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# A Variation of McShane's identity for punctured surface bundles

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## 1 Motivation from knot theory - cusp shape -

Let $K \subset S^3$ be a hyperbolic knot and $\rho : \pi_1(S^3 - K) \to \mathrm{PSL}(2, \mathbb{C})$ the holonomy representation. We may assume:

$$\rho(m) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(l) = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix},$$

for some complex number $\omega$, where $m$ and $l$ are the longitude and the meridian of $K$, respectively. Then the quotient of the horoball $\{(z,t) \in \mathbb{H}^3 \,|\, t \geq t_0\}$ with $t_0 > 1$ by the subgroup $\langle \rho(m), \rho(l) \rangle$ is identified with $N(K) - K$, where $N(K)$ is a regular neighborhood of $K$. The torus $\partial N(K)$ has an Euclidean structure and is called the *cusp torus*, and this Euclidean torus (modulo scale) is isomorphic to $\mathbb{C}/\langle 1, \omega \rangle$. We call the complex number $\omega$ the *modulus* of the cusp torus of $K$, or the *cusp shape* of $K$, and denote it by Modulus($K$). Since the complete hyperbolic structure on $S^3 - K$ is unique by Mostow's rigidity theorem, the cusp shape is a topological invariant of $K$.

**Example 1.1.** For a sequence of integers $(a_1, a_2, \cdots, a_n)$, let $S[a_1, a_2, \cdots, a_n]$ be the 2-bridge knot of type

$$\frac{q}{p} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}.$$

Then we have the following by using SnapPea [W]:

$$\begin{aligned} \mathrm{Modulus}(S[2,2]) &= 2\sqrt{3}i, \\ \mathrm{Modulus}(S[2,2,2,2]) &= 7.260774402783i, \\ \mathrm{Modulus}(S[2,2,2,-2]) &= 5.5768014361792 + 5.9133486946719i. \end{aligned}$$

The fact that the cusp shapes of the first two knots in the above example are pure imaginary reflects the amphicheirality of the knots. In fact, the following observation had been given by R. Riley [R].

**Proposition 1.2.** *If a hyperbolic knot $K$ is amphicheiral, then the cusp shape of $K$ is pure imaginary.*

It was proved by Nimershiem [N] that the cusp shapes of cusped hyperbolic manifolds form a dense subset of the moduli space of the Euclidean tori.

The Main Theorem 4.2 in this paper gives a formula of the cusp shape of a hyperbolic fibered knot in terms of the complex translation lengths of essential simple loops on the fiber surface.

# 2 McShane's identity

In this section we recall McShane's identity proved in [McS1] and [McS2]. Let $F$ be an orientable surface of finite type with at least one puncture, $p$. By $\mathcal{S}$ we denote the set of the isotopy classes of essential (unoriented) simple loops in the punctured surface $F$. A simple arc $\delta$ in $F$ with both ends in a puncture $p$ is said to be *essential* if it does not bound a monogon (i.e., a disk with one point removed from its boundary). By $\Delta$ (resp. $\vec{\Delta}$) we denote the set of the isotopy classes of unoriented (resp. oriented) essential simple arcs in $F$ with both ends in $p$. We shall abuse notation to denote a simple loop or an arc and its isotopy class by the same symbol. For each essential arc $\delta \in \Delta$ (or $\delta \in \vec{\Delta}$) there is a unique (up to isotopy) unordered pair of simple loops $\alpha(\delta)$ and $\beta(\delta)$ such that $\alpha(\delta) \cup \beta(\delta)$ bounds a punctured annulus containing $\delta$ (cf. [McS2, Proposition 1]). These loops determine a pair of elements of $\mathcal{S} \cup \mathcal{P}$, where $\mathcal{P}$ is the set of the isotopy classes of peripheral simple loops in $F$. We note the following facts.

1. If $F$ is a punctured torus, then $\alpha(\delta) = \beta(\delta) \in \mathcal{S}$. Otherwise, $\alpha(\delta) \neq \beta(\delta)$.
2. One of $\alpha(\delta)$ and $\beta(\delta)$ belongs to $\mathcal{P}$ if and only if $\delta$ bounds a once-punctured monogon.

Now assume that $F$ is endowed with a complete hyperbolic structure and let $\rho_0 : \pi_1(F) \to \mathrm{PSL}(2, \mathbb{R})$ be the holonomy. For an element $\delta \in \Delta$ (or $\delta \in \vec{\Delta}$), set

$$h_{\rho_0}(\delta) := \frac{1}{1 + e^{\frac{1}{2}(l_{\rho_0}(\alpha(\delta)) + l_{\rho_0}(\beta(\delta)))}},$$

where $l_{\rho_0}(\alpha)$ denotes the length of $\alpha$ with respect to the hyperbolic metric. Then the following theorem was proved by McShane (see [McS2, Theorem 2]).

**Theorem 2.1 (McShane [McS2]).** *For a punctured hyperbolic surface $F$ with holonomy $\rho_0$,*

$$\sum_{\delta \in \Delta} h_{\rho_0}(\delta) = \frac{1}{2}.$$

**Example 2.2.** (1) Suppose $F$ is a three-times punctured sphere. Then $\Delta$ consists of a single arc $\delta$, and both $\alpha(\delta)$ and $\beta(\delta)$ are peripheral. Hence we have

$$\sum_{\delta \in \Delta} h_{\rho_0}(\delta) = h_{\rho_0}(\delta) = \frac{1}{1 + e^{\frac{1}{2}(0+0)}} = \frac{1}{2}.$$

(2) Suppose $F$ is a once-punctured torus. Then $\alpha(\delta) = \beta(\delta)$ for any $\delta$ and McShane's identity is equivalent to the following identity.

$$\sum_{\gamma \in \mathcal{S}} \frac{1}{1 + e^{l_{\rho_0}(\gamma)}} = \frac{1}{2}. \tag{1}$$

