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Supersymmetric analysis of discrete magnetic Schrödinger operators

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Abstract

In the last symposium (Jul. 2001), T. Shirai talked about the spectrum of the infinitely extended Sierpinski lattice [7, 3]. Their results are based on some relations between the spectra of an infinite regular graph and its line-graph. In this report, we extend their results to the cases of discrete magnetic Schrödinger operators on infinite regular graphs.

1 Definitions

A graph $G = (V(G), A(G))$ is a pair of the vertex set $V(G)$ and the oriented edge set $A(G)$. We say that two vertices x, y are adjacent if there exists an edge

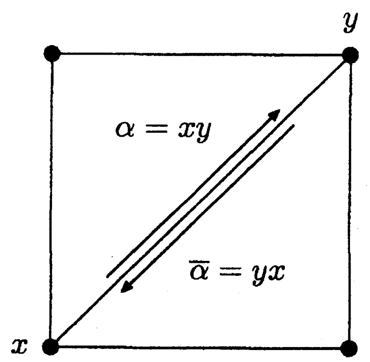

Figure 1: a graph G

which connects them. We denote $x \sim y$ if x and y are adjacent. Let $\alpha \in A(G)$, which has direction from the origin $x \in V(G)$ to the terminus $y \in V(G)$. Then, we denote $\alpha = xy$, $o(\alpha) = x$ and $t(\alpha) = y$. We denote $\overline{\alpha} = yx$ the reverse edge of α. We assume that $\overline{\alpha} \in A(G)$ provided $\alpha \in A(G)$.

Let

$$\begin{aligned} A_x(G) &= \{\alpha \in A(G); o(\alpha) = x\}, \\ \deg(x) &= \# A_x(G). \end{aligned}$$

We call $\deg(x)$ the degree of x. If there exists a constant d such that $\deg(x) = d$ for all $x \in V(G)$, then the graph G is called d-regular. Regularity has important role in this report.

Throughout this report, we assume that (i) G is locally finite, that is, $\deg(x) < \infty$ for all $x \in V(G)$; (ii) G has no loop and multiple edge.

We define the discrete Laplacian Δ_G on a graph G. We work on the Hilbert space

$$l^2(G) = \left\{ f: \ V(G) \to \mathbf{C}; \ \sum_{x \in V(G)} |f(x)|^2 < \infty \right\}.$$

The discrete Laplacian Δ_G acts $f \in l^2(G)$ as follow:

$$\begin{aligned} (\Delta_G f)(x) &= \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} [f(t(\alpha)) - f(x)] \\ &= \frac{1}{\deg(x)} \left[\sum_{\alpha \in A_x(G)} f(t(\alpha)) \right] - f(x) \\ &= \frac{1}{\deg(x)} \left[\sum_{y \sim x} f(y) \right] - f(x). \end{aligned}$$

We denote $\mathrm{Spec}\,(-\Delta_G)$ the spectrum of $-\Delta_G$.

Remark 1. It is a well-known fact that $\mathrm{Spec}\,(-\Delta_G) \subset [0, 2]$. We remark that all of the operators in this report is bounded.

We use three graphs associated with a given graph G. First one is the ***subdivision graph*** $S(G)$. See Figure 2. We make $S(G)$ by adding one vertex $|\alpha|$ at the midpoint of each edges $\alpha \in A(G)$. We note $|\overline{\alpha}| = |\alpha|$. Formally, we give

$$\begin{aligned} V(S(G)) &= V(G) \cup E(G), \\ A(S(G)) &= \{x\alpha, \alpha x; \ x \in V(G), \alpha \in A_x(G)\}. \end{aligned}$$

Here, we put

$$E(G) = \{|\alpha|; \ \alpha \in A(G)\}.$$

We call $E(G)$ the (unoriented) edge set of G.

