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$L^q - L^r$ estimate of the Stokes semigroup in a perturbed half-space

早稲田大学 久保 隆徹 (Kubo Takayuki)*
早稲田大学 柴田 良弘 (Shibata Yoshihiro)†

1 Introduction

Let Ω be a perturbed half-space with smooth boundary $\partial\Omega$ in $\mathbb{R}_+^n$ $(n \geq 3)$; to be precise, we call an open set Ω the exterior domain of $\mathbb{R}_+^n$ if there is a positive number R such that $\Omega \cap \{x \in \mathbb{R}_+^n;\ |x| > R\} = \{x \in \mathbb{R}_+^n;\ |x| > R\}$. In $\Omega \times (0, \infty)$, we consider the nonstationary Stokes initial boundary value problem concerning the velocity field $u(x,t)$ and the scalar pressure $p(x,t)$:

$$\begin{cases} u_t - \Delta u + \nabla p = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $u_t = \partial u/\partial t$, Δ is the Laplacian in $\mathbb{R}_+^n$, $\nabla = (\partial_1, \ldots, \partial_n)$ with $\partial_j = \partial/\partial x_j$ is gradient, and $\nabla \cdot u = \mathrm{div} u = \sum_{j=1}^n \partial_j u_j$ is the divergence of u.

To discuss our results more precisely, at first we outline at this point our notation used throughout the paper. To denote the special sets, we use the following symbols:

$$B_R = \{x \in \mathbb{R}_+^n;\ |x| < R\},\ \Omega_R = \Omega \cap B_R.$$

We will use the standard notations $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)}$ (or $\|\cdot\|_q$ if the underling domain is known from the context). We put

$$\begin{aligned} \mathbb{L}^p(\Omega) &= \{u = (u_1, \ldots, u_n);\ u_j \in L^p(\Omega), j = 1, \cdots n\}, \\ \mathbb{L}^p_R(\Omega) &= \{u \in \mathbb{L}^p(\Omega);\ u(x) = 0 \text{ for } |x| > R\}, \\ \mathbb{J}^p(\Omega) &= \text{the completion in } \mathbb{L}^p(\Omega) \text{ of the set } \{u \in C_0^\infty(\Omega);\ \nabla \cdot u = 0 \text{ in } \Omega\}, \\ \mathbb{J}^p_R(\Omega) &= \{u \in \mathbb{J}^p(\Omega);\ u(x) = 0 \text{ for } |x| > R\}, \\ \mathbb{G}^p(\Omega) &= \{\nabla p \in \mathbb{L}^p(\Omega);\ p \in L^p_{loc}(\Omega)\}. \end{aligned}$$

For Banach spaces X and Y, $B(X,Y)$ denotes the Banach space of all bounded linear operators from X to Y. We write $B(X) = B(X,X)$.

For the exterior domain Ω, R. Farwig and H. Sohr [7] proved that the Banach space $\mathbb{L}^p(\Omega)$ $(1 < p < \infty)$ admits the Helmholtz decomposition : $\mathbb{L}^p(\Omega) = \mathbb{J}^p(\Omega) \oplus \mathbb{G}^p(\Omega)$, where

*Department of Mathematical Sciences School of Science and Engineering, Waseda University
†Advanced Reserch Institute for Science and Engineering, Waseda University

$\oplus$ denotes the direct sum. Let $\mathbb{P}$ be a continuous projection from $\mathbb{L}^p(\Omega)$ to $\mathbb{J}^p(\Omega)$. The Stokes operator A is defined by $A = -\mathbb{P}\Delta$ with domain $D(A) = \{u \in \mathbb{J}^p(\Omega) \cap \mathbb{W}^{2,p}(\Omega);\ u|_{\partial\Omega} = 0\}$. It is proved by R. Farwig and H. Sohr that $-A$ generates an analytic semigroup e^{-tA} in $\mathbb{J}^p(\Omega)$.

Now we state our main results.

Theorem 1.1 (Local energy decay) *Let $1 < p < \infty$ and let m be a nonnegative integer. R is any positive number such that $\Omega \backslash B_R = \mathbb{R}^n_+ \backslash B_R$. Then, for any $t \geq 1$, $a \in \mathbb{J}^p_R(\Omega)$, there exists a positive constant $C_{p,m}$ such that*

$$\|\partial_t^m e^{-tA} a\|_{\mathbb{L}^p(\Omega_R)} \leq C_{p,m} t^{-\frac{n+1}{2}-m} \|a\|_{\mathbb{L}^p(\Omega)}. \tag{1.2}$$

Theorem 1.2 ($L^q - L^r$ estimates) *Let $n \geq 3$.*

1. *For all $t > 0$, $a \in \mathbb{J}^q(\Omega)$ and $1 \leq q \leq r \leq \infty (q \neq \infty,\ r \neq 1)$, there holds the estimate:*
$$\|e^{-tA} a\|_{\mathbb{L}^r(\Omega)} \leq C_{q,r} t^{-(n/q-n/r)/2} \|a\|_{\mathbb{L}^q(\Omega)}. \tag{1.3}$$
2. *For all $t > 0$, $a \in \mathbb{J}^q(\Omega)$ and $1 < q \leq r \leq n$, there holds the estimate:*
$$\|\nabla e^{-tA} a\|_{\mathbb{L}^r(\Omega)} \leq C_{q,r} t^{-(n/q-n/r)/2-\frac{1}{2}} \|a\|_{\mathbb{L}^q(\Omega)}. \tag{1.4}$$

2 The representation formula of the solution to the Stokes resolvent problem in $\mathbb{R}^n_+$

In this section, we shall give the solution formula of the Stokes resolvent problem:

$$\begin{cases} (\lambda - \Delta)u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ \qquad u = 0 & \text{on} \quad x_n = 0. \end{cases} \tag{2.1}$$

Let $P_\lambda f$ and $\pi_\lambda f$ be defined by $P_\lambda f = u$ and $\pi_\lambda f = p$ which satisfy (2.1). By Farwig-Sohr[7], we can construct $P_\lambda f \in W^{2,p}(\mathbb{R}^n_+)$ and $\pi_\lambda f \in \widehat{W}^{1,p}(\mathbb{R}^n_+)$ by using partial Fourier transform, which satisfies the estimate:

$$\|P_\lambda f\|_{W^{2,p}(\mathbb{R}^n_+)} + \|\nabla \pi_\lambda f\|_{L^p(\mathbb{R}^n_+)} \leq C_{\varepsilon,\lambda} \|f\|_{L^p(\mathbb{R}^n_+)},$$

provided that $\lambda \in \Sigma_\varepsilon$ and $|\lambda| \geq \lambda_0$. We shall investigate the property of $P_\lambda f$ and $\pi_\lambda f$ near $\lambda = 0$ when f has compact support.

