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Existence of Enhanced Binding in Quantum Field Models

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Abstract

Some results on existence of enhanced binding in a class of quantum field models are presented.

1 Introduction

In a quantum system whose Hamiltonian is described by a self-adjoint operator H bounded from below, a ground state is defined to be an eigenvector of H with eigenvalue equal to the infimum of the spectrum of H, the lowest energy of the quantum system. Physically the existence of a ground state ensures a stability or the persistency of the quantum system under consideration. But, generally speaking, it is not trivial if a quantum system has a ground state. It turns out that it is one of the most fundamental problems in mathematical analysis of quantum systems to prove or disprove the existence of a ground state.

The Hamiltonian H may be divided into two parts H_0, the unperturbed part, and H_I, the perturbation part: $H = H_0 + H_I$. In quantum field theory, one usually assumes that H_0 has a ground state and tries to prove the existence of a ground state of H. But, without that assumption, H may have a ground state. If such a structure exists, then we say that *enhanced binding* (with respect to ground state) exists or occurs in the quantum field system under consideration.

The phenomenon of enhanced binding, if it occurs, may be regarded as one of the evidences suppporting the view point that quantum fields are more fundamental objects underlying the material world. From this point of view as well as a purely mathematical one it is interesting to clarify whether or not enhanced binding *indeed* occurs in models of a quantum system —typically a system of nonrelativistic quantum particles— coupled to a quantum field.

The problem of enhanced binding was first discussed by Hiroshima and Spohn [11]. They discussed the Pauli-Fierz model in nonrelativistic quantum electrodynamics in the

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dipole approximation and proved that, under suitable hypotheses, enhanced binding occurs for large coupling constants. Hainzl, Vougalter and Vugalter [10] considered the Pauli-Fierz model without the dipole approximation showing that it has enhanced binding for small coupling constants. The results and the methods in [10] have been extended to the Pauli-Fierz model with spin [7, 8](cf. also [9]).

In a previous paper [6] the enhanced binding problem was considered for a general class of quantum field models, called the generalized spin-boson (GSB) model which describes an abstract quantum system coupled linearly to a Bose field [3, 4, 5], and proved, under suitable hypotheses, the existence of enhanced binding for a region of coupling constants. The GSB model was extended to a more general one in [2], whose Hamiltonian is obtained by adding quadratic self-interaction terms of the Bose field to the Hamiltonian of the GSB model, and it was shown that results similar to those in [2] hold also in the extended GSB model.

In this paper we consider a slightly more general model than the GSB model and show that, under suitable hypotheses, enhanced binding occurs in this model too.

The present paper is organized as follows. Section 2 is a preliminary section which recalls basic objects and elementary facts in the theory of the abstract boson Fock space. In Section 3 we describe the model considered in the present paper. The main theorems are stated in Section 4. The last section is devoted to sketches of proofs of them.

2 Bose fields

We denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$ respectively, where we use the convention that the inner product is antilinear (resp. linear) in the first (resp. second) variable. We sometimes omit the subscript $\mathcal{X}$ in $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$ if there is no danger of confusion.

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. For a subspace $D \subset D(T)$, $T|D$ denotes the restriction of T to D. If T is densely defined, then the adjoint of T is denoted T^*. For linear operators S and T on a Hilbert space, $D(S+T) := D(S) \cap D(T)$ unless otherwise stated.

For each complex Hilbert space $\mathcal{X}$, the boson Fock space $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$ over $\mathcal{X}$ is defined by

$$\mathcal{F}_{\mathrm{b}}(\mathcal{X}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathcal{X},$$

where $\otimes_{\mathrm{s}}^{n} \mathcal{X}$ denotes the n-fold symmetric tensor product of $\mathcal{X}$ with $\otimes_{\mathrm{s}}^{0} \mathcal{X} := \mathbf{C}$ (the set of complex numbers).