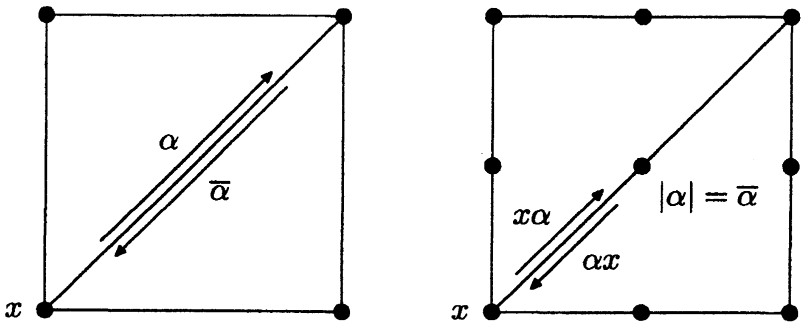

Figure 2: a graph G and its subdivision graph $S(G)$

Second one is the *line graph* $L(G)$. See Figure 3. A vertex of $L(G)$ is an edge of G;

$$V(L(G)) = E(G).$$

The vertices $|\alpha|$, $|\beta| \in V(L(G))$ are adjacent on $L(G)$ if and only if α, $\beta \in A(G)$ are adjacent on G;

$$A(L(G)) = \{\alpha\beta;\ \alpha, \beta \in A(G), \alpha \sim \beta\}.$$

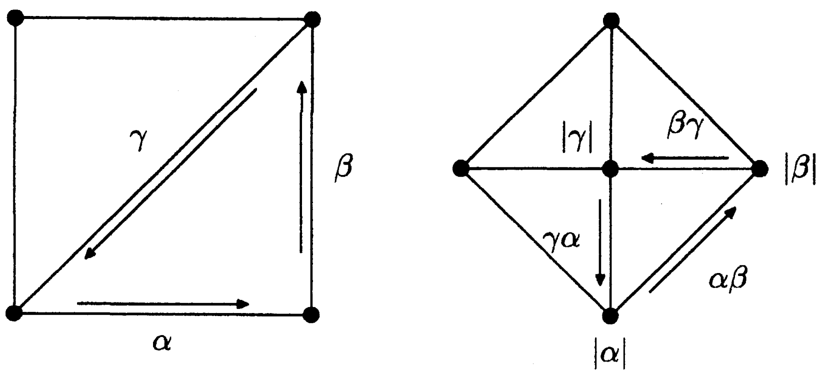

Figure 3: a graph G and its line graph $L(G)$

Last one is the *para-line graph* $P(G)$ introduced by Yu. Higuchi [2]. See Figure 4. To construct $P(G)$, we add two vertices x' and y' on each edges $xy \in A(G)$ in this order and then connect x' and y'. Moreover, if $o(\alpha) = o(\beta)$, then we connect $o(\alpha)'$ and $o(\beta)'$.

As a result, we have

$$P(G) = L(S(G)), \tag{1}$$

that is, the para-line graph is the line graph of the subdivision graph. It seems that this $P(G)$ has more information of G than $S(G)$ or $L(G)$.

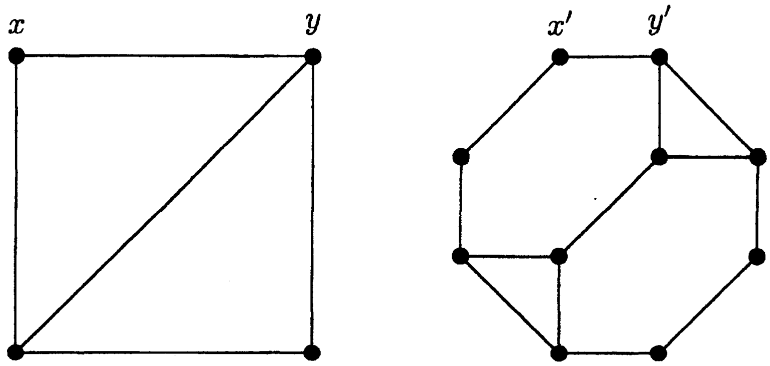

Figure 4: a graph G and its para-line graph $P(G)$

We have a natural question on the relation among the spectra of the four laplacians, Δ_G, $\Delta_{S(G)}$, $\Delta_{L(G)}$, $\Delta_{P(G)}$. Yu. Higuchi and T. Shirai gave the answer. In the next section, we review their results (See, [3, 7]).