In the course of our argument below, we shall use the following proposition which is proved by using the residue theorem.

Proposition 2.1 *For nonnegative integer k, the following equalities are valid:*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iz\xi_n} \xi_n^k}{\lambda + |\xi|^2} d\xi_n = \begin{cases} \dfrac{i^k (\lambda + |\xi'|^2)^{\frac{k-1}{2}}}{2} e^{-\sqrt{\lambda + |\xi'|^2}|z|} & z > 0, \\ \dfrac{(-i)^k (\lambda + |\xi'|^2)^{\frac{k-1}{2}}}{2} e^{-\sqrt{\lambda + |\xi'|^2}|z|} & z < 0, \end{cases}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iz\xi_n} \xi_n^k}{|\xi|^2 (\lambda + |\xi|^2)} d\xi_n = \begin{cases} \dfrac{i^k}{2\lambda} \left[|\xi'|^{k-1} e^{-|\xi'||z|} - (\lambda + |\xi'|^2)^{\frac{k-1}{2}} e^{-\sqrt{\lambda + |\xi'|^2}|z|} \right] & z > 0, \\ \dfrac{(-i)^k}{2\lambda} \left[|\xi'|^{k-1} e^{-|\xi'||z|} - (\lambda + |\xi'|^2)^{\frac{k-1}{2}} e^{-\sqrt{\lambda + |\xi'|^2}|z|} \right] & z < 0. \end{cases}$$

In order to obtain representation formula of solution $(P_\lambda f, \pi_\lambda f)$, first we extend a given external force $f = (f_1, \ldots, f_n)$ to $F = (f_1^e, \ldots, f_{n-1}^e, f_n^o)$, where $f^e(x)$ and $f^o(x)$ denote the even extension and the odd extension respectively defined by

$$f^e(x) = \begin{cases} f(x', x_n) & x_n > 0, \\ f(x', -x_n) & x_n < 0, \end{cases} \quad f^o(x) = \begin{cases} f(x', x_n) & x_n > 0, \\ -f(x', -x_n) & x_n < 0. \end{cases}$$

Let (U, Ψ) be the solution in a whole space corresponding to this F, namely (U, Ψ) is the solution of the equation:

$$(\lambda - \Delta)U + \nabla\Psi = F, \qquad \nabla \cdot U = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{2.2}$$

By the Fourier transform, (2.2) is reduced to the following equations:

$$(\lambda + |\xi|^2)\widehat{U}(\xi) + i\xi\widehat{\Psi}(\xi) = \widehat{F}(\xi) \qquad \text{for} \quad \mathbb{R}^n. \tag{2.3}$$

Since $\Delta\Psi = \nabla \cdot F$ follows from (2.2), by (2.3) we have

$$\Psi(x) = \mathcal{F}_\xi^{-1}\left[-\frac{i\xi \cdot \widehat{F}(\xi)}{|\xi|^2}\right](x), \tag{2.4}$$

$$U(x) = \mathcal{F}_\xi^{-1}\left[\frac{1}{\lambda + |\xi|^2}\left(\widehat{F}(\xi) - \frac{\xi\xi \cdot \widehat{F}(\xi)}{|\xi|^2}\right)\right](x), \tag{2.5}$$

where $\widehat{f}(\xi)$ and $\mathcal{F}_\xi^{-1}[f(\xi)](x)$ denote the Fourier transform of $f(x)$ and the Fourier inverse transform of $f(\xi)$ respectively defined by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}_\xi^{-1}[f(\xi)](x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

For the argument below, we caluculate $\widetilde{U_j}(\xi', 0)$, where $\widetilde{f}(\xi', x_n)$ denotes the partial Fourier transform of $f(x)$ with respect to $x' = (x_1, \ldots, x_{n-1})$ definded by

$$\widetilde{f}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} f(x', x_n)dx'.$$

Noticing the formula:

$$\widetilde{U_j}(\xi', 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_n\xi_n}\widehat{U_j}(\xi)d\xi_n\Big|_{x_n=0} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{U_j}(\xi', \xi_n)d\xi_n,$$

and applying Proposition 2.1 to (2.5), we obtain

$$\begin{aligned} \widetilde{U_j}(\xi', 0) &= \int_0^{+\infty} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}}{\sqrt{\lambda + |\xi'|^2}}\widetilde{f_j}(\xi', x_n) + \sum_{k=1}^{n-1}\xi_j\xi_k\widetilde{f_k}(\xi', x_n)\frac{1}{\lambda}\left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}}{\sqrt{\lambda + |\xi'|^2}} - \frac{e^{-|\xi'|x_n}}{|\xi'|}\right) \\ &\quad -i\xi_j\widetilde{f_n}(\xi', x_n)\frac{1}{\lambda}\left(e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-x_n|\xi'|}\right)dx_n \quad \text{for } j = 1, \cdots, n-1, \end{aligned} \tag{2.6}$$

$$\widetilde{U_n}(\xi', 0) = 0. \tag{2.7}$$

Secondary, we shall construct a Riemann function (v, θ). To this end, put $u = U + v$ and $\pi = \Phi + \theta$, then (v, θ) enjoys the equation:

$$\begin{cases} (\lambda - \Delta)v + \nabla\theta = 0, \quad \nabla \cdot v = 0 & \text{in} \quad \mathbb{R}^n_+, \\ v_j(x', 0) = -U_j(x', 0) & \text{for} \quad j = 1, \cdots, n-1, \\ v_n(x', 0) = 0. \end{cases} \tag{2.8}$$

