^*M)$ into Λ. This transformation will be explained in §5.

Concerning some parts of nations used in this section without definitions, we write a simple list for their definitions in [1]. In the following list, the right hand side corresponds to [1].

(a). $TX \cong TX^{\mathbf{R}}$ — Page 42
(b). $T^*X \cong T^*X^{\mathbf{R}}$ — (7.1.6) of page 43
(c). $\mathrm{Re}\,\theta$ and $\mathrm{Im}\,\theta$ — (7.1.7) of page 45

We give the definitions of (a), (b) and j_φ by using a coordinate. For simplicity, we assume that $Y = X = \mathbf{C}^n$. Let $z = (z_1 \dots z_n)$ be a holomorphic coordinate of $Y = X$. Then, a real analytic coordinate (x, y) is determined by $z = x + iy$. The identification $TX \cong TX^{\mathbf{R}}$ is given by

$$\partial/\partial z_k \leftrightarrow \partial/\partial x_k\ , \qquad i\partial/\partial z_k \leftrightarrow \partial/\partial y_k\ , \qquad 1 \leqq k \leqq n\ .$$

The identification $T^*X \cong T^*X^{\mathbf{R}}$ is given by

$$dz_k \leftrightarrow -dy_k\ , \qquad idz_k \leftrightarrow -dx_k\ , \qquad 1 \leqq k \leqq n\ .$$

The imbedding j_φ from Y to T^*Y is determined as follows:

$$j_\varphi(z) = (z\,, -2i\frac{\partial\varphi}{\partial z}(z))\ ; \qquad z \in Y\ .$$

§3. Sjöstrand space.

Let W be an open set of $\mathbf{C}^n$ and $\psi \in C(W, \mathbf{R})$. We define a functional space $H_\psi(W)$ as follows. Let $\mathbf{R}_+ = \{\lambda \in \mathbf{R}\ :\ \lambda > 0\}$.

Definition 3.1. Let f be a function from $W \times \mathbf{R}_+$ to $\mathbf{C}$. We denote by $(z, \lambda) \in W \times \mathbf{R}_+$ the independent variables of f. We say that $f \in H_\psi(W)$ if the following (i) and (ii) hold.
(i) f is holomorphic in W with respect to z.
(ii) For all $B \subset W$: compact and all $\varepsilon > 0$ there exists $K > 0$ such that for all $z \in B$ and all $\lambda \geqq 1$ we have

$$|f(z, \lambda)| \leqq K \exp(\lambda(\psi(z) + \varepsilon)) \ . \tag{3.1}$$

$H_\psi(W)$ is called Sjöstrand space with weight function ψ.

Definition 3.2. Let f, $g \in H_\psi(W)$ and $a \in W$. We say that $f \sim g$ at a if there exist an open neighborhood $W_0 \subset W$ of a and $c > 0$ such that we have $f - g \in H_{\psi - c}(W_0)$.

Let $f \in H_\psi(W)$. We define $SSf \subset W$ as follows.

Definition 3.3. Let $z \in W$. We say that $z \notin SSf$ if we have $f \sim 0$ at z.

SSf is called singular support of f.

§4. Mother functions.

At first, we prepare some notations.

Let Y, X be complex manifolds and $b \in Y$, $a \in X$. $B \overset{\text{def}}{=} Y \times X$ and $E \overset{\text{def}}{=} T^*Y \times T^*X$. Let $\delta \in T^*X$, $\delta = (z, w)$, $z \in X$ and $w \in T^*_zX$. We define $\delta^{anti} \in T^*X$ by $\delta^{anti} = (z, -w)$. We define a map A from E to E by $A(\rho, \delta) = (\rho, \delta^{anti})$. Here, $\rho \in T^*Y$ and $\delta \in T^*X$.

Let g be a holomorphic funcion from B to $\mathbf{C}$. We define $Lg \subset T^*B$ by

$$Lg = \{(s, (dg)_s) \ : \ s \in B\} \ . \tag{4.1}$$

We define also $\Sigma g \subset E$ by $\Sigma g = ALg$. Here we identify canonically T^*B with E. We identify canonically $T_{(b,a)}B$ with $T_bY \times T_aX$. We denote by $\widetilde{v}$ the element of $T_bY \times T_aX$ which corresponds to $v \in T_{(b,a)}B$, concerning $T_{(b,a)}B \cong T_bY \times T_aX$. We define complex bilinear map Sg from $T_bY \times T_aX$ to $\mathbf{C}$ by

$$(Sg)(\widetilde{v}) = \langle (dg)_{(b,a)}, v \rangle_{T^*_{(b,a)}B \times T_{(b,a)}B} \ , \qquad v \in T_{(b,a)}B \ . \tag{4.2}$$

Let Y, X be open sets of $\mathbf{C}^n$, M be an open set of $\mathbf{R}^n$ and $M \subset X$, $b \in Y$, $a \in M$. Let g be a holomorphic function from B to $\mathbf{C}$. Here, $B = Y \times X$. We define $f \in C^\omega(M, \mathbf{C})$ by $f(t) = g(b, t)$, $t \in M$.