2 Higuchi and Shirai's results

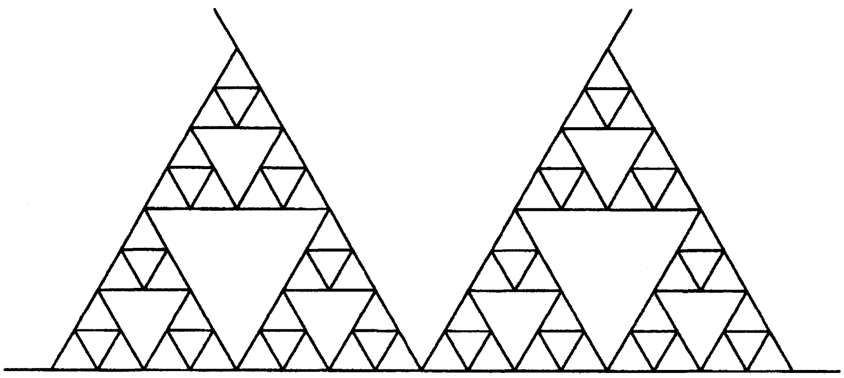

Figure 5: n-dim. infinitely extended Sierpinski lattice S_n

In the last symposium (Jul. 2001), T. Shirai talked about the spectrum of the infinitely extended Sierpinski lattice S_n [3, 7]. On the spectrum, the following theorem by Fukushima and Shima, and Teplyaev are known.

Theorem 2 (Fukushima and Shima (1992), Teplyaev (1998)).

$$\mathrm{Spec}\,(-\Delta_{S_n}) = \overline{\bigcup_{k=0}^{\infty}\left[g^{-k}\left(\frac{n+1}{2n}\right) \cup g^{-k}\left(\frac{n+3}{2n}\right)\right]} \cup \{\frac{n+1}{n}\}$$

Here, $g(x) = -2nx^2 + (n+3)x$.

They proved this theorem using approximation by finite lattices. Higuchi and Shirai gave new proof based on some relations between the spectra of an infinite regular graph and its line-graph, without any approximation. They proved Theorem 2 as a conclusion of the following four lemmas.

Lemma 3 (Shirai [6]). *Let G be a d-regular graph with $d \geq 3$. We have*

$$\mathrm{Spec}\,(-\Delta_{S(G)}) = \psi^{-1}(\mathrm{Spec}\,(-\Delta_G)) \cup \{1\}$$

Here, $\psi(x) = 2(2x - x^2)$.

Lemma 4 (Shirai [6]). *Let G be a d-regular graph with $d \geq 3$. We have*

$$\mathrm{Spec}\,(-\Delta_{L(G)}) = \frac{2}{2d-2}\mathrm{Spec}\,(-\Delta_G) \cup \{\frac{d+2}{d}\}.$$

Shirai proved Lemma 3 and Lemma 4 using the weak Weyl criterion on essential spectrum; He constructed a weak sequence for $-\Delta_{S(G)}$ from a eigenvector of $-\Delta_G$ and vice versa.

Lemma 5 (Shirai [6]). *Let G be d-regular with $d \geq 3$. We have*

$$\mathrm{Spec}\,(-\Delta_{P(G)}) = \phi^{-1}(\mathrm{Spec}\,(-\Delta_G)) \cup \{1\} \cup \{\frac{d+2}{d}\}.$$

Here, $\phi(x) = -dx^2 + (d+2)x$.

This Lemma 5 can be obtained from Lemmas 3 and 4 and the fact Eq. (1).

Lemma 6 (Higuchi and Shirai [3]). *Let S_n be n-dim. Sierpinski lattice. Then, there exists a $(n+1)$-regular graph G_n such that*

$$P(G_n) = G_n \qquad \text{and} \qquad S_n = L(G_n).$$

Outline of HS's proof of Theorem 2. By Lemmas 5 and 6, we have the equation of the set $\mathrm{Spec}\,(-\Delta_{G_n})$,

$$\mathrm{Spec}\,(-\Delta_{G_n}) = \phi^{-1}(\mathrm{Spec}\,(-\Delta_{G_n})) \cup \{1\} \cup \{\frac{n+3}{n+1}\}.$$

Since the map from $\mathrm{Spec}\,(-\Delta_{G_n})$ to RHS is a contraction map, there exists a unique solution of this equation and we can derive $\mathrm{Spec}\,(-\Delta_{G_n})$ exactly. Since $S_n = L(G_n)$, Lemmas 4 implies the desired result. For more detail, see Refs. [3, 7, 6]. □

Our goals in this report are (i) we give another simplest proof of Lemma 3 and Lemma 4 using supersymmetry; (ii) we extend these to magnetic Schrödinger case. We devote the next section to (i) and do the last section to (ii).