A solution (v,θ) to (2.8) is given by the following formula:

$$\begin{aligned}
\widetilde{v_j}(\xi',x_n) &= -\widetilde{U_j}(\xi',0)e^{-\sqrt{\lambda+|\xi'|^2}x_n} \\
&\quad -\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}-e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|}\frac{\xi_j}{|\xi'|}\xi'\cdot\widetilde{U'}(\xi',0), && (2.9)\\
\widetilde{v_n}(\xi',x_n) &= -\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}-e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|}i\xi'\cdot\widetilde{U'}(\xi',0), && (2.10)\\
\widetilde{\theta}(\xi',x_n) &= \frac{\sqrt{\lambda+|\xi'|^2}+|\xi'|}{|\xi'|}e^{-|\xi'|x_n}i\xi'\cdot\widetilde{U'}(\xi',0), && (2.11)
\end{aligned}$$

where $\xi'\cdot\widetilde{U'}(\xi',0)=\sum_{j=1}^{n-1}\xi_j\widetilde{U_j}(\xi',0)$. In fact, since $\Delta\theta=0$ by (2.8), applying the Laplacian reduces (2.8) to the following equation:

$$\begin{cases}(\lambda-\Delta)\Delta v_n=0,\\ v_n(x',0)=0,\\ \partial_n v_n(x',0)=\nabla'\cdot U'(x',0),\end{cases} \tag{2.12}$$

where $U'(x)=(U_1,\dots,U_{n-1})(x)$ and $\nabla'=(\partial_1,\dots,\partial_{n-1})$. Application of the partial Fourier transform converts (2.12) into

$$\begin{cases}(\partial_n^2-\lambda-|\xi'|^2)(\partial_n^2-|\xi'|^2)\widetilde{v_n}(\xi',x_n)=0,\\ \widetilde{v_n}(\xi',0)=0,\\ \partial_n\widetilde{v_n}(\xi',0)=i\xi'\cdot\widetilde{U'}(\xi',0).\end{cases} \tag{2.13}$$

Also application of the partial Fourier transform converts $(\lambda-\Delta)v_n+\partial_n\theta=0$ in (2.8) and $\Delta\theta=0$ into

$$\begin{cases}(|\xi'|^2-\partial_n^2)\widetilde{\theta}(\xi',x_n)=0,\\ \partial_n\widetilde{\theta}(\xi',0)=-\left(\lambda+|\xi'|^2-\partial_n^2\right)\widetilde{v_n}(\xi',x_n)\big|_{x_n=0}.\end{cases} \tag{2.14}$$

Solving (2.13)-(2.14), we obtatin (2.10)-(2.11).
Next we shall show (2.9). Application of (2.8) implies that $\widetilde{v_j}(\xi',x_n)$ satisfies the equation:

$$\begin{cases}-(\partial_n^2-\lambda-|\xi'|^2)\widetilde{v_j}(\xi',x_n)=i\xi_j\widetilde{\theta}(\xi',x_n),\\ \widetilde{v_j}(\xi',0)=-\widetilde{U_j}(\xi',0),\end{cases} \tag{2.15}$$

for $j=1,2,\cdots,n-1$. Solving (2.15), we obatin (2.9). Summing up, we have obtatined the solution formulas of (u,p):

$$
\begin{aligned}
\widetilde{u_j}(\xi',x_n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_n\xi_n}\left\{\frac{1}{\lambda+|\xi|^2}\left[\widehat{f_j^e}(\xi)-\sum_{k=1}^{n-1}\frac{\xi_j\xi_k}{|\xi|^2}\widehat{f_k^e}(\xi)-\frac{\xi_j\xi_n}{|\xi|^2}\widehat{f_n^o}(\xi)\right]\right\}d\xi_n \\
&\quad -e^{-\sqrt{\lambda+|\xi'|^2}x_n}\widetilde{U_j}(\xi',0)-\sum_{k=1}^{n-1}\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}-e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi|}\frac{\xi_j\xi_k}{|\xi'|}\widetilde{U_k}(\xi',0), \\
&\qquad\qquad \text{for } j=1,2,\cdots,n-1 \qquad (2.16) \\
\widetilde{u_n}(\xi',x_n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_n\xi_n}\left\{\frac{1}{\lambda+|\xi|^2}\left[\widehat{f_n^o}(\xi)-\sum_{k=1}^{n-1}\frac{\xi_n\xi_k}{|\xi|^2}\widehat{f_k^e}(\xi)-\frac{\xi_n^2}{|\xi|^2}\widehat{f_n^o}(\xi)\right]\right\}d\xi_n \\
&\quad -i\sum_{k=1}^{n-1}\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}-e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi|}\xi_k\widetilde{U_k}(\xi',0), \qquad (2.17) \\
\widetilde{p}(\xi',x_n) &= \frac{-i}{2\pi}\int_{-\infty}^{\infty} e^{ix_n\xi_n}\left\{\sum_{k=1}^{n-1}\frac{\xi_k}{|\xi|^2}\widehat{f_k^e}(\xi)+\frac{\xi_n}{|\xi|^2}\widehat{f_n^o}(\xi)\right\}d\xi_n \\
&\quad +i\frac{\sqrt{\lambda+|\xi'|^2}+|\xi'|}{|\xi'|}e^{-|\xi'|x_n}\sum_{k=1}^{n-1}\xi_k\widetilde{U_k}(\xi',0). \qquad (2.18)
\end{aligned}
$$

When we consider the expansion formula of (u,p) when $|\lambda|\le\frac{1}{2}$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, we devide the solution formula of (2.16)-(2.18) into the parts where $|\xi'|\le 2$ and where $|\xi'|\ge 1$ by using the cut-off function. And to analyse the part where $|\xi'|\le 2$, we use the following more detailed formulas obatined by applying Proposition 2.1 to (2.16) - (2.18);