Definition 4.1. We say that g is mother function of mirror symmetry if g satisfies the following (H.1) and (H.2).
(H.1) $(d(\mathrm{Re}\, f))_a \neq 0$, $(d(\mathrm{Im}\, f))_a = 0$, $(\mathrm{Hess}\ (\mathrm{Im}\, f))_a > 0$.
(H.2) The complex bilinear map Sg from $T_bY \times T_aX$ to $\mathbf{C}$ is non-degenerate.

Remark. (H.1) and (H.2) in Definition 4.1 is same as (H.1) and (H.2) in [1, page 51].

Remark. Σg does not appear in Definition 4.1. But it is used in §5.

At the end of this section, we give an example of mother function.

Example 4.2. Let $Y = X = \mathbf{C}^n$, $M = \mathbf{R}^n$, $b \in Y$, $\operatorname{Im} b \neq 0$, $a \in M$ and $a = \operatorname{Re} b$. We define a function g as follows.

$$g(z\,,\,t) = \frac{i}{2}(z-t)^2 \ ; \qquad (z\,,\,t) \in Y \times X \tag{4.3}$$

Then g satisfies (H.1) and (H.2) in Definition 4.1. Therefore, g is mother function of mirror symmetry.

§5. Mirror symmetry.

The main claim of this section is as follows.

Claim 5.1. Assume that mother function of mirror symmetry is given. Then, B model is determined. Moreover, mirror symmetry holds between A model and B model.

For simplicity, we write the items which describe the mathematical contents of Claim 5.1 under the assumption that mother function of mirror symmetry is the function defined by (4.3) in §4.

Let $L = Y$ or X. We denote by σ_L the canonical 2-form of T^*L. Let ω be the canonical 2-form of T^*M. Let g be the function defined by (4.3) in §4.

(I). A holomorphic isomorphism $\tau^{\mathbf{C}}$ from T^*Y to T^*X is determined by Σg. That is to say, Σg is graph of $\tau^{\mathbf{C}}$.
(II). $\varphi \in C^\omega(Y\,,\,\mathbf{R})$ is determied as follows:

$$\varphi(z) = \sup\{-\operatorname{Im} g(z\,,\,t) \ : \ t \in M\ \} \ ; \quad z \in Y \ . \tag{5.1}$$

(III). $\varphi \in C^\omega(Y\,,\,\mathbf{R})$ is strictly pluri-subharmonic on Y.
(IV). An imbedding j_φ from Y to T^*Y is determined by φ.
(V). There exists uniquely real analytic isomorphism τ from Y to T^*M such that we have $\tau^{\mathbf{C}} \circ j_\varphi = \iota \circ \tau$ on Y. Here, ι is canonical imbedding from T^*M to T^*X.

Let $\Lambda = j_\varphi(Y)$, $\theta = -2i\bar{\partial}\partial\varphi$ and π be the projection from T^*Y to Y.

(VI). Λ is I-lagrangian and R-symplectic in T^*Y.
(VII). T^*Y is complexification of Λ.
(VIII). $\iota(T^*M)$ is I-lagrangian and R-symplectic in T^*X.
(IX). T^*X is complexification of $\iota(T^*M)$.
(X). $(j_\varphi)^*(\operatorname{Re}\sigma_Y) = \theta$ and $\iota^*(\operatorname{Re}\sigma_X) = \omega$ hold.

(XI). $(\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda)$ and $(Y^{\mathbf{R}}, \theta)$ are symplectic manifolds.
(XII). π is symplectiomorphism from $(\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda)$ to $(Y^{\mathbf{R}}, \theta)$.
(XIII). $\tau^{\mathbf{C}}$ is symplectiomorphism from (T^*Y, σ_Y) to (T^*X, σ_X).
(XIV). τ is symplectiomorphism from $(Y^{\mathbf{R}}, \theta)$ to (T^*M, ω).
(XV). $j_\varphi \circ \tau^{-1}$ and $(\tau^{\mathbf{C}})^{-1} \circ \iota$ are symplectiomorphism from (T^*M, ω) to $(\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda)$.