The following interpretation of the above identity was brought to us from T. Jorgensen. Suppose the hyperbolic once-punctured torus contains a very short simple closed geodesic $\gamma_0$. Then $F$ contains a very long tube with core $\gamma_0$ and any simple closed geodesic $\gamma$ different from $\gamma_0$ must pass through the long tube and hence it is very long. Hence

$$\begin{aligned}\sum_{\gamma \in \mathcal{S}} \frac{1}{1 + e^{l_{\rho_0}(\gamma)}} &= \frac{1}{1 + e^{l_{\rho_0}(\gamma_0)}} + \sum_{\gamma \in \mathcal{S} - \{\gamma_0\}} \frac{1}{1 + e^{l_{\rho_0}(\gamma)}} \\ &\sim \frac{1}{1+e^0} + \frac{1}{1+e^\infty} + \frac{1}{1+e^\infty} + \cdots \\ &= \frac{1}{2} + 0 + 0 + \cdots = \frac{1}{2}.\end{aligned}$$

In the remainder of this section, we explain an idea for the proof of McShane's identity. Let $\mathcal{G}$ be the set of the oriented complete simple geodesics in $F = \mathbb{H}^2/\rho_0(\pi_1(F))$ emanating from the puncture $p$. Then the set $\vec{\Delta}$ is regarded as a subset of $\mathcal{G}$. Let $\widetilde{\mathcal{G}}$ be the set of oriented complete geodesics in $\mathbb{H}^2$ emanating from $\infty$ which projects to a simple geodesic in $F$. Then $\widetilde{\mathcal{G}}$ is identified with a subset of $\mathbb{R} = \partial\mathbb{H}^2 - \{\infty\}$ by associating each element $\widetilde{\mu} \in \widetilde{\mathcal{G}}$ with its endpoint $z_{\rho_0}(\widetilde{\mu}) \in \mathbb{R}$. This induces an identification of $\mathcal{G}$ with a subset of the circle $S^1_p := \mathbb{R}/\langle\rho_0(m)\rangle = \mathbb{R}/\mathbb{Z}$. Here $S^1_p$ inherits the standard metric from that of $\mathbb{R}$. In particular, the total length of $S^1_p$ is 1. Then the following result has been proved by McShane [McS2, Theorem 4 and Proposition 3].

**Proposition 2.3.** *(1) $\vec{\Delta}$ consists of the isolated points of $\mathcal{G}$, and $\mathcal{G} - \vec{\Delta}$ is a Cantor set of measure 0.*

*(2) For $\delta \in \vec{\Delta}$, let $J(\delta)$ be the maximal open interval in $S^1_p$ such that $J(\delta) \cap \mathcal{G} = \{\delta\}$. Then, generically, the two boundary points of $J(\delta)$ correspond to the elements of $\mathcal{G}$ which spiral to the oriented simple closed geodesics $\alpha(\delta)$ and $\beta(\delta)$, respectively. Here $\alpha(\delta)$ and $\beta(\delta)$ are oriented so that they are homologous to $\delta$ in the annulus obtained from the punctured annulus bounded by $\alpha(\delta) \cup \beta(\delta)$ through one point compactification. In the special case when $\alpha(\delta)$ or $\beta(\delta)$ is a peripheral circle around a puncture $q$, then the corresponding boundary point of $J(\delta)$ is a simple oriented geodesic joining $p$ to $q$.*

*(3) The length of $J(\delta)$ is equal to $h_{\rho_0}(\delta)$ for every $\delta \in \vec{\Delta}$.*

McShane's original identity [McS2, Theorem 2] is obtained from the above proposition as follows. Since the measure of $\mathcal{G} - \vec{\Delta}$ is 0, the length of $S^1_p$ is equal to the infinite sum of the lengths of $J(\delta)$ where $\delta$ runs over all elements of $\vec{\Delta}$.

Hence

$$1 = \sum_{\delta \in \vec{\Delta}} (\text{length of } J(\delta)) = \sum_{\delta \in \vec{\Delta}} h_{\rho_0}(\delta) = 2 \sum_{\delta \in \Delta} h_{\rho_0}(\delta).$$

By a similar argument, we obtain the following corollary.

**Corollary 2.4.** *Let $\mu_1$ and $\mu_2$ be elements of $\mathcal{G} - \vec{\Delta} \subset S^1_p$. Let $[\mu_1, \mu_2]$ be the interval of $S^1_p$ such that $\partial[\mu_1, \mu_2] = \mu_2 - \mu_1$ with respect to the orientation induced from the natural orientation of $S^1_p$. Then the length of $[\mu_1, \mu_2]$ is equal to*

$$\sum_{\delta \in [\mu_1, \mu_2] \cap \vec{\Delta}} h_{\rho_0}(\delta).$$

# 3 Quasifuchsian groups and complex translation lengths

**Quasifuchsian representations.** Let $F$ be an orientable complete hyperbolic surface of finite type with at least one puncture. Let $\rho_0 : \pi_1(F) \to \mathrm{PSL}(2, \mathbb{R})$ be the holonomy representation and $\Gamma_0 := \rho_0(\pi_1(F))$ the holonomy group. A representation $\rho : \pi_1(F) \to \mathrm{PSL}(2, \mathbb{C})$ is said to be *type-preserving* if $\rho$ sends the peripheral elements to parabolic transformations and $\rho$ is irreducible. Two representations $\rho$ and $\rho'$ are said to be *equivalent* if $\rho'$ is equal to the composition of $\rho$ and an inner-automorphism of $\mathrm{PSL}(2, \mathbb{C})$. A type-preserving representation $\rho$ is said to be *fuchsian* if it is equivalent to a discrete faithful representation into $\mathrm{PSL}(2, \mathbb{R})$. If $\rho$ is fuchsian, the limit set of the image of $\rho$ is a round circle. A type-preserving representation $\rho$ is said to be *quasifuchsian* if it is quasiconformally equivalent to a fuchsian representation. Let $\mathcal{QF}$ (resp. $\mathcal{F}$) be the space of the equivalence classes of quasifuchsian (resp. fuchsian) representations of $\pi_1(F)$. Then the complex structure of $\mathrm{PSL}(2, \mathbb{C})$ descends to the complex structure on $\mathcal{QF}$, and $\mathcal{F}$ is a totally real analytic submanifold of $\mathcal{QF}$ with $\dim_{\mathbb{R}} \mathcal{F} = \dim_{\mathbb{C}} \mathcal{QF}$. By Bers' simultaneous uniformization, the quasifuchsian space $\mathcal{QF}$ is canonically identified with the product space $\mathrm{Teich}(F) \times \mathrm{Teich}(F)$ as complex manifold. In particular $\mathcal{QF}$ is contractible.