The annihilation operator $a(f)$ $(f \in \mathcal{X})$ on $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$ is defined to be a densely defined closed linear operator such that, for all $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\psi)^{(0)} = 0$ and

$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n \left(f \otimes \psi^{(n-1)} \right), \quad n \geq 1,$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{X}$. The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$ $(g \in \mathcal{X})$ obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle_{\mathcal{X}}, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{X}$ on the dense subspace

$$\mathcal{F}_0(\mathcal{X}) := \{\psi \in \mathcal{F}_{\rm b}(\mathcal{X}) | \text{there exists a number } n_0 \text{ such that} \psi^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[X, Y] := XY - YX$.

Let

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}, \quad f \in \mathcal{X},$$

which is called the Segal field operator. It is shown that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_0(\mathcal{X})$ [12, §X.7]. *We denote its closure by the same symbol* $\phi(f)$. The "conjugate momentum" of $\phi(f)$ is defined by

$$\pi(f) := \phi(if), \quad f \in \mathcal{X}.$$

We have

$$[\phi(f), \pi(g)] = i\mathrm{Im}\,(i\langle f, g\rangle_{\mathcal{X}})\,.$$

For every symmetric operator S on $\mathcal{X}$, one can define a closed symmetric operator $d\Gamma(S)$, called the second quantization of S, by

$$d\Gamma(S) := \oplus_{n=0}^{\infty} S^{(n)},$$

with $S^{(0)} = 0$ and $S^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\breve{S}} \otimes \cdots \otimes I\right) \Big| \otimes_{\rm alg}^{n} D(S),$$

where I denotes identity and $\otimes_{\rm alg}^{n}$ algebraic tensor product. If S is self-adjoint, then so is $d\Gamma(S)$.

3 Definition of the model

We consider a model of an abstract quantum system S coupled to an N-component Bose field over $\mathbf{R}^d$ ($d, N \in \mathbf{N}$). We denote the Hilbert space of the system S by $\mathcal{H}$, which is taken to be an arbitrary separable complex Hilbert space. In concrete realizations, S may be a system of nonrelativistic quantum particles or a quantum field system.

The one-particle Hilbert space of the Bose field is taken to be

$$\mathcal{M} := \oplus^N L^2(\mathbf{R}^d),$$

the N direct sum of $L^2(\mathbf{R}^d)$. Then the Hilbert space for the Bose field is given by the Fock space $\mathcal{F}_{\rm b}(\mathcal{M})$ over $\mathcal{M}$.

Let ω be a Borel measurable function on $\mathbf{R}^d$ such that $0 < \omega(k) < \infty$ for almost everywhere (a.e.) $k \in \mathbf{R}^d$ with respect to the Lebesgue measure on $\mathbf{R}^d$. Physically ω denotes a dispersion relation of a boson. The function ω defines a multiplication operator on $L^2(\mathbf{R}^d)$, which is nonnegative, injective and self-adjoint. We denote it by the same symbol.

We define an operator

$$\hat{\omega} := \oplus^N \omega$$

acting in $\mathcal{M}$.

The Hilbert space of the coupled system of S and the Bose field is given by the tensor product

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b(\mathcal{M}).$$

Let A be a self-adjoint operator on $\mathcal{H}$, which denotes physically the Hamiltonian of the quantum system S.

The Hamiltonian of the model we consider in the present paper is defined by

$$H := A \otimes I + I \otimes d\Gamma(\hat{\omega}) + \sum_{j=1}^{J} B_j \otimes \phi(g_j) + \sum_{j=1}^{J} K_j \otimes \pi(h_j),$$

where B_j $(j = 1, \cdots, J;\ J \in \mathbb{N})$ is a symmetric operator on $\mathcal{H}$ such that $\cap_{j=1}^{J} D(B_j)$ is dense in $\mathcal{H}$, K_j $(j = 1, \cdots, J)$ is a bounded self-adjoint operator on $\mathcal{H}$ and $g_j, h_j \in \mathcal{M}$, $j = 1, \cdots, J$.

Remark 1 The case where $h_j = 0$ or $K_j = 0$ $(j = 1, \cdots, J)$ is the original GSB model [3]:

$$H_{\mathrm{GSB}} := A \otimes I + I \otimes d\Gamma(\hat{\omega}) + \sum_{j=1}^{J} B_j \otimes \phi(g_j).$$

The existence of ground states of H_{GSB} with $N = 1$ was discussed in [3] under the assumption that A has a ground state (cf. also [4] for further extensions). The problem of enhanced binding in H_{GSB} was considered in [6]. For the absence of ground states of H_{GSB}, see [5].