Remark 7. Shirai proved that 1 is always the infinitely degenerate eigenvalue of $-\Delta_{S(G)}$ and $(d+2)/d$ is always the infinitely degenerate eigenvalue of $-\Delta_{L(G)}$. Though we are interested in these eigenvalues, we omit the discussions on these (Remark 18).

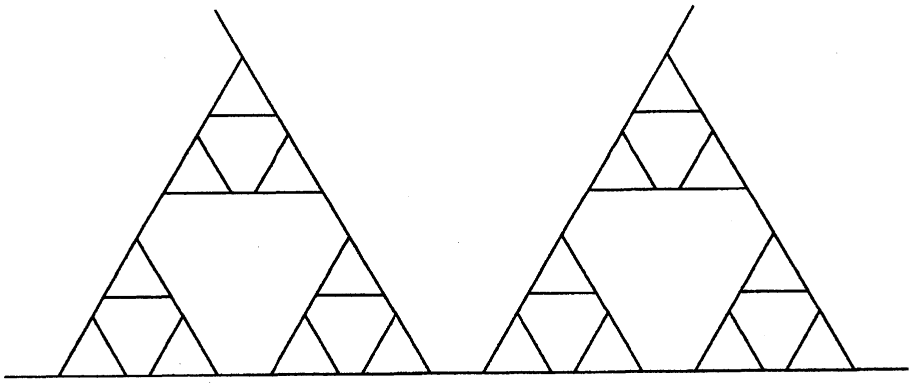

Figure 6: Sierpinski pre-lattice G_n for S_n

3 Supersymmetry

In this section, we give our new proof on the relations among the spectra of G, $S(G)$ and $L(G)$. We summarize the facts on supersymmetry. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and A be a densely defined closed linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Theorem 8 (Deift [1]). *We have*

$$\mathrm{Spec}\,(AA^*) \setminus \{0\} = \mathrm{Spec}\,(A^*A) \setminus \{0\}$$

with taking account of multiplicity.

Corollary 9 (I. Shigekawa [5]). *Let*

$$D = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{H}_1 \oplus \mathcal{H}_2$$

and

$$H = D^2 = \begin{pmatrix} A^*A & 0 \\ 0 & A^*A \end{pmatrix}.$$

Then, we have that

$$\mathrm{Spec}\,(H) \setminus \{0\} = \mathrm{Spec}\,(A^*A) \setminus \{0\} = \mathrm{Spec}\,(AA^*) \setminus \{0\}$$

and

$$\mathrm{Spec}\,(D) \setminus \{0\} = (\sqrt{\mathrm{Spec}\,(H)} \cup -\sqrt{\mathrm{Spec}\,(H)}) \setminus \{0\}.$$

In physics literatures, D is called a superchage and H is called a SUSY-Hamiltonian. We remark that we can ignore the condition, which A must be densely defined and closed, since all of our operators is bounded (Remark 1).

3.1 the spectra of bipartite graph

We start to prove a well-known fact on spectrum of graph using supersymmetry. Let G be bipartite, that is,

$$\begin{aligned} &V(G) = V_1 \cup V_2, \\ &V_1 \cap V_2 = \emptyset, \\ &x \not\sim y \quad \text{for all } x, y \in V_i \quad (i = 1, 2). \end{aligned}$$