Let $p$ be a puncture and $m$ be a *meridian* around $p$, i.e., a peripheral simple loop around $p$. We choose a base point for $\pi_1(F)$ on the circle $m$ and denote the element of $\pi_1(F)$ represented by $m$ by the same symbol. Pick an element $\gamma_0 \in \pi_1(F)$ represented by a non-peripheral loop. Then each element of $\mathcal{QF}$ has a unique representative $\rho \in \mathrm{Hom}(\pi_1(F), \mathrm{PSL}(2, \mathbb{C}))$ which satisfies the following conditions:

$$\rho(m) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{Fix}^+ \rho(\gamma_0) = 0, \quad \mathrm{Fix}^+ \rho(m\gamma_0 m^{-1}) = 1.$$

Here $\mathrm{Fix}^+$ denotes the attractive fixed point of a loxodromic transformation. The correspondence

$$\mathcal{QF} \ni [\rho] \mapsto \rho \in \mathrm{Hom}(\pi_1(F), \mathrm{PSL}(2,\mathbb{C}))$$

gives a holomorphic cross section of $\mathcal{QF}$. Throughout this paper, we identify the space $\mathcal{QF}$ with its image by the holomorphic section.

**Complex translation length.** Recall that the *complex translation length* $\lambda(A)$ of a loxodromic element $A \in \mathrm{PSL}(2,\mathbb{C})$ is defined to be the unique element of $\mathbb{C}/2\pi i\mathbb{Z}$ satisfying the following conditions:

1. The real part $\Re(\lambda(A)) > 0$ is the translation length along the axis of $A$. Thus $\Re(\lambda(A)) = \min_{x\in\mathbb{H}^3} d(x, A(x))$, where $d$ is the hyperbolic metric.
2. The imaginary part $\Im(\lambda(A))$ is the rotation angle of $A$ around the axis of $A$.

If $A$ is parabolic, then $\lambda(A)$ is defined to be $0 \in \mathbb{C}/2\pi i\mathbb{Z}$. Then $\lambda(A) \in \mathbb{C}/2\pi i\mathbb{Z}$ is characterized by

$$2\cosh\frac{\lambda(A)}{2} = \pm\mathrm{tr}(A), \quad \Re(\lambda(A)) > 0.$$

Let $\alpha$ be an *essential* simple loop in $F$, i.e., a simple loop in $F$ which does not bound a disk nor a once-punctured disk in $F$. We abuse notation to denote an element of $\pi_1(F)$ represented by $\alpha$ by the same symbol. Then, for any quasifuchsian representation $\rho$ of $\pi_1(F)$, $\rho(\alpha)$ is a loxodromic transformation. The correspondence $\rho \mapsto \lambda(\rho(\alpha))$ determines a holomorphic function $\mathcal{QF} \to \mathbb{C}/2\pi i\mathbb{Z}$. Since $\mathcal{QF}$ is contractible, this map lifts to a holomorphic function $\mathcal{QF} \to \mathbb{C}$ which sends $\mathcal{F}$ into $\mathbb{R}$. We denote by $\lambda_\rho(\alpha)$ the complex number obtained as the image of $\rho \in \mathcal{QF}$ by the holomorphic function, and continue to call it the *complex translation length* of $\rho(\alpha)$. If $\alpha$ is a *peripheral* simple loop, i.e., $\alpha$ bounds a once-punctured disk in $F$, then we define $\lambda_\rho(\alpha) = 0 \in \mathbb{C}$.

Then the following theorem generalizes McShane's identity for fuchsian punctured surface groups (see Theorem 2.1) and Bowditch's generalization [B3, Theorem 3] of (1) for quasifuchsian punctured torus groups.

**Theorem 3.1.** *For any $\rho \in \mathcal{QF}$, we have*

$$\sum_{\delta\in\Delta} h_\rho(\delta) = \frac{1}{2}.$$

# 4 Hyperbolic punctured surface bundles

In this section, we present a generalization of Bowditch's result [B2] on hyperbolic once-punctured torus bundles. Let $\varphi : F \to F$ be a pseudo-Anosov homeomorphism preserving a puncture $p$, and let $B_\varphi$ be the $F$-bundle over $S^1$

with monodromy $\varphi$. Then $B_\varphi$ admits a unique complete hyperbolic structure of finite volume, and each cusp torus carries a Euclidean structure. Let $\partial_p B_\varphi$ be the cusp torus around the suspension of $p$. A *meridian* $m$ of $\partial_p B_\varphi$ is defined as the meridian around $p$ of a fiber. We shall specify a *longitude* $l$ of $\partial_p B_\varphi$ in Definition 4.4. Then, as in Section 1, the *modulus* $\mathrm{Modulus}(\partial_p B_\varphi)$ of the cusp torus $\partial_p B_\varphi$, with respect to the meridian-longitude pair $(m,l)$, is defined as follows: Let $\rho : \pi_1(B_\varphi) \to \mathrm{PSL}(2,\mathbb{C})$ be the holonomy representation of the hyperbolic manifold $B_\varphi$ such that $\rho(m)$ is the parallel translation $z \mapsto z+1$. Then $\rho(l)$ is the parallel translation $z \mapsto z + \mathrm{Modulus}(\partial_p B_\varphi)$.