4 Main results

For a self-adjoint operator L on a Hilbert space, we denote its spectrum (resp. essential spectrum) by $\sigma(L)$ (resp. $\sigma_{\mathrm{ess}}(L)$).

Definition 1 Let L be a self-adjoint operator on a Hilbert space bounded from below and set

$$E_0(L) := \inf \sigma(L),$$

which is called the *lowest energy* of L. We say that L has a ground state if $E_0(L)$ is an eigenvalue of L. In that case, each non-zero vector in $\ker(L - E_0(L))$ is called a *ground state* of L.

To state the main results of this paper, we formulate additional hypotheses. For this purpose, we first recall an important notion on commutativity of self-adjoint operators:

Definition 2 We say that two self-adjoint operators S_1 and S_2 on a Hilbert space strongly commute (or S_1 strongly commutes with S_2) if their spectral measures commute.

A family of self-adjoint operators $\{S_j\}_{j=1}^{n}$ on a Hilbert space is said to be strongly commuting if S_j strongly commutes with S_l for all $j, l = 1, \cdots, n$ with $j \neq l$.

In what follows, we assume that A is of the form

$$A = A_0 + A_1$$

with A_0 a nonnegative self-adjoint operator and A_1 a symmetric operator on $\mathcal{H}$.

Hypothesis (I) g_j, $g_j/\omega^{3/2}, h_j, h_j/\omega \in \mathcal{M}$ $(j = 1, \cdots, J)$ and

$$\langle g_j(k), g_l(k)\rangle_{\mathbf{C}^N}, \langle g_j(k), h_l(k)\rangle_{\mathbf{C}^N} \in \mathbf{R}, \text{ a.e.} k \in \mathbf{R}^d \ (j, l = 1, \cdots, J).$$

Remark 2 Hypothesis (I) implies the following (i) and (ii):

(i) the set $\{\phi(ig_j/\omega)\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators and each $\phi(ig_j/\omega)$ strongly commutes with each $\pi(h_l)$ $(j, l = 1, \cdots, J)$.

(ii)

$$[\phi(g_j), \pi(h_l)] = i\langle g_j, h_l\rangle_{\mathcal{M}}$$

on $\mathcal{F}_0(\mathcal{M})$.

Hypothesis (II) The operator A_1 is A_0-bounded, i.e., $D(A_0) \subset D(A_1)$ and there exist constants $a, b \geq 0$ such that, for all $u \in D(A_0)$,

$$\|A_1 u\|_{\mathcal{H}} \leq a\|A_0 u\|_{\mathcal{H}} + b\|u\|_{\mathcal{H}}.$$

Hypothesis (III) The operator A_0 strongly commutes with each B_j $(j = 1, \cdots, J)$ and

$$D(A_0) \subset \cap_{j,l=1}^J D(B_j B_l).$$

Moreover, there exist constants $c_j, d_j \geq 0$ such that, for all $u \in D(A_0^{1/2})$,

$$\|B_j u\|_{\mathcal{H}} \leq c_j\|A_0^{1/2} u\|_{\mathcal{H}} + d_j\|u\|_{\mathcal{H}} \quad (j = 1, \cdots, J).$$

Hypothesis (IV) The set $\{B_j\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators.

Hypothesis (V) $D(A_0) \subset \cap_{j=1}^J D(B_j A_1) \cap D(A_1 B_j)$ and $[B_j, A_1]|D(A_0)$ is bounded $(j = 1, \cdots, J)$. We denote the operator norm of $[B_j, A_1]$ by $\|[B_j, A_1]\|$.

We introduce an operator

$$R_B := \frac{1}{2}\sum_{j,l=1}^J \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle_{\mathcal{M}} B_j B_l.$$

and define

$$A_{\text{ren}} := A - R_B.$$

Under Hypotheses (I)–(III), we have $D(A_{\text{ren}}) = D(A_0)$.

