The initial value problem for Schrödinger equations on the torus

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This note is a summary of a paper [2]. We are concerned with the initial value problems for linear Schrödinger-type equations of the form

\[ Lu \equiv \partial_t u - i \Delta u + \tilde{b}(x) \cdot \nabla u + c(x) u = f(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{T}^n, \]
\[ u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{T}^n, \] (1) (2)

and for semilinear Schrödinger equations of the form

\[ \partial_t u - i \Delta u = F(u, \nabla u, \bar{u}, \nabla \bar{u}) \quad \text{in} \quad \mathbb{R} \times \mathbb{T}^n, \]
\[ u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{T}^n, \] (3) (4)

where \( u(t, x) \) is a complex valued unknown function of \( (t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R} \times \mathbb{T}^n, \)
\( \mathbb{T}^n = \mathbb{R}^n/2\pi \mathbb{Z}^n, \)
\( \alpha = \sqrt{-1}, \partial_t = \partial/\partial t, \partial_j = \partial/\partial x_j (j = 1, \ldots, n), \nabla = (\partial_1, \ldots, \partial_n), \)
\( \Delta = \nabla \cdot \nabla, \) and \( \tilde{b}(x) = (b_1(x), \ldots, b_n(x)), \)
\( c(x), f(t, x) \) and \( u_0(x) \) are given functions.

Suppose that \( b_1(x), \ldots, b_n(x) \) and \( c(x) \) are smooth functions on \( \mathbb{T}^n, \)
and that \( F(u, v, \bar{u}, \bar{v}) \)
is a smooth function on \( \mathbb{R}^{2+2n}, \)
and

\[ F(u, v, \bar{u}, \bar{v}) = O(|u|^2 + |v|^2) \quad \text{near} \quad (u, v) = 0. \]

In [7], Mizohata proved that, when \( x \in \mathbb{R}^n, \)
if the initial value problem (1)-(2) is \( L^2 \)-well-posed, then it follows that

\[ \sup_{(t, x, \omega) \in \mathbb{R}^{1+n} \times S^{n-1}} \left| \int_0^t \text{Im} \tilde{b}(x - \omega s) \cdot \vec{\omega} ds \right| < +\infty, \] (5)

where \( \tilde{b} \cdot \xi = b_1 \xi_1 + \cdots + b_n \xi_n. \) Moreover, he gave sufficient condition for \( L^2 \)-well-posedness which is slightly stronger than (5). In particular, (5) is also sufficient condition for \( L^2 \)-well-posedness when \( n = 1. \) Roughly speaking, (5) gives an upper bound of the strength of the real vector field \( (\text{Im} \tilde{b}(x)) \cdot \nabla. \) In other words, if \( (\text{Im} \tilde{b}(x)) \cdot \nabla \)
can be dominated by so-called local smoothing effect of \( e^{it\Delta}, \)
then (5) must holds. After his results, many authors investigated the necessary and sufficient condition, and some weaker sufficient conditions were discovered. Unfortunately, however, the characterization of \( L^2 \)-well-posedness for (1)-(2) remains open except for one-dimensional case. Such linear theories were applied to solving (1)-(2) in case \( x \in \mathbb{R}^n. \) See, e.g., [3] for linear equations, [1], for nonlinear equations, and references therein.

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On the other hand, the periodic case is completely different from the Euclidean case. The local smoothing effect of $e^{it\Delta}$ fails because the Hamiltonian flow generated by the Hamiltonian vector field $2\xi \cdot \nabla$ is completely trapped. See [4] for the relationship between the global behavior of the Hamiltonian flow and the local smoothing effect.

The purpose of this note is to present the necessary and sufficient condition of $L^2$-well-posedness of (1)-(2), and apply this condition to (3)-(4). To state a definition and our results, we here introduce notation. Let $s \in \mathbb{R}$. $H^s(\mathbb{T}^n)$ denotes the set of all distributions on $\mathbb{T}^n$ satisfying

$$\|u\|_s^2 = \int_{\mathbb{T}^n} \|(1 - \Delta)^{s/2}u(x)\|^2dx < +\infty.$$ 

Set $L^2(\mathbb{T}^n) = H^0(\mathbb{T}^n)$, and $\|\cdot\| = \|\cdot\|_0$ for short. Let $I$ be an interval in $\mathbb{R}$. $C(I; H^s(\mathbb{T}^n))$ denotes the set of all $H^s(\mathbb{T}^n)$-valued continuous functions on $I$. Similarly $L^1(I; H^s(\mathbb{T}^n))$ is the set of $H^s(\mathbb{T}^n)$-valued integrable functions on $I$.

We here give the definition of $L^2$-well-posedness.