To recall Bowditch's theorem, suppose for a while that $F$ is a once-punctured torus $T$. Then the monodromy $\varphi$ induces a self-homeomorphism of the projective measured lamination space $\mathcal{PML}(T) \cong S^1$ preserving the subset $\mathcal{S}$. This homeomorphism has two fixed points in $\mathcal{PML}(T)$, namely the stable and unstable laminations, $\nu^+$ and $\nu^-$, of the monodromy. Since $\nu^+$ and $\nu^-$ are irrational, they determine a natural partition of $\mathcal{S}$ into two subsets $\mathcal{S}_L$ and $\mathcal{S}_R$. This in turn gives a partition of the quotient set $\mathcal{S}/\langle\varphi\rangle$ (which is identified with the set of essential simple loops on a fiber $F$ modulo isotopy in the ambient 3-manifold $B_\varphi$) into two subsets $\mathcal{S}_L/\langle\varphi\rangle$ and $\mathcal{S}_R/\langle\varphi\rangle$. For two elements $\alpha$ and $\alpha'$ of $\mathcal{S}$ representing the same element in $\mathcal{S}/\langle\varphi\rangle$, the complex translation lengths of $\rho(\alpha)$ and $\rho(\alpha')$ coincide. So, the complex translation length $\lambda_\rho(\alpha) \in \mathbb{C}/2\pi i\mathbb{Z}$ is well-defined for $\alpha \in \mathcal{S}/\langle\varphi\rangle$. It should be noted that $e^{\lambda_\rho(\alpha)}$ is a well-defined complex number. Then the following theorem was proved by Bowditch [B2].

**Theorem 4.1 (Bowditch [B2]).** *Let $B_\varphi$ be a complete hyperbolic 3-manifold which fiber over the circle with fiber a once-punctured torus $T$ with monodromy $\varphi$. Then the modulus $\mathrm{Modulus}(\partial_p B_\varphi)$ of the cusp torus $\partial_p B_\varphi$, with respect to a suitable choice of a longitude $l$, is given by the following formula.*

$$\pm\mathrm{Modulus}(\partial_p B_\varphi) = \sum_{\alpha\in\mathcal{S}_L/\langle\varphi\rangle} \frac{1}{1+e^{\lambda_\rho(\alpha)}} = -\sum_{\alpha\in\mathcal{S}_R/\langle\varphi\rangle} \frac{1}{1+e^{\lambda_\rho(\alpha)}}.$$

In the general punctured surface bundle case, we study the action of the monodromy $\varphi$ on the sets $\mathcal{G}$ and $\vec{\Delta}$, and specify a certain subset, $\vec{\Delta}_\varphi$, of $\vec{\Delta}$ in Definition 4.6 (cf. Remark 4.7). Then our generalization of Bowditch's result can be stated as follows.

**Theorem 4.2.** *Let $B_\varphi$ be a complete hyperbolic 3-manifold which fiber over the circle with fiber $F$ with monodromy $\varphi$ that preserves the puncture $p$. Then the modulus $\mathrm{Modulus}(\partial_p B_\varphi)$ of the cusp torus $\partial_p B_\varphi$, with respect to a suitable choice of a longitude $l$, is given by the following formula.*

$$\mathrm{Modulus}(\partial_p M_p) = \pm \sum_{\delta\in\vec{\Delta}_\varphi} h_\rho(\delta).$$

In the above theorem, $h_\rho(\delta)$ is defined by

$$h_\rho(\delta) := \frac{1}{1+e^{\frac{1}{2}(\lambda_\rho(\alpha(\delta))+\lambda_\rho(\beta(\delta)))}},$$

where $\lambda_\rho(\alpha)$ with $\alpha \in \mathcal{S} \cup \mathcal{P}$ denotes a lift to $\mathbb{C}$ of the complex translation length of $\rho(\alpha)$ specified by Definition 4.9 below. In the remainder of this section, we give explicit definitions of the longitude $l$, the subset $\vec{\Delta}_\varphi \subset \vec{\Delta}$ and the complex translation length $\lambda_\rho(\alpha) \in \mathbb{C}$.

**Behavior of $\varphi$ on the boundary.** Since $\varphi$ is a pseudo-Anosov homeomorphism, there are measured foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ satisfying the following conditions (cf. e.g. [K, Section 11.4]).

1. $\mathcal{F}^+$ and $\mathcal{F}^-$ are transversal, that is, their singular sets are equal, and $\mathcal{F}^+$ is transversal to $\mathcal{F}^-$ away from the singular set.
2. $\varphi(\mathcal{F}^+) = k\mathcal{F}^+$ and $\varphi(\mathcal{F}^-) = k^{-1}\mathcal{F}^-$ for some $k > 1$. Namely, $\varphi$ preserves the singular foliations $\mathcal{F}^+$ and $\mathcal{F}^-$, and multiplies the measures by $k$ and $k^{-1}$ respectively.

For each puncture $q$ of $F$, there is a neighborhood of $q$ that is identified with a neighborhood of 0 of a complex plane, such that the $\mathcal{F}^+$ and $\mathcal{F}^-$ are given by $|\Im(z^{d/2}dz)|$ and $|\Re(z^{d/2}dz)|$, respectively, for some integer $d \geq -1$ (cf. e.g. [G, Section 11.1], [K, Section 11.3]). In particular, each of $\mathcal{F}^+$ and $\mathcal{F}^-$ has $d+2$ ($\geq 1$) singular leaves landing at the puncture $q$. The number $d+2$ is called the *degree* of $\mathcal{F}^\pm$ at $q$.