Hypothesis (VI) The operator A_{ren} is self-adjoint and bounded from below.

One can prove the following fact:

Theorem 3 *Assume Hypotheses (I)–(VI). Then H is self-adjoint and bounded from below.*

We set

$$\omega_0 := \text{ess.}\inf_{k \in \mathbf{R}^d} \omega(k),$$

where ess. inf means essential infimum.

Theorem 4 *Assume Hypotheses (I)–(VI). Suppose that*

$$\{\omega(k) | k \in \mathbf{R}^d\} = [\omega_0, \infty) \tag{4.1}$$

Then the following (i) and (ii) hold.

(i) If $\omega_0 > 0$, then

$$[E_0(H) + \omega_0, \infty) \subset \sigma_{\mathrm{ess}}(H).$$

(ii) If $\omega_0 = 0$, then

$$\sigma(H) = [E_0(H), \infty).$$

To establish an existence theorem of a ground state of H *without* the assumption that A has a ground state, we need additional conditions.

Hypothesis (VII) The function ω is continuous on $\mathbf{R}^d$ with

$$\lim_{|k| \to \infty} \omega(k) = \infty$$

and there exist constants $\gamma > 0$ and $c_0 > 0$ such that

$$|\omega(k) - \omega(k')| \leq c_0 |k - k'|^\gamma (1 + \omega(k) + \omega(k')), \quad k, k' \in \mathbf{R}^d.$$

For $s \geq 0$, we introduce constants $C_s(g), D_s(h)$ $(g := (g_1, \cdots, g_J), h := (h_1, \cdots, h_J))$ by

$$\begin{aligned} C_s(g) &:= \sqrt{2} \sum_{j=1}^{J} \|[B_j, A_1]\| \left\| \frac{g_j}{\omega^s} \right\|_{\mathcal{M}}, \\ D_s(h) &:= \sqrt{2} \sum_{j=1}^{J} \|K_j\| \left\| \frac{h_j}{\omega^s} \right\|_{\mathcal{M}}. \end{aligned}$$

provided that $g_j/\omega^s \in \mathcal{M}$ and $h_j/\omega^s \in \mathcal{M}$ $(j = 1, \cdots, J)$ respectively. We define constants F_α $(\alpha = 1, 2, 3)$ by

$$F_1 := C_1(g) + D_0(h), \quad F_2 := C_2(g) + \frac{1}{2} D_1(h), \quad F_3 := C_{3/2}(g) + D_{1/2}(h).$$

We set

$$\Sigma(A_{\mathrm{ren}}) := \inf \sigma_{\mathrm{ess}}(A_{\mathrm{ren}}).$$

Generally speaking, the existence of a ground state of H may depend on whether ω_0 is positive or zero [5]. We first state a result on the existence of enhanced binding in the case $\omega_0 > 0$.

Theorem 5 (Enhanced binding in the case $\omega_0 > 0$). *Consider the case $\omega_0 > 0$. Assume Hypotheses (I)–(VII) and that*

$$\Sigma(A_{\rm ren}) - E_0(A_{\rm ren}) > \omega_0 + \frac{1}{2}F_3^2 + F_1. \tag{4.2}$$

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \omega_0)$. In particular, H has has a ground state.

Remark 3 Condition (4.2) implies that $E_0(A_{\rm ren})$ is a discrete eigenvalue of $A_{\rm ren}$ and hence $A_{\rm ren}$ has a finite number of ground states. But A *does not necessarily have a ground state.*

Corollary 6 *Under the assumption of Theorem 5 and condition (4.1),*

$$\sigma_{\rm ess}(H) = [E_0(H) + \omega_0, \infty).$$

Theorem 7 (Enhanced binding in the case $\omega_0 = 0$). *Consider the case $\omega_0 = 0$. Assume Hypotheses (I)–(VII) with*

$$g_j/\omega^2 \in \mathcal{M}, \quad j = 1, \cdots, J$$

in addition. Suppose that

$$\Sigma(A_{\rm ren}) - E_0(A_{\rm ren}) > \frac{1}{2}F_3^2 + F_1. \tag{4.3}$$

and

$$\frac{F_1^2}{[\Sigma(A_{\rm ren}) - E_0(H)]^2} + \left\{\frac{2F_1^2}{[\Sigma(A_{\rm ren}) - E_0(H)]^2} + 1\right\}\frac{1}{2}F_2^2 < 1. \tag{4.4}$$

Then H has a ground state.