Remark. From (V) we see that $j_\varphi \circ \tau^{-1} = (\tau^{\mathbf{C}})^{-1} \circ \iota$.

Let T be an integral operator of the following form.

(5.2)
$$(Tu)(z, \lambda) = \int \exp(i\lambda g(z, t))\, u(t)\, dt$$
$$u \in \mathcal{E}'(M), \quad z \in Y, \quad \lambda > 0$$

T is called FBI transformation whose phase function is g.

Let P be a partial differential operator on M. We assume that the coefficients of P are analytic.

(XVI). T is map from $\mathcal{E}'(M)$ to $H_\varphi(Y)$.
(XVII). There exists pseudo differential operator $\widetilde{P}$ on $H_\varphi(Y)$ such that $TP = \widetilde{P}T$ (module analytic) holds. That is to say, for all $u \in \mathcal{E}'(M)$ the singular support of $(TP - \widetilde{P}T)u$ in $H_\varphi(Y)$ is empty.

Remark. Concerning pseudo differential operators on $H_\varphi(Y)$, see [2] or [1, §11-2].

Remark. (XVII) is not true in general. To revise (XVII), we need some local description. However, we write the claim by global description to simplify the notations.

Let p be the principal symbol of P and $\widetilde{p}$ be the principal symbol of $\widetilde{P}$. Let $\gamma = j_\varphi \circ \tau^{-1} = (\tau^{\mathbf{C}})^{-1} \circ \iota$.

(XVIII). p is function on T^*M and $\widetilde{p}$ is function on Λ.
(XIX). The relation between p and $\widetilde{p}$ is compatible with the isomorphism $T^*M \cong \Lambda$ by γ. That is to say, $p = \widetilde{p} \circ \gamma$ holds.

Remark. As we have already said in (XV), γ is symplectiomorphism from (T^*M, ω) to $(\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda)$.

Remark. Concerning the definitions of principal symbols, see [1, page 15 and page 70].

In the following (XX) – (XXIV), we explain the senses of several words.

(XX). (T^*M, ω) is symplectic manifold of A model.
(XXI). $(\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda) \cong (Y^{\mathbf{R}}, \theta)$ is symplectic manifold of B model.

Remark. As we have already said in (XII), $(\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda)$ and $(Y^{\mathbf{R}}, \theta)$ are symplectiomorphism by π. Here $\theta = -2i\overline{\partial}\partial\varphi$.

Remark. We have said in §1 that symplectic manifold of B model is complex manifold with a weight function. Here, complex manifold means Y and weight function means φ.

(XXII). $[p \to P]$ is Quantization I at A model.
(XXIII). $[\widetilde{p} \to \widetilde{P}]$ is Quantization I at B model.
(XXIV). $[\gamma \to T]$ is Quantization II. We also say that $[\tau^{-1} \to T]$ is Quantization II.

Remark. From the definition we see that $\pi \circ \gamma = \tau^{-1}$. We say again that γ , τ, π are symplectiomorphisms between the following symplectic manifolds.

$$\gamma : (T^*M, \omega) \cong (\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda)$$
$$\tau : (Y^{\mathbf{R}}, \theta) \cong (T^*M, \omega)$$
$$\pi : (\Lambda, (\mathrm{Re}\,\sigma_Y)\,|_\Lambda) \cong (Y^{\mathbf{R}}, \theta).$$

The items mentined above are the outline of mirror symmetry. We do not have explained wave front sets in this note. Concerning wave front sets, see [1, §2], [1, Definition 7.3.1 in §7 (page 52) and Proposition 7.4.4 in §7 (page 53)], [2, Proposition 7.2 in §7 (page 46)] etc.

Finally we write briefly a review of this section. We conserve the notations.

Review. Assume that mother function g of mirror symmetry is given. Then, integral operator T and symplectiomorhism γ is determined from g. T generates the transformation of operator $[P \to \widetilde{P}]$ and γ generates the transformation of principal symbol $[p \to \widetilde{p}]$. Thus the transformation of operator generated by T and the transformation of principal symbol generated by γ are compatible. This relation is mirror symmetry.

References

1. T. Morioka, *Basics of second microlocalization* (unpublished), Lecture Note of Ryukoku University (2000).
2. J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque **95** (1982).