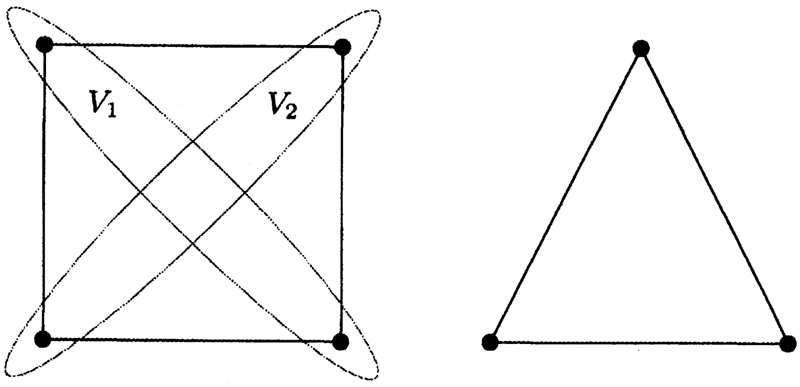

Figure 7: bipartite graph and non-bipartite graph

Lemma 10. *If G is bipartite, then* $\mathrm{Spec}\,(-\Delta_G)$ *is symmetric w.r.t. 1.*

Proof. We have $l^2(G) = l^2(V_1) \oplus l^2(V_2)$. Let ϕ_{12} be an operator from $l^2(V_1)$ to $l^2(V_2)$ defined by

$$(\phi_{12} f)(y) = \frac{1}{\deg(y)} \sum_{x \sim y} f(x).$$

Then, we have

$$\Delta_G + 1 = \begin{pmatrix} 0 & \phi_{12}^* \\ \phi_{12} & 0 \end{pmatrix}.$$

Thus, $\mathrm{Spec}\,(-\Delta_G - 1)$ is symmetric w.r.t. 0. □

Similarly, we define the operator $\phi_{21}$from $l^2(V_2)$ to $l^2(V_1)$ by

$$(\phi_{21} g)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} g(y).$$

Then we have $\phi_{21} = \phi_{12}^*$. These operators ϕ_{12} and ϕ_{21} are used in our new proofs in the following.

3.2 the spectra of subdivision graph

We consider the relation between G and $S(G)$.

Lemma 11 (SUSY version of Lemma 3). *For arbitrary graph G, we have*

$$\mathrm{Spec}\,(-\Delta_{S(G)}) = \psi^{-1}(\mathrm{Spec}\,(-\Delta_G)) \cup \{1\}$$

Here, $\psi(x) = 2(2x - x^2)$.

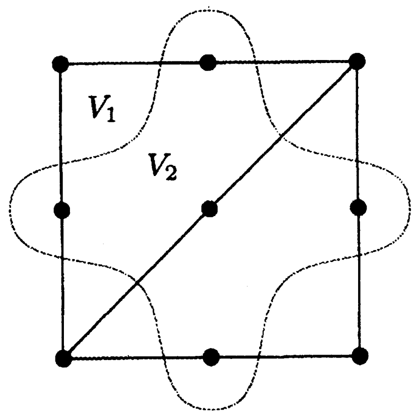

Figure 8: the subdivision graph as a bipartite graph

Proof. $S(G)$ is bipartite (See, Fig. 8). In fact, we can take $V_1 = V(G)$ and $V_2 = E(G)$;

$$V(S(G)) = V(G) \cup E(G) = V_1 \cup V_2.$$

Thus, we have

$$\Delta_{S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix}.$$

Moreover, we can see

$$(\Delta_{S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(\Delta_G + 2) & 0 \\ 0 & \phi_{12}\phi_{21} \end{pmatrix}.$$

Indeed, we can write ϕ_{12} and ϕ_{21} as follows:

$$(\phi_{12} f)(|\alpha|) = \frac{1}{2}[f(t(\alpha)) + f(o(\alpha))],$$
$$(\phi_{21} g)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} g(|\alpha|).$$

Therefore,

$$
\begin{aligned}
(\phi_{21}\phi_{12}f)(x) &= \frac{1}{\deg(x)} \sum_{\alpha\in A_x(G)} [\phi_{12}f](|\alpha|) \\
&= \frac{1}{\deg(x)} \sum_{\alpha\in A_x(G)} \frac{1}{2}[f(t(\alpha)) + f(o(\alpha))] \\
&= \frac{1}{2}\left(\frac{1}{\deg(x)}\left[\sum_{\alpha\in A_x(G)} f(t(\alpha))\right] + f(x)\right) \\
&= \frac{1}{2}(\Delta_G + 2)f(x)
\end{aligned}
$$

Thus, we obtain

$$
\mathrm{Spec}\,(\Delta_{S(G)} + 1) \setminus \{0\} = \pm\sqrt{\mathrm{Spec}\,(\frac{1}{2}(\Delta_G + 2)) \setminus \{0\}}.
$$

Thus, the spectral mapping theorem implies Lemma 1. □

Via supersymmetry, we can not see that 1 is an infinitely degenerate eigenvalue of $-\Delta_{S(G)}$. We need another discussion, but omit it here (Remark 18).