Remark 4 In Theorems 5 and 7, *the existence of a ground state of A is not assumed.*

5 Proofs of the main theorems

We give only sketches of proofs of the main theorems stated in the preceding section.

5.1 Proof of Theorem 3

We introduce a unitary operator

$$U := \prod_{j=1}^{J} e^{-iB_j \otimes \phi(ig_j/\omega)}.$$

Let

$$H_0 := A_{\rm ren} \otimes I + I \otimes d\Gamma(\hat{\omega}),$$

$$V_1 := U(A_1 \otimes I)U^{-1} - A_1 \otimes I, \quad V_2 := \sum_{j=1}^{J} \left(U(K_j \otimes I)U^{-1}\right) I \otimes \pi(h_j).$$

and

$$\widetilde{H} := H_0 + V_1 + V_2.$$

Lemma 8 *Assume Hypotheses (I)–(VI). Then* $UD(H_0) = D(H_0)$ *and, for all* $\Psi \in D(H_0)$,

$$UHU^{-1}\Psi = \widetilde{H}\Psi.$$

Proof. Similar to the proof of [6, Lemma 3.7]. ∎

Using [6, Lemma 3.10] and the well known estimates

$$\|a(f)\Psi\| \leq \left\|\frac{f}{\sqrt{\omega}}\right\|_{\mathcal{M}} \|d\Gamma(\hat{\omega})^{1/2}\Psi\|,$$

$$\|a(f)^*\Psi\| \leq \left\|\frac{f}{\sqrt{\omega}}\right\|_{\mathcal{M}} \|d\Gamma(\hat{\omega})^{1/2}\Psi\| + \|f\|_{\mathcal{M}}\|\Psi\|$$

holding for all $\Psi \in D\left(d\Gamma(\hat{\omega})^{1/2}\right)$ and $f, f/\sqrt{\omega} \in \mathcal{M}$, one can easily see that V_1 and V_2 are infinitesimally small with respect to H_0. Hence, by the Kato-Rellich theorem, $\widetilde{H}$ is self-adjoint with $D(\widetilde{H}) = D(H_0)$ and bounded from below. By this fact and Lemma 8, H is self-adjoint with $D(H) = D(H_0)$ and bounded from below.

5.2 Proof of Theorem 4

This follows from an application of [1, Theorem 3.3].

5.3 Proofs of Theorems 5 and 7

By Theorem 3 and Lemma 8, it is sufficient to prove that $\widetilde{H}$ has a ground state. One sees that the methods developed in [6] work in the present case too (in [6], $\widetilde{H}$ with $V_2 = 0$ is considered). This is due to the fact that the new perturbation term V_2 has properties similar to those of V_1, e.g.,

$$\|V_2\Psi\| \leq D_{1/2}\|I \otimes d\Gamma(\hat{\omega})^{1/2}\Psi\| + \frac{1}{2}D_0\|\Psi\|, \quad \Psi \in D\left(I \otimes d\Gamma\left(\hat{\omega}\right)^{1/2}\right),$$

$$[V_2, I \otimes a(f)]\Phi = -\frac{i}{\sqrt{2}}\sum_{j=1}^{J} U(K_j \otimes I)U^{-1}\langle f, h_j\rangle_{\mathcal{M}}\Phi, \quad \Phi \in D(I \otimes N_b),$$

where $N_b := d\Gamma(I)$ is the number operator on $\mathcal{F}_b(\mathcal{M})$. It turns out that we need only to shift the constants $c_s(g)$ $(s = 1, 3/2, 2)$ used in Theorems 2.2 and 2.3 in [6], which yields conditions (4.2) –(4.4) in the present context.

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