Let $\bar{F}$ be the compact surface with boundary obtained by adding the circle of rays from $q$ for each puncture $q$. We denote this boundary circle by $\partial_q F$. Then the measured foliations $\mathcal{F}^\pm$ extend to measured foliations $\bar{\mathcal{F}}^\pm$ of $\bar{F}$. Each of $\bar{\mathcal{F}}^\pm$ has $b$ ($\geq 1$) singular leaves landing in $\partial_q F$, where $b$ is the degree of $\mathcal{F}^\pm$ at $q$. Moreover $\varphi$ extends to a homeomorphism of $\bar{F}$, which we continue to denote by $\varphi$.

Since $\varphi$ preserves the puncture $p$, $\varphi : \bar{F} \to \bar{F}$ induces a homeomorphism of the boundary circle $\partial_p F$. Let $b$ be the degree of $\mathcal{F}^\pm$ at the puncture $p$, and let $\{x_1^\pm, x_2^\pm, \cdots, x_b^\pm\}$ be the endpoints of the singular leaves of $\mathcal{F}^\pm$ in $\partial_p F$. We assume that they are arranged on $\partial_p F$ in this cyclic order. Since $\varphi$ preserves the singular leaves, there is a unique integer $c$ with $0 \leq c < b$ such that $\varphi$ acts on the sets $\{x_1^\pm, x_2^\pm, \cdots, x_b^\pm\}$ as the shift of indices by $c$. Set $n_0 = b/\gcd(b, c)$. Since $\varphi$ is affine with respect to the singular Euclidean metric determined by the mutually transversal measured laminations $\mathcal{F}^+$ and $\mathcal{F}^-$, we have the following lemma.

**Lemma 4.3.** *The sets $\{x_1^+, x_2^+, \cdots, x_b^+\}$ and $\{x_1^-, x_2^-, \cdots, x_b^-\}$, respectively, are equal to the attractive and the repulsive fixed point sets of $\varphi^{n_0} : \partial_p F \to \partial_p F$, and they are arranged on $\partial_p F$ alternatively.*

**Specifying the longitude.** Let $\bar{B}_\varphi = \bar{F} \times [0,1]/(x,0) \sim (\varphi(x), 1)$ be the $\bar{F}$-bundle over $S^1$ with monodromy $\varphi$, and let $\partial_p \bar{B}_\varphi$ be the boundary component of $\bar{B}_\varphi$ corresponding to the puncture $p$ of $F$. Namely, $\partial_p \bar{B}_\varphi = \partial_p F \times [0,1]/(x,0) \sim (\varphi(x), 1)$. Then $B_\varphi$ is identified with the interior of $\bar{B}_\varphi$, and the cusp torus $\partial_p B_\varphi$ is identified with $\partial_p \bar{B}_\varphi$.

**Definition 4.4.** *By the longitude $l$ of $\partial_p B_\varphi$, we mean the isotopy class of the simple loop in $\partial_p \bar{B}_\varphi$ obtained as the image of $\cup_{j=0}^{n_0-1} \varphi^j(x) \times [0,1]$, where $n_0$ is the natural number in Lemma 4.3 and $x$ is a fixed point of $\varphi^{n_0}$.*

Note that the meridian-longitude pair $(m, l)$ defined in the above forms a basis of $H_1(\partial_p B_\varphi; \mathbb{Z})$ if and only if $n_0 = 1$. However, it always forms a basis of $H_1(\partial_p B_\varphi; \mathbb{Q})$ and hence the modulus of $\partial_p B_\varphi$ with respect to any basis of $H_1(\partial_p B_\varphi; \mathbb{Z})$ can be calculated from Modulus$(\partial_p B_\varphi)$.

**The action of $\varphi$ on $\vec{\Delta}$ and $\mathcal{G}$.** We may assume $\varphi$ is the Teichmüller map and the conformal structure on $F = \mathbb{H}^2/\Gamma_0$ is absolutely $\varphi$-minimal in the sense of Bers, that is, it lies in the axis of the invariant axis of the action of $\varphi$ on the Teichmüller space (see e.g. [IT, Section 5.2]). Then $\varphi : F \to F$ is quasiconformal, and hence its lift to $\mathbb{H}^2$ extends to a homeomorphism of $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. Let $\widetilde{\varphi}$ be such a homeomorphism of $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ which stabilizes $\infty = \mathrm{Fix}(\rho_0(m))$. We denote by $\varphi_p$ the homeomorphism of $S_p^1 = \mathbb{R}/\langle \rho_0(m) \rangle$ induced by the restriction of $\widetilde{\varphi}$ to $\mathbb{R} = \partial\mathbb{H}^2 - \{\infty\}$.

For each $\widetilde{\mu} \in \widetilde{\mathcal{G}}$, consider the geodesic in $\mathbb{H}^2$ emanating from $\infty$ and ending at $\widetilde{\varphi}(z_{\rho_0}(\widetilde{\mu}))$, the image by $\widetilde{\varphi}$ of the endpoint $z_{\rho_0}(\widetilde{\mu})$ of $\widetilde{\mu}$. Then it also belongs to $\widetilde{\mathcal{G}}$. This determines a bijection $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$, which in tern induces a bijection $\mathcal{G} \to \mathcal{G}$. After identifying $\mathcal{G}$ with a subset of $S_p^1$, the bijection is identified with the restriction of $\varphi_p : S_p^1 \to S_p^1$ to $\mathcal{G}$. The following lemma describes the dynamics of $\varphi_p$.