We remark that we do not need the regularity condition as in Lemma 3.

3.3 the spectra of line graph

We consider the relation between G and $L(G)$.

Lemma 12 (SUSY version of Lemma 4). *Let G be d-regular with $d \geq 3$. Then, we have*

$$
\mathrm{Spec}\,(-\Delta_{L(G)}) = \frac{2}{2d-2}\mathrm{Spec}\,(-\Delta_G) \cup \{\frac{d+2}{d}\}.
$$

Proof. We use same ϕ_{21} and ϕ_{12} as in Lemma 1 and we have

$$
\Delta_{S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix} \text{ on } l^2(V_1) \oplus l^2(V_2).
$$

Here, $V_1 = V(G)$, $V_2 = E(G)$. We can identify $E(G)$ and $V(L(G))$. (See, Fig. 9.) If G is d-regular, then $L(G)$ is $2d-2$-regular. Therefore, $l^2(L(G))$ and $l^2(V_2)$ is unitary equivalent through the unitary operator U defined by

$$
U : l^2(L(G)) \to l^2(V_2), \qquad Uf = \sqrt{d-1}\,f.
$$

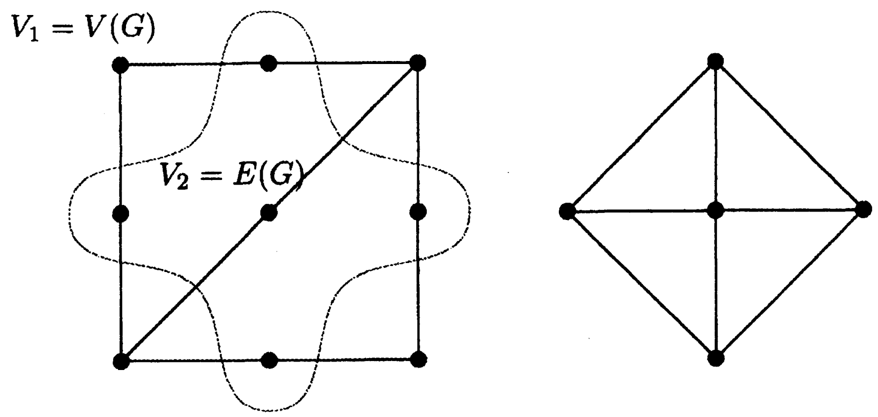

Figure 9: Identification between $E(G)$ and $V(L(G))$

Using this U, we obtain

$$(\Delta_{S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(\Delta_G + 2) & 0 \\ 0 & U\left[\frac{d-1}{d}(\Delta_{L(G)} + \frac{d}{d-1})\right]U^* \end{pmatrix}$$

by direct computations. Thus,

$$\mathrm{Spec}\left(\frac{1}{2}(\Delta_G + 2)\right) \setminus \{0\} = \mathrm{Spec}\left(\frac{d-1}{d}(\Delta_{L(G)} + \frac{d}{d-1})\right) \setminus \{0\}.$$

□

Via supersymmetry, we can not see that $(d+2)/d$ is an infinitely degenerate eigenvalue of $-\Delta_{L(G)}$. We need another discussion, but omit it here (Remark 18).

4 discrete magnetic Schrödinger operator

For simplicity, we assume that the transition probability on G is isotropic. We can remove this restriction.