**Definition 1.** The initial-boundary value problem (1)-(2) is said to be $L^2$-well-posed if for any $u_0 \in L^2(\mathbb{T}^n)$ and $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{T}^n))$, (1)-(2) has a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^n))$.

It follows from Banach's closed graph theorem that the condition required in Definition 1 is equivalent to a seemingly stronger condition, that is, for any $u_0 \in L^2(\mathbb{T}^n)$ and any $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{T}^n))$, (1)-(2) has a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^n))$, and for any $T > 0$ there exists $C_T > 0$ such that

$$\|u(t)\| \leq C_T \left(\|u_0\| + \left|\int_0^t \|f(s)\|ds\right|\right), \quad t \in [-T, T]. \quad (6)$$

Firstly, we present $L^2$-well-posedness results for linear equations.

**Theorem 2.** The following conditions are mutually equivalent:

1. (1)-(2) is $L^2$-well-posed.

2. For $x \in \mathbb{T}^n$ and $\alpha \in \mathbb{Z}^n$

$$\int_0^{2\pi} \text{Im} \vec{b}(x - \alpha s) \cdot \alpha ds = 0. \quad (7)$$

3. There exists a scalar function $\phi(x) \in C^\infty(\mathbb{T}^n)$ such that $\nabla \phi(x) = \text{Im} \vec{b}(x)$.
When $n = 1$, set $b(x) = b_1(x)$. The condition (7) is reduced to
\[ \int_0^{2\pi} \Im b(x) dx = 0. \] (8)

The condition (7) is the natural torus version of (5). More precisely, (7) is a special case of Ichinose's necessary condition of $L^2$-well-posedness discovered in [5]. On the other hand, the condition 3 corresponds to Ichinose's sufficient condition of $L^2$-well-posedness discovered in [6]. Theorem 2 makes us expect analogous results for nonlinear equations. In fact, we have local existence and local ill-posedness results as follows.

**Theorem 3.** Let $s > n/2 + 2$. Suppose that there exists a smooth real-valued function $\Phi(u, \overline{u})$ on $\mathbb{R}^2$ such that for any $u \in C^1(\mathbb{T}^n)$
\[ \nabla \Phi(u, \overline{u}) = \Im \nabla_x F(u, \nabla u, \overline{u}, \nabla \overline{u}). \] (9)

Then for any $u_0 \in H^s(\mathbb{T}^n)$, there exists $T > 0$ depending on $\|u_0\|_s$ such that (3)-(4) possesses a unique solution $u \in C([-T, T]; H^s(\mathbb{T}^n))$. Furthermore, Let $\{u_{0,k}\}$ be a sequence of initial data belonging to $H^s(\mathbb{T}^n)$, and let $\{u_k\}$ be a sequence of corresponding solutions. If
\[ u_{0,k} \rightarrow u_0 \text{ in } H^s(\mathbb{T}^n) \text{ as } k \rightarrow \infty, \]
then for any $m < s$
\[ u_k \rightarrow u \text{ in } C([-T, T]; H^m(\mathbb{T}^n)) \text{ as } k \rightarrow \infty. \] (10)

**Theorem 4.** Suppose that there exists a holomorphic $n$-vector function
\[ \vec{G}(u) = (G_1(u), \cdots, G_n(u)), \quad u \in \mathbb{C} \]
such that $G(u) \neq 0$, and
\[ F(u, \nabla u, \overline{u}, \nabla \overline{u}) = \nabla \cdot \vec{G}(u) \] (11)

for any $u \in C^1(\mathbb{T}^n)$. Then (3)-(4) is not locally well-posed in the sense of Theorem 3.

It seems to be hard to show the continuous dependence of the solution on the initial data because the gain of derivative of $e^{it\Delta}$ fails when $x \in \mathbb{T}^n$. To prove Theorem 4, we construct a sequence of solutions which are real-analytic in $x$ by using the idea of the abstract Cauchy-Kowalewski theorem. Hence it is essential that $G(u)$ is holomorphic.

In what follows we give the sketch the proofs of Theorems 2 and 4. We omit the sketch of the proof of Theorem 3.