$$
\begin{aligned}
\widetilde{u_j}(\xi',x_n) &= \frac{1}{2}\int_0^{\infty}\left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}|x_n-y_n|}}{\sqrt{\lambda+|\xi'|^2}}-\frac{e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)}}{\sqrt{\lambda+|\xi'|^2}}\right)\widetilde{f_j}(\xi',y_n)dy_n \\
&+ \sum_{k=1}^{n-1}\frac{\xi_j\xi_k}{2\lambda}\left\{\int_0^{\infty}\left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}|x_n-y_n|}}{\sqrt{\lambda+|\xi'|^2}}-\frac{e^{-|\xi'||x_n-y_n|}}{|\xi'|}\right)\widetilde{f_k}(\xi',y_n)dy_n\right. \\
&\quad +\int_0^{\infty}\left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)}}{\sqrt{\lambda+|\xi'|^2}}-\frac{e^{-|\xi'|(x_n+y_n)}}{|\xi'|}\right)\widetilde{f_k}(\xi',y_n)dy_n \\
&\quad \left.-2e^{-\sqrt{\lambda+|\xi'|^2}x_n}\int_0^{\infty}\left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}}-\frac{e^{-|\xi'|y_n}}{|\xi'|}\right)\widetilde{f_k}(\xi',y_n)dy_n\right\} \\
&+\frac{i\xi_j}{2\lambda}\left\{\int_0^{x_n}\left(e^{-\sqrt{\lambda+|\xi'|^2}(x_n-y_n)}-e^{-|\xi'|(x_n-y_n)}\right)\widetilde{f_n}(\xi',y_n)dy_n\right. \\
&\quad -\int_{x_n}^{\infty}\left(e^{-\sqrt{\lambda+|\xi'|^2}(y_n-x_n)}-e^{-|\xi'|(y_n-x_n)}\right)\widetilde{f_n}(\xi',y_n)dy_n \\
&\quad -\int_0^{\infty}\left(e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)}-e^{-|\xi'|(x_n+y_n)}\right)\widetilde{f_n}(\xi',y_n)dy_n \\
&\quad \left.-2e^{-\sqrt{\lambda+|\xi'|^2}x_n}\int_0^{\infty}\left(e^{-\sqrt{\lambda+|\xi'|^2}y_n}-e^{-|\xi'|y_n}\right)\widetilde{f_n}(\xi',y_n)dy_n\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{k=1}^{n-1} \frac{\xi_j\xi_k}{|\xi'|} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \int_0^\infty \frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \widetilde{f_k}(\xi', y_n)\,dy_n \\
&- \sum_{k=1}^{n-1} \frac{\xi_j\xi_k|\xi'|}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|}\right) \widetilde{f_k}(\xi', y_n)\,dy_n \\
&+ \frac{i\xi_j|\xi'|}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \int_0^\infty \left(e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}\right) \widetilde{f_n}(\xi', y_n)\,dy_n, \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{u_n}(\xi', x_n) &= \frac{1}{2}\int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}|x_n-y_n|}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)}}{\sqrt{\lambda+|\xi'|^2}}\right) \widetilde{f_n}(\xi', y_n)\,dy_n \\
&+ \sum_{k=1}^{n-1} \frac{i\xi_k}{2\lambda} \left\{ \int_0^{x_n} \left(e^{-\sqrt{\lambda+|\xi'|^2}(x_n-y_n)} - e^{-|\xi'|(x_n-y_n)}\right) \widetilde{f_k}(\xi', y_n)\,dy_n \right. \\
&\qquad - \int_{x_n}^\infty \left(e^{-\sqrt{\lambda+|\xi'|^2}(y_n-x_n)} - e^{-|\xi'|(y_n-x_n)}\right) \widetilde{f_k}(\xi', y_n)\,dy_n \\
&\qquad \left. + \int_0^\infty \left(e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)} - e^{-|\xi'|(x_n+y_n)}\right) \widetilde{f_k}(\xi', y_n)\,dy_n \right\} \\
&- \frac{1}{2\lambda} \left\{ \int_0^\infty \left(\sqrt{\lambda+|\xi'|^2}e^{-\sqrt{\lambda+|\xi'|^2}|x_n-y_n|} - |\xi'|e^{-|\xi'||x_n-y_n|}\right) \widetilde{f_n}(\xi', y_n)\,dy_n \right. \\
&\qquad \left. - \int_0^\infty \left(\sqrt{\lambda+|\xi'|^2}e^{-\sqrt{\lambda+|\xi'|^2}(x_n+y_n)} - |\xi'|e^{-|\xi'|(x_n+y_n)}\right) \widetilde{f_n}(\xi', y_n)\,dy_n \right\} \\
&- \sum_{k=1}^{n-1} i\xi_k \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \int_0^\infty \frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \widetilde{f_k}(\xi', y_n)\,dy_n \\
&- \sum_{k=1}^{n-1} \frac{i|\xi'|^2\xi_k}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|}\right) \widetilde{f_k}(\xi', y_n)\,dy_n \\
&- \frac{|\xi'|}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \int_0^\infty \left(e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}\right) \widetilde{f_n}(\xi', y_n)\,dy_n, \qquad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{p}(\xi', x_n) &= -i\sum_{k=1}^{n-1} \frac{\xi_k}{2|\xi'|} \left\{ \int_0^\infty \left(e^{-|\xi'||x_n-y_n|} + e^{-|\xi'|(x_n+y_n)}\right) \widetilde{f_k}(\xi', y_n)\,dy_n \right\} \\
&- \frac{i}{2} \left\{ \int_0^{x_n} e^{-|\xi'|(x_n-y_n)} \widetilde{f_n}(\xi', y_n)\,dy_n - \int_{x_n}^\infty e^{-|\xi'|(y_n-x_n)} \widetilde{f_n}(\xi', y_n)\,dy_n \right. \\
&\qquad \left. - \int_0^\infty e^{-|\xi'|(x_n+y_n)} \widetilde{f_n}(\xi', y_n)\,dy_n \right\} \\
&+ \sum_{k=1}^{n-1} \frac{i\xi_k(\sqrt{\lambda+|\xi'|}+|\xi'|)}{|\xi'|} e^{-|\xi'|x_n} \int_0^\infty \frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \widetilde{f_k}(\xi', y_n)\,dy_n \\
&+ \sum_{k=1}^{n-1} \frac{i\xi_k|\xi'|(\sqrt{\lambda+|\xi'|}+|\xi'|)}{\lambda} e^{-|\xi'|x_n} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|}\right) \widetilde{f_k}(\xi', y_n)\,dy_n \\
&+ \frac{(\sqrt{\lambda+|\xi'|^2}+|\xi'|)|\xi'|}{\lambda} e^{-|\xi'|x_n} \int_0^\infty \left(e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}\right) \widetilde{f_n}(\xi', y_n)\,dy_n. \qquad (2.21)
\end{aligned}
$$