**Lemma 4.5.** *Let $n_0$ be as in Lemma 4.3. Then $\varphi_p^{n_0}$ has finitely many attractive fixed points and repulsive fixed points, which are arranged on $S_p^1$ alternatively. Moreover, for any component $J$ of $S_p^1 - (\mathrm{Fix}^+(\varphi_p^{n_0}) \cup \mathrm{Fix}^-(\varphi_p^{n_0}))$ bounded by an attractive fixed point $A^+$ and a repulsive fixed point $A^-$, $\varphi_p^{n_0}$ maps every point $X \in J$ to a point strictly closer to $A^+$, and we have $\lim_j (\varphi_p^{n_0})^j(X) = A^+$ and $\lim_j (\varphi_p^{n_0})^{-j}(X) = A^-$.*

Now the subset $\vec{\Delta}_\varphi$ of $\vec{\Delta}$ in Theorem 4.2 is defined as follows.

**Definition 4.6.** *Let $n_0$ be the natural number in Lemma 4.5. Pick a connected component, $J$, of $S_p^1 - \mathrm{Fix}(\varphi_p^{n_0})$ and an element $\mu \in J \cap (\mathcal{G} - \vec{\Delta})$, and let $[\mu, \varphi_p^{n_0}(\mu)]$ be the closed sub-interval of $J$ bounded by $\mu$ and $\varphi_p^{n_0}(\mu)$. Then we define $\vec{\Delta}_\varphi := [\mu, \varphi_p^{n_0}(\mu)] \cap \vec{\Delta}$.*

**Remark 4.7.** *(1) There is a one-to-one correspondence between $\vec{\Delta}_\varphi$ and the quotient set $(J \cap \vec{\Delta})/\langle \varphi_p^{n_0} \rangle$, which in tern is a subset of $\vec{\Delta}/\langle \varphi_p^{n_0} \rangle$. Moreover $h_\rho(\varphi_p^{n_0}(\delta)) = h_\rho(\delta)$ for every $\delta \in \vec{\Delta}$. Thus we may identify $\vec{\Delta}_\varphi$ with the subset $(J \cap \vec{\Delta})/\langle \varphi_p^{n_0} \rangle$ of $\vec{\Delta}/\langle \varphi_p^{n_0} \rangle$. So the choice of $\mu$ in the definition of $\vec{\Delta}_\varphi$ is not essential.*

*(2) Throughout the remainder of this section and Section 5, we assume that $\mu$ and $\varphi_p^{n_0}(\mu)$ lie in this order with respect to the orientation of $J$ induced by that of $\mathbb{R}$.*

**Complex translation length in the fiber group.** Let $\rho$ be the holonomy representation of the fiber group $\pi_1(F)$ in the hyperbolic manifold $B_\varphi$. Pick a point $\sigma \in \text{Teich}(F)$, and let $\rho_n$ be the element of $\mathcal{QF}$ uniformizing $(\varphi_*^{-n}(\sigma), \varphi_*^n(\sigma))$ for each natural number $n$. Here $\varphi_*$ denotes the automorphism of Teich$(F)$ induced by $\varphi$. Then $\rho_n$ converges to $\rho$ strongly, because we know from the proof of Theorem 0.1 in [Th] (see [Th, §5]) that any subsequence of $\{\rho_n\}$ contains a subsequence converging to $\rho$ strongly.

**Lemma 4.8.** *Under the above situation, the sequence of the complex translation lengths* $\lambda_{\rho_n}(\alpha)$ *in* $\mathbb{C}$ *converges for each* $\alpha \in \mathcal{S}$. *Moreover the limit* $\lim \lambda_{\rho_n}(\alpha)$ *does not depend on the choice of* $\sigma$.

**Definition 4.9.** *Let* $\rho$ *be the holonomy representation of the fiber group* $\pi_1(F)$ *in the hyperbolic manifold* $B_\varphi$. *Then for an essential simple loop* $\alpha \in \mathcal{S}$, $\lambda_\rho(\alpha)$ *denotes* $\lim \lambda_{\rho_n}(\alpha) \in \mathbb{C}$ *in Lemma 4.8.*

# 5 Outline of the proof

**Step 1.** Let $\rho_\infty : \pi_1(B_\varphi, x_0) \to \text{PSL}(2,\mathbb{C})$ be the holonomy representation of the complete hyperbolic structure of $B_\varphi$. Then we may assume that the restriction of $\rho_\infty$ to $\pi_1(F, x_0)$, which we continue to denote by the same symbol, lies in the closure of the image of the holomorphic section $\mathcal{QF} \to \text{Hom}(\pi_1(F), \text{PSL}(2,\mathbb{C}))$ (which we fixed in Section 3) and that the following identities hold.

$$\rho_\infty(m) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \rho_\infty(l) = \begin{pmatrix} 1 & \text{Modulus}(\partial_p B_\varphi) \\ 0 & 1 \end{pmatrix}.$$

Now pick a non-peripheral element $\gamma \in \pi_1(F, x_0)$. Then we have:

$$\text{Modulus}(\partial_p B_\varphi) = \text{Fix}^+(\rho_\infty(\varphi_*(\gamma))) - \text{Fix}^+(\rho_\infty(\gamma)).$$

Because

$$\begin{aligned} \text{Fix}^+(\rho_\infty(\varphi_*(\gamma))) &= \text{Fix}^+(\rho_\infty(l\gamma l^{-1})) \\ &= \rho_\infty(l)(\text{Fix}^+(\rho_\infty(\gamma))) \\ &= \text{Fix}^+(\rho_\infty(\gamma)) + \text{Modulus}(\partial_p B_\varphi). \end{aligned}$$

**Step 2.** Let $\{\rho_n\}$ be a sequence in $\mathcal{QF}$ which converges strongly to $\rho_\infty$. Then, by Step 1, we have:

$$\text{Modulus}(\partial_p B_\varphi) = \lim \left(\text{Fix}^+(\rho_n(\varphi_*(\gamma))) - \text{Fix}^+(\rho_n(\gamma))\right)$$