**Proof of Theorem 2.** To prove $1 \Rightarrow 2$, we suppose that the condition 2 fails, and construct a sequence of approximate solutions $\{u_0(t, x)\}$ which break an energy inequality (6). Suppose that there exist $x_0 \in \mathbb{T}^n$ and $\alpha \in \mathbb{Z}^n \setminus \{0\}$ such that
\[ \int_0^{2\pi} \Im \tilde{b}(x_0 - \alpha s) \cdot \alpha ds \equiv 4\pi b_0 \neq 0. \]
Without loss of generality, we can assume that $b_0 > 0$. It follows that there exists a small positive constant $\delta$ such that

$$\int_0^{2\pi} \text{Im} \bar{b}(x - s\alpha) \cdot \alpha ds \geq 2\pi b_0$$

(12)

for any $x \in D$, which is defined by

$$D = \bigcup_{\beta \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \mid |x - x_0 - 2\pi\beta - \alpha\alpha| \leq 2\delta\}.$$

Fix an arbitrary $T > 0$. We construct a sequence $\{u_l\}_{l=1,2,3,\ldots}$ by

$$u_1(t, x) = \exp(i\phi_1(t, x))\psi(x),$$

$$\phi_1(t, x) = -l^2t\alpha \cdot \alpha + l\alpha \cdot x - \frac{1}{2} \int_0^{2l(t-T)} \bar{b}(x - \alpha s) \cdot \alpha ds,$$

where the amplitude function $\psi$ is a smooth function on $\mathbb{T}^n$ and supported on $D/2\pi\mathbb{Z}^n$. It is easy to see that

$$\|u_l(T)\| = 1, \quad \|u_l(0)\| = O(\exp(-lb_0T)), \quad \|Lu_l(t)\| = O(\exp(-lb_0(T-t)/2)),$$

which means that the energy inequality fails for $\{u_l\}$.

Next we give the sketch of the proof $2 \Rightarrow 3$ in case $n \geq 2$. Suppose (7). Since $\text{Im} \bar{b} \in (C(\mathbb{T}^n))^n$, $\text{Im} \bar{b}(x)$ is represented by a Fourier series

$$\text{Im} \bar{b}(x) = \sum_{\beta \in \mathbb{Z}^n} \tilde{b}_{1,\beta} e^{i\beta \cdot x}, \quad \tilde{b}_{1,\beta} \in \mathbb{C}^n.$$  

(13)

The substitution of (13) into (7) gives

$$0 = \sum_{\beta \in \mathbb{Z}^n} \tilde{b}_{1,\beta} \cdot \alpha e^{i\beta \cdot x} \int_0^{2\pi} e^{-i\alpha \cdot \beta s} ds = 2\pi \sum_{\beta \cdot \alpha = 0} \tilde{b}_{1,\beta} \cdot \alpha e^{i\beta \cdot x}.$$  

(14)

Then it follows that $\tilde{b}_{1,\beta} \cdot \alpha = 0$ for any $\alpha \in \mathbb{Z}^n$. Since the orthogonal complement of $\beta \neq 0$ is spanned by some $\alpha^1, \ldots, \alpha^{n-1} \in \mathbb{Z}^x$, there exists $a_\beta \in \mathbb{C}$ such that $\tilde{b}_{1,\beta} = a_\beta \beta$ for $\beta \neq 0$. On the other hand, (14) implies $\tilde{b}_{1,0} = 0$ since $V_0 = \mathbb{R}^n$ is spanned by $e_1, \ldots, e_n \in \mathbb{Z}^n$. Then we have

$$\text{Im} \bar{b}(x) = \sum_{\beta \neq 0} a_\beta \beta e^{i\beta \cdot x}.$$  

If we set

$$\phi(x) = -i \sum_{\beta \neq 0} a_\beta e^{i\beta \cdot x},$$

then $\nabla \phi(x) = \text{Im} \bar{b}(x)$. 


It is easy to prove $3 \Rightarrow 1$. Since $\exp(\pm \phi(x)/2)$ is a smooth function on $\mathbb{T}^n$, a mapping $u \mapsto v = \exp(-\phi(x)/2)u$ is automorphic on $L^2(\mathbb{T}^n)$. Multiplying $Lu = f$ by $\exp(\phi(x)/2)$, we have

\[
(\partial_t - i\Delta + \text{Re} \overline{b}(x) \cdot \nabla + \overline{c}(x))v = g(t, x),
\]

where $\overline{c}(x) \in C^\infty(\mathbb{T}^n)$ and $g(t, x) = \exp(-\phi(x)/2)f(t, x)$. It is easy to obtain forward and backward energy inequalities in $t$. The duality arguments proves that (1)-(2) is $L^2$-well-posed.