3 An expansion of Solution Operator of (2.1)

The aim of this section is to get an expansion of solution operator of (2.1). To this end, we choose $\varphi_0(r) \in C^\infty(\mathbb{R}^{n-1})$ so that

$$\varphi_0(r) = 1 \quad \text{for} \quad r \leq 1 \quad \text{and} \quad \varphi_0(r) = 0 \quad \text{for} \quad r \geq 2. \tag{3.1}$$

Put $\varphi_\infty(r) = 1 - \varphi_0(r)$ and set

$$\begin{aligned} R_j^\infty(\lambda)f &= \mathcal{F}_{\xi'}^{-1}\left[\varphi_\infty(|\xi'|)\widetilde{u_j}(\xi', x_n)\right](x'), \qquad j = 1, 2, \cdots, n & (3.2)\\ \Pi^\infty(\lambda)f &= \mathcal{F}_{\xi'}^{-1}\left[\varphi_\infty(|\xi'|)\widetilde{p}(\xi', x_n)\right](x'), \\ R_j^0(\lambda)f &= \mathcal{F}_{\xi'}^{-1}\left[\varphi_0(|\xi'|)\widetilde{u_j}(\xi', x_n)\right](x'), \qquad j = 1, 2, \cdots, n & (3.3)\\ \Pi^0(\lambda)f &= \mathcal{F}_{\xi'}^{-1}\left[\varphi_0(|\xi'|)\widetilde{p}(\xi', x_n)\right](x'), \end{aligned}$$

where $\mathcal{F}_{\xi'}^{-1}[f(\xi', x_n)](x)$ denotes the partial Fourier inverse transform with respect to ξ' defined by

$$\mathcal{F}_{\xi'}^{-1}[f(\xi', x_n)](x) = \left(\frac{1}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} f(\xi', x_n)d\xi'.$$

In particular, setting

$$\begin{aligned} R(\lambda)f &= {}^t(R_1^\infty(\lambda)f, \ldots, R_n^\infty(\lambda)f) +^t (R_1^0(\lambda)f, \ldots, R_n^0(\lambda)) & (3.4)\\ \Pi(\lambda)f &= \Pi^\infty(\lambda)f + \Pi^0(\lambda)f, & (3.5) \end{aligned}$$

$(u, p) = (R(\lambda)f, \Pi(\lambda)f)$ is the solution operator which gives us the solution of (2.1). We get the following theorem for resolvent expansion around the origin .

Theorem 3.1 *Let $n \geq 2$ and $\Omega = \mathbb{R}_+^n$. We put $U_{1/2} = \{\lambda \in \mathbb{C}; |\lambda| < 1/2\}$. Then $(R(\lambda), \Pi(\lambda))$ has the following expansion of with respect $\lambda \in U_{1/2}\backslash(-\infty, 0]$:*

$$(R(\lambda), \Pi(\lambda)) = \begin{cases} G_1(\lambda)\lambda^{\frac{n-1}{2}} + G_2(\lambda)\lambda^{\frac{n}{2}}\log\lambda + G_3(\lambda) & n \text{ is even,} \\ G_1(\lambda)\lambda^{\frac{n}{2}} + G_2(\lambda)\lambda^{\frac{n-1}{2}}\log\lambda + G_3(\lambda) & n \text{ is odd,} \end{cases} \tag{3.6}$$

where $G_1(\lambda), G_2(\lambda)$ and $G_3(\lambda)$ are $B(L_R^p(\mathbb{R}_+^n), W^{2,p}(B_R^+) \cap W^{1,p}(B_R^+))$-valued holomorophic function in $U_{1/2}$.

In order to prove Theorem 3.1, we introduce the following proposition.

Proposition 3.2 *Let $\varphi_0(r)$ be the same as (3.1). Then, the following assertion are valid. For any non-negative integers a and b, we have the formulas:*

$$\int_0^\infty \varphi_0(r)r^a\left(\sqrt{\lambda + r^2}\right)^{b-1}dr = H_{a,b}(\lambda) + B_{a,b}^1\lambda^{\frac{a+b}{2}} + B_{a,b}^2\lambda^{\frac{a+b}{2}}\log\lambda, \tag{3.7}$$

$$\int_0^\infty \varphi_0(r)\frac{r^a\left(\sqrt{\lambda + r^2}\right)^{b-1}}{\sqrt{\lambda + r^2} + r}dr = \widetilde{H_{a,b}}(\lambda) + \widetilde{B_{a,b}^1}\lambda^{\frac{a+b}{2}} + \widetilde{B_{a,b}^2}\lambda^{\frac{a+b}{2}}\log\lambda, \tag{3.8}$$

where $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{1}{2}$. Here $H_{a,b}(\lambda)$ and $\widetilde{H_{a,b}}(\lambda)$ are the holomorophic functions in $\lambda \in U_{1/2} = \{\lambda \in \mathbb{C}; |\lambda| \leq \frac{1}{2}\}$; $B_{a,b}^1$, $B_{a,b}^2$, $\widetilde{B_{a,b}^1}$ and $\widetilde{B_{a,b}^2}$ are the real numbers. Moreover, they have the following properties:

$$\begin{aligned} &\widetilde{B_{a,b}^2} = 0 \quad \textit{when } a + b \textit{ is an odd number;} \\ &B_{a,b}^2 = 0 \quad \textit{when } a + b \textit{ is an even number;} \end{aligned}$$

$$|B^1_{a,b}| \leq 1, \quad |\widetilde{B^1_{a,b}}| \leq 1, \quad |B^2_{a,b}| \leq \frac{1}{2}, \quad |\widetilde{B^2_{a,b}}| \leq \frac{1}{2},$$

and there exist constants C and L independence of a and b such that

$$|H_{a,b}(\lambda)| \leq CL^{a+b}, \qquad |\widetilde{H_{a,b}}(\lambda)| \leq CL^{a+b} \qquad \lambda \in U_{\frac{1}{2}}.$$