**Step 3.** Recall that the set $\vec{\Delta}_\varphi$ is defined to be $[\mu, \varphi_p(\mu)] \cap \vec{\Delta}$, where $[\mu, \varphi_p(\mu)]$ is the closed sub-interval of a component, $J$, of $S^1_p - \text{Fix}(\varphi_p)$ bounded by $\mu$ and $\varphi_p(\mu)$ (see Definition 4.6). Here $\mu$ is an arbitrary element of $J \cap (\mathcal{G} - \vec{\Delta})$. So we may assume that $\mu$ spirals to an oriented simple closed geodesic $\gamma$. Let $\widetilde{\mu}$

be a lift of the oriented geodesic $\mu \subset F = \mathbb{H}^2/\Gamma_0$ to $\mathbb{H}^2$ emanating from $\infty$. Then there is an element of $\pi_1(F, x_0)$, denoted by the same symbol $\gamma$, which represents the closed geodesic $\gamma$ such that $z_{\rho_0}(\widetilde{\mu}) = \mathrm{Fix}^+(\rho_0(\gamma))$, where $z_{\rho_0}(\widetilde{\mu})$ is the endpoint of $\widetilde{\mu}$ in $\mathbb{R} = \partial\mathbb{H}^2 - \{\infty\}$. Then, by Corollary 2.4, for any $\rho \in \mathcal{F} \subset \mathrm{Hom}(\pi_1(F), \mathrm{PSL}(2, \mathbb{C}))$, we have:

$$\mathrm{Fix}^+(\rho(\varphi_*(\gamma))) - \mathrm{Fix}^+(\rho(\gamma)) = \sum_{\delta \in \vec{\Delta}_\varphi} h_\rho(\delta).$$

**Step 4.** For any $\rho \in \mathcal{QF}$, we have:

$$\mathrm{Fix}^+(\rho(\varphi_*(\gamma))) - \mathrm{Fix}^+(\rho(\gamma)) = \sum_{\delta \in \vec{\Delta}_\varphi} h_\rho(\delta).$$

This is a consequence of Step 3 and the following lemma, which implies that the correspondence $\mathcal{QF} \ni \rho \mapsto \sum_{\delta \in \vec{\Delta}} h_\rho(\delta)$ is a well-defined holomorphic mapping.

**Lemma 5.1.** *For each $\rho \in \mathcal{QF}$, the infinite sum $\sum_{\delta \in \vec{\Delta}} h_\rho(\delta)$ converges absolutely and uniformly on every compact subset of $\mathcal{QF}$.*

This lemma is proved by using the facts

1. The number of simple closed geodesics with length $\leq L$ in a given hyperbolic surface is bounded by a polynomial function of $L$ (see [BS] or [Miz2]).

2. For any compact subset $C$ of $\mathcal{QF}$, there is a constant $k = k(C) > 1$ such that

$$\frac{1}{k} l_{\rho_0}(\gamma) \leq l_\rho(\gamma) \leq k l_{\rho_0}(\gamma),$$

for any $\gamma \in \mathcal{S}$ and $\rho \in C$. Here $\rho_0$ is a fixed element of $\mathcal{F}$. (See [JM, Lemma 3] or [K, Theorem 8.57].)

**Step 5.** The infinite sum $\sum_{\delta \in \vec{\Delta}_\varphi} h_{\rho_\infty}(\delta)$ converges absolutely, and

$$\sum_{\delta \in \vec{\Delta}_\varphi} h_{\rho_\infty}(\delta) = \lim \sum_{\delta \in \vec{\Delta}_\varphi} h_{\rho_n}(\delta).$$

The above key fact is proved as follows:

1. We show that there is a compact submanifold $K_\infty$ of $M_\infty$ which contains the closed geodesic $\gamma_{\rho_\infty}$ for every $\gamma \in \mathcal{S}(\vec{\Delta}_\varphi)$.

2. Since $\rho_n(\pi_1(F))$ converges to $\rho_\infty(\pi_1(F))$ geometrically, there are smooth embeddings $f_n : K_\infty \to M_n$, defined for all $n$ sufficiently large, such that $f_n$ sends $\omega_\infty$ to $\omega_n$ and $f_n$ tends to an isometry in the $C^\infty$-topology. Namely, the lift $\tilde{f}_n : \tilde{K}_\infty \to \mathbb{H}^3$ of $f_n$ to the inverse image $\tilde{K}_\infty$ of $K_\infty$ in $\mathbb{H}^3$, sending the standard frame at the origin to itself, tends to the identity map in the compact-open $C^\infty$-topology (see [BP, Theorem E.1.13], [McM, Section 2.2]). Since $\rho_n$ converges to $\rho_\infty$ algebraically, we may assume that the following diagram is commutative, where the vertical arrows represent homomorphisms induced by the inclusion maps and $o_n$ denotes the origin of the frame $\omega_n$ for $n \in \mathbb{N} \cup \{\infty\}$:

$$\begin{array}{ccc}
\pi_1(M_n, o_n) & \xleftarrow{\rho_n \circ \rho_\infty^{-1}} & \pi_1(M_\infty, o_\infty) \\
\uparrow & & \uparrow \\
(f_n)_*(\pi_1(K_\infty, o_\infty)) & \xleftarrow{(f_n)_*} & \pi_1(K_\infty, o_\infty)
\end{array}$$

3. Fix a positive real number $r > 0$. Then there is a natural number $N_1$ which satisfies the following condition. For any $n \geq N_1$ and for any closed geodesic $\gamma^*$ in $M_\infty$ which lies in $K_\infty$, the $r$-neighborhood of $f_n(\gamma^*)$ contains its geodesic representative in $M_n$. This is proved by using the following fact: If a loop in a hyperbolic manifold is faraway from its geodesic representative, then the "geodesic curvature" of the loop at the point where the distance attains the maximum should be large.