We introduce the space of 1-forms (vector potentials) on graph G.

$$C^1(G) = \{\theta : A(G) \to \mathbf{R};\ \theta(\overline{\alpha}) = -\theta(\alpha)\}.$$

We define the discrete magnetic Schrödinger operator $H_{\theta,G}$ with a 1-form θ by

$$\begin{aligned} H_{\theta,G}f(x) &= \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} \left[e^{i\theta(\alpha)} f(t(\alpha)) - f(x)\right]. \\ &= \frac{1}{\deg(x)} \left[\sum_{\alpha \in A_x(G)} e^{i\theta(\alpha)} f(t(\alpha))\right] - f(x). \end{aligned}$$

Our problem is whether we can extend Lemma 11 and Lemma 12 for $H_{\theta,G}$.

Remark 13. In ordinary, $H_{\theta,G}$ is defined with the opposite sign. Then $H_{\theta,G}$ is non-negative. But, in this report, we want to compare it to the discrete Laplacian, so we choose this sign.

For later use, we introduce a quantity related to 1-form. Let C be an oriented cycle on G, *i.e.*,

$$C = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \subset A(G)$$

such that $t(\alpha_i) = o(\alpha_{i+1})$ $(\alpha_n = \alpha_0)$. For this cycle C and θ, we set

$$\Psi(\theta, C) = \sum_{\alpha \in C} \theta(\alpha).$$

We call this $\Psi(\theta, C)$ the magnetic flux through the cycle C.

4.1 the spectra of subdivision graph

Lemma 14 (magnetic case of Lemma 11). *Let G be an arbitrary graph. Assume that $\theta \in C^1(G)$ and $\theta_S \in C^1(S(G))$ satisfy that*

$$\theta(\alpha) = \theta_S(o(\alpha)|\alpha|) + \theta_S(|\alpha|t(\alpha)) \quad \textit{for all } \alpha \in A(G).$$

Then,

$$\mathrm{Spec}\,(-H_{\theta_S, S(G)}) = \psi^{-1}(\mathrm{Spec}\,(-H_{\theta,G})) \cup \{1\}$$

Here, $\psi(x) = 2(2x - x^2)$.

Proof. Let

$$(\phi_{12} f)(|\alpha|) = \frac{1}{2} \sum_{\beta \in \{\alpha, \bar{\alpha}\}} e^{i\theta_s(|\alpha| t(\beta))} f(t(\beta)),$$

$$(\phi_{21} g)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} e^{i\theta_s(x|\alpha|)} g(|\alpha|).$$

Then, by direct computations, we obtain that

$$H_{\theta_S, S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix},$$

$$(H_{\theta_S, S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(H_{\theta,G} + 2) & 0 \\ 0 & \phi_{12}\phi_{21} \end{pmatrix}.$$

This implies the desired result. $\square$

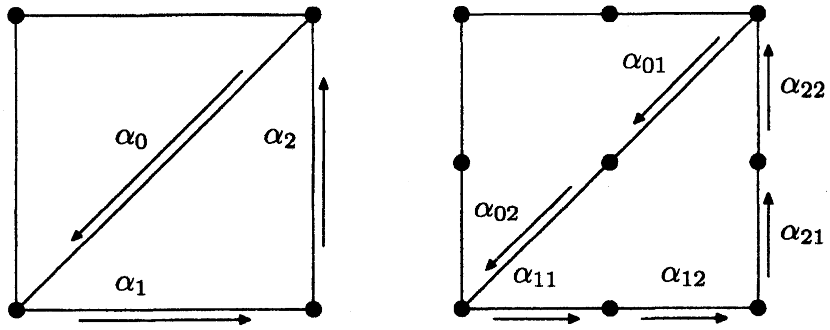

Figure 10: Same cycle on G and $S(G)$

Remark 15. The assumption of this Lemma 14 is natural. These θ and θ_S has same magnetic flux for same cycle. Let

$$C = \{\alpha_0, \alpha_1, \dots, \alpha_n\},$$
$$C_S = \{\alpha_{01}, \alpha_{02}, \alpha_{10}, \alpha_{11}, \dots, \alpha_{n,0}, \alpha_{n,1}\}.$$

See Figure 10. Then, we have

$$\begin{aligned}\Psi(\theta_S, C_S) &= \sum_{\alpha \in C_S} \theta_S(\alpha) = \sum_{i=0}^{n}(\theta_S(\alpha_{i,0}) + \theta_S(\alpha_{i,1})) \\ &= \sum_{i=0}^{n} \theta(\alpha_i) = \sum_{\alpha \in C} \theta(\alpha) = \Psi(\theta, C).\end{aligned}$$

Of course, via supersymmetry, we can not see that 1 is an infinitely degenerate eigenvalue of $-H_{\theta_s,S(G)}$. We need another discussion, but omit it here. (cf. Remark 18).