[Proof of Proposition 3.2] By substituting $t = r + \sqrt{\lambda + r^2}$, we see easily the formula (3.7) and (3.8). In fact,

$$\begin{aligned}
H_{a,b}(\lambda) &= \int_1^\infty \varphi_0(r) r^a \left(\sqrt{\lambda + r^2}\right)^{b-1} dr \\
&+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0,\ell+m\neq\frac{a+b}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{\lambda^{\ell+m}\left(1+\sqrt{1+\lambda}\right)^{a+b-2(\ell+m)}}{a+b-2(\ell+m)} \\
&+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0,\ell+m=\frac{a+b}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m} \lambda^{\frac{a+b}{2}} \log\left(1+\sqrt{1+\lambda}\right), \\
B^1_{a,b} &= -\left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0,\ell+m\neq\frac{a+b}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{1}{a+b-2(\ell+m)}, \\
B^2_{a,b} &= -\left(\frac{1}{2}\right)^{a+b+1} \sum_{\ell=0,\ell+m=\frac{a+b}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m}, \\
\widetilde{H_{a,b}}(\lambda) &= \int_1^\infty \varphi_0(r) \frac{r^a\left(\sqrt{\lambda+r^2}\right)^{b-1}}{\sqrt{\lambda+r^2}+r} dr \\
&+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0,\ell+m\neq\frac{a+b-1}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{\lambda^{\ell+m}\left(1+\sqrt{1+\lambda}\right)^{a+b-1-2(\ell+m)}}{a+b-1-2(\ell+m)} \\
&+ \left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0,\ell+m=\frac{a+b-1}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m} \lambda^{\frac{a+b-1}{2}} \log\left(1+\sqrt{1+\lambda}\right), \\
\widetilde{B^1_{a,b}} &= -\left(\frac{1}{2}\right)^{a+b} \sum_{\ell=0,\ell+m\neq\frac{a+b-1}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m} \frac{1}{a+b-1-2(\ell+m)}, \\
\widetilde{B^2_{a,b}} &= -\left(\frac{1}{2}\right)^{a+b+1} \sum_{\ell=0,\ell+m=\frac{a+b-1}{2}}^{a} \sum_{m=0}^{b} (-1)^\ell \binom{a}{\ell} \binom{b}{m}.
\end{aligned}$$

■

[Proof of Theorem 3.1] The assertion of Theorem 3.1 for $|\xi'| \geq 1$ follows from Fourier multiplier theorem, but to get the expansion for $|\xi'| \leq 1$, we need the different method from the $|\xi'| \geq 1$ case. We explain how to expand the $v_n(x)$ only because we can expand all the other term by the same method. To develope the expansion of $v_n(x)$, we separate $v_n(x)$

into three parts :

$$
\begin{aligned}
v_n(x) &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \varphi_0(|\xi'|) e^{ix'\cdot\xi'} \\
&\quad \left\{ -\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \sum_{j=1}^{n-1} \frac{i\xi_j}{\sqrt{\lambda+|\xi'|^2}} \int_0^\infty e^{-\sqrt{\lambda+|\xi'|^2}y_n} \widetilde{f}_j(\xi', y_n) dy_n \right. \\
&\quad + \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \sum_{j=1}^{n-1} \frac{|\xi'|^2}{\lambda} i\xi_j \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) \widetilde{f}_j(\xi', y_n) dy_n \\
&\quad \left. - \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \frac{|\xi'|^2}{\lambda} \int_0^\infty \left(e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n} \right) \widetilde{f}_n(\xi', y_n) dy_n \right\} d\xi' \\
&= : I + II + III
\end{aligned}
$$

where $\varphi_0(r)$ is the same as in (3.1). At first, we consider the first term I:

$$
I = -\left(\frac{1}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{n-1}} \varphi_0(|\xi'|) e^{ix'\cdot\xi'} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} \sum_{j=1}^{n-1} \frac{i\xi_j e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} \int_0^\infty \widetilde{f}_j(\xi', y_n) dy_n.
$$

By using Maclaurin expansion of e^x, we can expand the first factor and the second factor as follows:

$$
\begin{aligned}
\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2}-|\xi'|} &= -x_n \int_0^1 e^{-|\xi'|(1-\theta)x_n} e^{-\theta\sqrt{\lambda+|\xi'|^2}x_n} d\theta \\
&= \sum_{p,t=0}^\infty \frac{(-x_n)^{p+t+1}}{(p+t+1)!} |\xi'|^t \left(\sqrt{\lambda+|\xi'|^2}\right)^p. \qquad (3.9) \\
\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} &= \sum_{q=0}^\infty \frac{(-y_n)^q}{q!} \left(\sqrt{\lambda+|\xi'|^2}\right)^{q-1}. \qquad (3.10)
\end{aligned}
$$

Moreover, we rewrite $e^{ix'\cdot\xi'}\widetilde{f}_j(\xi', x_n)$ as follows:

$$
e^{ix'\cdot\xi'}\widetilde{f}_j(\xi', x_n) = \int_{\mathbb{R}^{n-1}} e^{i(x'-y')\cdot\xi'} f_j(y', y_n) dy' = \sum_{s=0}^\infty \int_{\mathbb{R}^{n-1}} \frac{(i(x'-y')\cdot\xi')^s}{s!} f_j(y) dy'. (3.11)
$$

By substituting (3.9)-(3.11) into I, we can rewrite I as follows:

$$
\begin{aligned}
I &= -\left(\frac{1}{2\pi}\right)^{n-1} \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} f_j(y) dy \\
&\quad \times \sum_{p,q,s,t} \frac{(-x_n)^{p+t+1}(-y_n)^q}{(p+t+1)!q!} \int_{\mathbb{R}^{n-1}} \varphi_0(|\xi'|) i\xi_j |\xi'|^t \left(\sqrt{\lambda+|\xi'|^2}\right)^{p+q-1} (i(x'-y')\cdot\xi')^s d\xi' \\
&= -\left(\frac{1}{2\pi}\right)^{n-1} \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} f_j(y) dy \\
&\quad \times \sum_{p,q,s,t} \frac{(-x_n)^{p+t+1}(-y_n)^q}{(p+t+1)!q!} \int_{S^{n-1}} i\omega_j \frac{(i(x'-y')\cdot\omega')^s}{s!} d\sigma \int_0^\infty \varphi_0(r) r^{s+t+n-1} \left(\sqrt{\lambda+r^2}\right)^{p+q-1} dr
\end{aligned}
$$