**Proof of Theorem 4.** We will construct a sequence which fails to satisfy (19). It suffices to do it for one dimensional case since a one dimensional counter example is also an any dimensional counter example. Suppose that there exists a nonconstant holomorphic function $G(u)$ in $\mathbb{C}$ such that for $u \in C^1(\mathbb{T})$

\[
F(u, \nabla u, \overline{u}, \nabla \overline{u}) = \frac{\partial}{\partial x}G(u) = G'(u)u_x.
\]

Set $g = G'$ for short. If $u$ is a smooth solution to (3), then

\[
\frac{d}{dt} \int_{\mathbb{T}} u(t, x)dx = \int_{\mathbb{T}} \partial_t u(t, x)dx = \int_{\mathbb{T}} \frac{\partial}{\partial x}(u_x(t, x) + G(u(t, x)))dx = 0.
\]

(16)

We here express $u$ by a Fourier series

\[
u(t, x) = \sum_{i \in \mathbb{Z}} u_i(t)e^{ix}.
\]

Then (16) implies $u_0(t) = u_0(0)$. Set $u_0(0) = z_0$ and $v(t, x) = u(t, x) - z_0$ for short. Since $g(0) = 0$ and $u_x = v_x$, there exists an appropriate complex constant $z_0$ such that

\[
g(u)u_x = -(\mu + i\lambda)v_x + h(v)v_x,
\]

where $\mu \in \mathbb{R}$, $\lambda > 0$, and $h$ is holomorphic in $\mathbb{C}$. Then, $v$ solves

\[
v_t - iv_{xx} + (\mu + i\lambda)v_x = h(v)v_x.
\]

In what follows, fix $z_0$. Note that $u(t, x) \equiv z_0$ is a solution to (3)-(4). Suppose that the conclusion of Theorem 3 holds. Consider the initial value problem of the form $v^{(m)}$ solves the initial value problem of the form

\[
v^{(m)}_t - iv^{(m)}_{xx} + (\mu + i\lambda)v^{(m)}_x = h(v^{(m)})v^{(m)}_x \quad \text{in} \quad (0, T) \times \mathbb{T},
\]

(17)

\[
v^{(m)}(0, x) = \frac{e^{imx}}{(1 + m)^s} \quad \text{in} \quad \mathbb{T},
\]

(18)
where $s > 5/2$, $m = 1, 2, 3, \ldots$. Since $\{v^{(m)}(0, x)\}$ is bounded in $H^s(\mathbb{T})$ and

$$v^{(m)}(0, x) \to 0 \text{ in } H^s(\mathbb{T}) \text{ as } m \to \infty$$

for any $\sigma < s$, it follows from the hypothesis that

$$v^{(m)} \to 0 \text{ in } C([0, T]; H^s(\mathbb{T})) \text{ as } m \to \infty$$

(19)

for any $\sigma < s$. We investigate a formal Fourier series solution to (17)-(18) of the form

$$w^{(m)}(t, x) = \sum_{l=1}^{\infty} w_{l}^{(m)}(t)e^{ilmx}. \quad (20)$$

The substitution of (20) into (17)-(18) gives

$$\frac{d}{dt}w_{l}^{(m)}(t) + (il^2m^2 + i\mu lm - \lambda lm)w_{l}^{(m)}(t)
= \sum_{p=1}^{\infty} h_p \sum_{l_0 + \ldots + l_p = l} \prod_{j=0}^{p} w_{l_j}^{(m)}(t), \quad (21)$$

$$w_{l}^{(m)}(0) = \begin{cases} (1 + m)^{-s} \text{ if } l = 1 \\ 0 \text{ otherwise} \end{cases} \quad (22)$$

For $l = 1$, (21)-(22) is concretely solved by

$$w_{1}^{(m)}(t) = (1 + m)^{-s} \exp(-i(m^2 + \mu m)t + \lambda mt). \quad (23)$$

For $l \geq 2$, we apply the idea of the abstract Cauchy-Kowalewski theorem to (21)-(22). We can show that there exists $T_m \in (0, T)$ such that the formal series (20) converges in $C([0, T_m]; H^s(\mathbb{T}))$. Then it follows from the hypothesis that

$$v^{(m)} = w^{(m)} \text{ in } C([0, T_m]; H^s(\mathbb{T})).$$

Finally we can find $\delta > 0$, $\alpha \in (0, 1)$ and $t_m \in (0, T_m)$ such that

$$\sup_{t \in [0, T]} \|v^{(m)}(t)\|_{(1-\alpha)s} \geq \|v^{(m)}(t_m)\|_{(1-\alpha)s} = \|w^{(m)}(t_m)\|_{(1-\alpha)s} = \left(\sum_{l=1}^{\infty} (1 + lm)^{2(1-\alpha)s}|w_{l}^{(m)}(t_m)|^2\right)^{1/2} \geq (1 + m)^{(1-\alpha)s}|w_{1}^{(m)}(t_m)| = (1 + m)^{-s\alpha}\exp(\lambda mt_m) = \delta,$$

which contradicts (19). Here we omit the detail.
References