4. By the above assertion, we see that the real lengths $l_\rho(\gamma)$, where $\rho$ runs over all elements in $\{\rho_n \,|\, n \geq 1\} \cup \{\rho_\infty\}$, are comparable. Namely, there exists $k \geq 1$ such that

$$\frac{1}{k} l_{\rho_0}(\gamma) \leq l_\rho(\gamma) \leq k l_{\rho_0}(\gamma)$$

for every $\rho \in \{\rho_n \,|\, n \geq 1\} \cup \{\rho_\infty\}$ and $\gamma \in \mathcal{S}(\vec{\Delta}_\varphi)$. By using this estimate, we can prove the desired result.

We can now complete the proof of Theorem 4.2 as follows.

$$\begin{aligned}
\text{Modulus}(\partial_p B_\varphi) &= \text{Fix}^+(\rho_\infty(\varphi_*(\gamma))) - \text{Fix}^+(\rho_\infty(\gamma)) && \text{by Step 1} \\
&= \lim \left(\text{Fix}^+(\rho_n(\varphi_*(\gamma))) - \text{Fix}^+(\rho_n(\gamma))\right) && \text{by Step 2} \\
&= \lim \sum_{\delta \in \vec{\Delta}_\varphi} h_{\rho_n}(\delta) && \text{by Step 4} \\
&= \sum_{\delta \in \vec{\Delta}_\varphi} h_{\rho_\infty}(\delta) && \text{by Step 5.}
\end{aligned}$$

# 6 Further variations and questions.

In [AMS1] the authors refined the identity in [B3] to a "width" formula of the limit set of a geometrically finite punctured torus group. The refinement was generalized to an identity for every quasifuchsian punctured torus groups in [AMS2]. The full proof of The Main Theorem 4.2 of this note is also contained in [AMS2]. Another possible 3-dimensional variation of MacShane's identity for punctured torus groups was announced by the third author [S]: a conjecture was proposed that the modulus of the cusp torus of a 2-bridge knot is expressed by the complex translation lengths of geodesics which are homotopic to essential simple loops on the 4-times punctured bridge sphere. The conjecture is equivalent to a conjecture concerning certain conjugacy problem for 2-bridge knot groups, which in tern is valid for the figure-eight knot and $5_2$ knot. Therefore, we would like to propose the following question.

**Question 6.1.** Is there a variation of McShane's identity for minimal bridge decompositions for hyperbolic knots?

In [Miz1], M. Mirzakhani has generalized the identity for bordered hyperbolic surfaces and found beautiful applications of the identity.

**Theorem 6.1 (Mirzakhani [Miz1]).** *For any hyperbolic surface $X$ with $n$ boundary components $\beta_1, \cdots, \beta_n$ of lengths $L_1, \cdots, L_n$,*

$$\sum_{(\alpha_1,\alpha_2)} D(L_1, l_X(\alpha_1), l_X(\alpha_2)) + \sum_{i=2}^{n}\sum_{\gamma} R(L_1, L_2, l_X(\gamma)) = L_1.$$

*Here*

$$D(x, y, z) = 2\log\left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{y+z}{2}}}\right),$$

$$R(x, y, z) = x - \log\left(\frac{\cosh(\frac{y}{2}) + \cosh(\frac{x+z}{2})}{\cosh(\frac{y}{2}) + \cosh(\frac{x-z}{2})}\right),$$

*the first sum is over all oriented pairs of simple closed geodesics $(\alpha_1, \alpha_2)$ bounding a pair of pants with $\beta_1$, and the second sum is over simple closed geodesics $\gamma$ bounding a pair of pants with $\beta_1$ and $\beta_i$.*

She used the above identity to obtain a recursive formula for the Weil-Petersson volume of moduli spaces. The starting point is to regard the identity as a constant function on the modulie space and use the fact that the integral over the moduli space gives (the constant times) the volume of the moduli space. She gives an ingenious way to calculate the integral, and proves that the Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}(L_1, \cdots, L_n)$, of hyperbolic Riemann surfaces of genus $g$ with $n$ geodesic boundary components of length $L_1, \cdots, L_n$, is a polynomial in $L_1, \cdots, L_n$ of total degree $3g-3+n$, such that the coefficient of a term of degree $d$ lies in $\pi^{6g-6+2n-2d}\mathbb{Q}$. For example,

$$\mathrm{Vol}(\mathcal{M}_{1,1}(L)) = \frac{L^2}{24} + \frac{\pi^2}{6}.$$

Moreover, in [Miz2], she studied the asymptotic behavior of $s_X(L)$, the number of simple closed geodesics of hyperbolic length less than $L$ in a hyperbolic bordered Riemann surface $X$, and proved the following theorem.

**Theorem 6.2 (Mirzakhani [Miz2]).** *For any hyperbolic bordered Riemann surface $X$, there exists a constant $n_X > 0$, such that*

$$s_X(L) \sim n_X \cdot L^{6g-6+2n}$$

*as $L \to \infty$. Namely,*

$$\lim_{L\to\infty} \frac{n_X \cdot L^{6g-6+2n}}{s_X(L)} = 1.$$

We would like to propose the following question:

**Question 6.2.** (1) Let $B_\varphi$ be a hyperbolic 3-manifold of finite volume which fibers over the circle, and let $s_\varphi(L)$ be the number of closed geodesics in $B_\varphi$ of length less than $L$ which are isotopic to simple loops on the fiber surface. Then how does $s_\varphi(L)$ behave asymptotically as $L \to \infty$?

(2) Let $F$ be a Heegaard surface of a hyperbolic 3-manifold $M$, and let $s_{(M,F)}(L)$ be the number of closed geodesics in $M$ of length less than $L$ which are isotopic to simple loops on the Heegaard surface $F$. Then how does $s_{(M,F)}(L)$ behave asymptotically as $L \to \infty$?

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