4.2 the spectra of line graph

Lemma 16 (magnetic case of Lemma 12 (Lemma 4)). *Let G be d-regular with $d \geq 3$. Assume that $\theta \in C^1(G)$, $\theta_S \in C^1(S(G))$, $\theta_L \in C^1(L(G))$ satisfy that*

$$\theta(\alpha) = \theta_S(o(\alpha)|\alpha|) + \theta_S(|\alpha|t(\alpha)) \quad \textit{for all } \alpha \in A(G),$$
$$\theta_L(\alpha\beta) = \theta_S(|\alpha|x) + \theta_S(x|\beta|) \quad \textit{for all } \alpha\beta \in A((L(G)).$$

Then

$$\mathrm{Spec}\,(-H_{\theta_L,L(G)}) = \frac{2}{2d-2}\mathrm{Spec}\,(-H_{\theta,G}) \cup \{\frac{d+2}{d}\}.$$

Proof. We use same identification between $E(G)$ and $V(L(G))$, $l^2(V_2)$ and $l^2(L(G))$ using U. Then, using same ϕ_{12} and ϕ_{21} in the proof of Lemma 14, we have

$$(-H_{\theta_s,S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(-H_{\theta,G} + 2) & 0 \\ 0 & U\left[\frac{d-1}{d}\left(-H_{\theta_L,L(G)} + \frac{d}{d-1}\right)\right]U^* \end{pmatrix}.$$

Thus, we can obtain the desired result. □

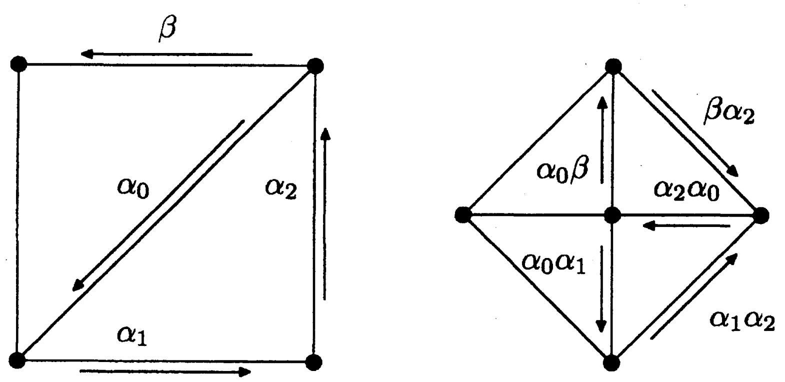

Figure 11: Same cycle on G and $L(G)$

Remark 17. The assumption of this Lemma 16 is natural. These θ and θ_S has same magnetic flux for same cycle. Let

$$C = \{\alpha_0, \alpha_1, \ldots, \alpha_n\},$$
$$C_L = \{\alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_n\alpha_1\}.$$

Then, it holds that $\Psi(\theta_L, C_L) = \Psi(\theta, C)$. The pair of $C = \{\alpha_0, \alpha_1, \alpha_2\}$ and $C_L = \{\alpha_0\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_0\}$ in Figure 11 is an example. But, $L(G)$ maybe has some cycles, which has no corresponding cycles on G. The cycle $\{\alpha_2\alpha_0, \alpha_0\beta, \beta\alpha_2\}$ in Figure 11 is an example. These cycles have zero magnetic flux.

Remark 18. As in Remark 7, though we omit the discussions on the eigenvalue 1 of $-H_{\theta,G}$ and the eigenvalue $(d+2)/d$ of $-H_{\theta_L,L(G)}$, these are corresponding to $\ker\phi_{12}$ and $\ker\phi_{21}$. In other words, these eigenvalues are zero-modes in SUSY context. So, we must investigate these states in detail [4].

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