Applying Propositon 3.2, when n is odd we have

$$I = G^I(\lambda)f + \lambda^{\frac{n-1}{2}}G^{II}(\lambda)f + \lambda^{\frac{n}{2}}G^{III}(\lambda)f + \lambda^{\frac{n-1}{2}}\log\lambda G^{IV}(\lambda)f,$$

where we have put

$$\begin{aligned}
G_{p,q,s,t}(\lambda)f &= -\left(\frac{1}{2\pi}\right)^{n-1}\sum_{j=1}^{n-1}\int_{\mathbb{R}^n_+} f_j(y)dy\int_{S^{n-1}} i\omega_j(-i(x'-y')\cdot\omega')^s d\sigma\frac{(-x_n)^{p+t+1}(-y_n)^q}{s!(p+t+1)!q!},\\
G^I(\lambda)f &= \sum_{p,q,s,t}(G_{p,q,s,t}(\lambda)f)h_{s+t+n-1,p+q}(\lambda),\\
G^{II}(\lambda)f &= \sum_{m=0}^{\infty}\sum_{p+q+s+t=2m}(G_{p,q,s,t}(\lambda)f)B^1_{s+t+n-1,p+q}\lambda^m,\\
G^{III}(\lambda)f &= \sum_{m=0}^{\infty}\sum_{p+q+s+t=2m+1}(G_{p,q,s,t}(\lambda)f)B^1_{s+t+n-1,p+q}\lambda^m,\\
G^{IV}(\lambda)f &= \sum_{m=0}^{\infty}\sum_{p+q+s+t=2m}(G_{p,q,s,t}(\lambda)f)B^2_{s+t+n-1,p+q}\lambda^m,
\end{aligned}$$

and when n is even we have

$$I = G^I(\lambda)f + \lambda^{\frac{n-1}{2}}G^{II}(\lambda)f + \lambda^{\frac{n}{2}}G^{III}(\lambda)f + \lambda^{\frac{n}{2}}\log\lambda G^{IV}(\lambda)f,$$

where $G_{p,q,s,t}(\lambda)f, G^I(\lambda)f, G^{II}(\lambda)f$ and $G^{III}(\lambda)f$ are the same as in the odd dimension case, but

$$G^{IV}(\lambda)f = \sum_{m=0}^{\infty}\sum_{p+q+s+t=2m+1}(G_{p,q,s,t}(\lambda)f)B^2_{s+t+n-1,p+q}\lambda^m.$$

When $|x| < R$ and supp $f \subset \{y \in \mathbb{R}^n_+; |y| < R\}$, we can show that the right hand sides of $G^i(\lambda)f$ $(i = I, II, III, IV)$ are absolutely convergent when $\lambda \in U_{\frac{1}{2}}$. In fact, since

$$\sup_{|x|\le R}|G_{p,q,s,t}(\lambda)f| \le C_R\|f\|_{L^p(\mathbb{R}^n_+)}\frac{R^{p+q+t+1}(2R)^s}{s!(p+t+1)!q!},$$

by Proposition 3.2, we have

$$\begin{aligned}
\sup_{|x|\le R}|G^I(\lambda)f| &\le \sum_{p,q,s,t}\frac{R^{p+q+t+1}(2R)^s L^{s+t+n-1+p+q}}{s!(p+t+1)!q!}\times CC_R\|f\|_{L^p(\mathbb{R}^n_+)}\\
&\le CC_R\|f\|_{L^p(\mathbb{R}^n_+)}L^{n-1}e^{2LR}e^{LR}\sum_{p,t}\frac{(LR)^{p+t}}{(p+t+1)!}\\
&\le CC_R\|f\|_{L^p(\mathbb{R}^n_+)}L^{n-1}e^{2LR}e^{LR}\sum_{p,t}\frac{(LR)^{p+t}}{p!t!}\\
&\le CC_R\|f\|_{L^p(\mathbb{R}^n_+)}L^{n-1}e^{5LR}e^{LR},
\end{aligned}$$

where we have used the fact that $(p+t+1)! \geq p!t!$. In the same manner, by Proposition 3.2 we have

$$
\begin{aligned}
|G^{II}(\lambda)f| &\leq C_R\|f\|_{L^p(\mathbb{R}^n_+)} \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m} \frac{1}{s!(p+t+1)!q!} \\
&= C_R\|f\|_{L^p(\mathbb{R}^n_+)} \sum_{m=0}^{\infty} \sum_{p+t\leq 2m} \frac{1}{(p+t+1)!} \sum_{s=0}^{2m-p-t} \frac{1}{s!(2m-p-t-s)!} \\
&= C_R\|f\|_{L^p(\mathbb{R}^n_+)} \sum_{m=0}^{\infty} \sum_{p+t\leq 2m} \frac{2^{2m-p-t}}{(p+t+1)!(2m-p-t)!} \\
&\leq C_R\|f\|_{L^p(\mathbb{R}^n_+)} \sum_{m=0}^{\infty} \sum_{\ell=0}^{2m} \frac{2^{2m}}{\ell!(2m-\ell)!} \leq C_R\|f\|_{L^p(\mathbb{R}^n_+)} \sum_{m=0}^{\infty} \frac{2^{2m}}{(2m)!} = C_R e^4\|f\|_{L^p(\mathbb{R}^n_+)},
\end{aligned}
$$

and also we have

$$
\begin{aligned}
|G^{III}(\lambda)f| &\leq C_R\|f\|_{L^p(\mathbb{R}^n_+)} \sum_{m=0}^{\infty} \sum_{p+q+s+t=2m+1} \frac{1}{s!(p+t+1)!q!} \leq C_R e^4\|f\|_{L^p(\mathbb{R}^n_+)}, \\
|G^{IV}(\lambda)f| &\leq C_R\frac{e^4}{2}\|f\|_{L^p(\mathbb{R}^n_+)}.
\end{aligned}
$$

Therefore, $G^i(\lambda)f$ $(i = I, II, III, IV)$ are the holomorphic with respect to $\lambda \in U_{\frac{1}{2}}$. In the same way, we obtain the similar expansion formula of II and III. This completes the proof of the proposition. ■

4 Continuity property of $(R(\lambda), \Pi(\lambda))$ near $\lambda = 0$

In this section we shall prove the following theorem:

Theorem 4.1 *Let $1 < p < \infty$ and $f = (f_1, \ldots, f_n) \in L^p(\mathbb{R}^n_+)$ with supp $f \subset B_R$. Let P_λ and π_λ be the operators defined in section 2. If we put $u = P_0 f$ and $\pi = \pi_0 f$, then $(u, \pi) \in W^{2,p}_{loc}(\mathbb{R}^n_+)^n \times W^{1,p}_{loc}(\mathbb{R}^n_+)$, and (u, π) satisfies the equation:*

$$-\Delta u + \nabla \pi = f, \ \nabla \cdot u = 0, \quad \text{in } \mathbb{R}^n_+, \qquad u(x', 0) = 0,$$

Moreover (u, π) satisfies the estimates:

$$
\begin{aligned}
\|u\|_{W^{2,p}(B^+_L)} + \|\pi\|_{W^{1,p}(B^+_L)} \leq C_{R,L}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad \text{for} \quad L > 0 \\
\sup_{|x|\geq 1, x\in\mathbb{R}^n_+} \left[|x|^{n-2}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|\pi(x)|\right] \leq C_R\|f\|_{L^p(\mathbb{R}^n_+)}.
\end{aligned}
$$

and the formula:

$$\lim_{\lambda\to 0} \|P_\lambda f - P_0 f\|_{W^{2,p}(B^+_L)} + \|\pi_\lambda f - \pi_0 f\|_{W^{1,p}(B^+_L)} = 0 \quad \text{for any} \quad L > 0,$$

when $n \geq 3$.

In order to show Theorem 4.1, we consider the limit of $P_\lambda f$ and $\pi_\lambda f$ as $\lambda \to 0$. To this end, we only consider $\tilde{v}_j(\xi', x_n)$ $(j = 1, \cdots, n)$ defined in (2.19) and (2.20). The estimates of $U(x)$ follow from the Fourier multiplier theorem, so it suffices to investigate the Riemann fuction $v_j(x)$ $(j = 1, \cdots, n)$. Let $v_j^0(x)$, $\theta^0(x)$ $(j = 1, \cdots, n)$ be limits to which $v_j(x), \theta(x)$ $(j = 1, \cdots, n)$ converge formally as $\lambda \to 0$. Below, we consider only the term $I_\lambda(x)$ defined by

$$\widetilde{I_\lambda}(\xi', x_n) = \sum_{k=1}^{n-1} \frac{i|\xi'|^2 \xi_k}{\lambda} \frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} \int_0^\infty \left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) \widetilde{f_k}(\xi', y_n) dy_n,$$

which appears as the 7th. term of $\widetilde{u_n}(\xi', x_n)$ (cf . (2.20)). Put $I^0(x) = \lim_{\lambda\to 0} I_\lambda(x)$. Noticing that as $\lambda \to 0$,

$$\begin{aligned}
\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n} - e^{-|\xi'|x_n}}{\sqrt{\lambda+|\xi'|^2} - |\xi'|} &= \int_0^1 e^{[|\xi'|+\theta(\sqrt{\lambda+|\xi'|^2}-|\xi'|)]x_n} d\theta x_n \to x_n e^{-|\xi'|x_n} \\
\frac{1}{\lambda}\left(\frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n}}{\sqrt{\lambda+|\xi'|^2}} - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) &= \frac{1}{\lambda}\left\{ \frac{e^{-\sqrt{\lambda+|\xi'|^2}y_n} - e^{-|\xi'|y_n}}{\sqrt{\lambda+|\xi'|^2}} + \left(\frac{1}{\sqrt{\lambda+|\xi'|^2}} - \frac{1}{|\xi'|} \right) e^{-|\xi'|y_n} \right\} \\
&= \frac{\sqrt{\lambda+|\xi'|^2} - |\xi'|}{\lambda\sqrt{\lambda+|\xi'|^2}} \left(y_n \int_0^1 e^{-(\theta\sqrt{\lambda+|\xi'|^2}+(1-\theta)|\xi'|)y_n} d\theta - \frac{e^{-|\xi'|y_n}}{|\xi'|} \right) \\
&\to \frac{e^{-|\xi'|y_n}}{2|\xi'|^3}(|\xi|y_n - 1).
\end{aligned}$$

$I^0(x)$ has the representation formula:

$$\begin{aligned}
I^0(x) &= \frac{1}{2}\left(\frac{1}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} x_n e^{-|\xi'|x_n} \\
&\left(\sum_{k=1}^{n-1} \frac{i\xi_k}{|\xi'|} \int_0^\infty -e^{-|\xi'|y_n} \widetilde{f_k}(\xi', y_n) dy_n + \sum_{k=1}^{n-1} i\xi_k \int_0^\infty e^{-|\xi'|y_n} y_n \widetilde{f_k}(\xi', y_n) dy_n \right) d\xi'.
\end{aligned}$$

In order to obtain the decay with respect to $|x|$, we introduce the following lemma.

Lemma 4.2 *Let B be a Banach space and $|\cdot|_B$ its corresponding norm. Let α be a number $> -n$ and set $\alpha = N + \sigma - n$ where $N \geq 0$ is an integer and $0 < \sigma \leq 1$. Let $f(\xi)$ be a function in $C^\infty(\mathbb{R}^n \backslash \{0\}; B)$ such that*

$$\partial_\xi^\gamma f(\xi) \in L^1(\mathbb{R}^n; B), \tag{4.1}$$

$$|\partial_\xi^\gamma f(\xi)|_B \leq C_\gamma |\xi|^{\alpha-|\gamma|} \quad {}^\forall\xi \neq 0 \quad {}^\forall\gamma, \tag{4.2}$$

$$g(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi) d\xi.$$

Then, we have

$$|g(x)|_B \leq C_{n,\alpha} \left(\max_{|\gamma|\leq N+2} C_\gamma \right) |x|^{-(n+\alpha)} \quad {}^\forall x \neq 0. \tag{4.3}$$

[proof of Lemma 4.2] See Shibata and Shimizu [13] ■

At first, we consider the following function $g(x)$ corresponding to $v^0(x)$:

$$g(x) = \left(\frac{1}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \frac{\xi_j}{|\xi'|} x_n e^{-|\xi'|x_n} \int_0^\infty y_n |\xi'| e^{-|\xi'|y_n} \widetilde{f_j}(\xi', y_n) dy_n d\xi'.$$

There holds the estimate:

$$|\partial_{\xi'}^{\alpha'} e^{-|\xi'|x_n}| \le C_{\alpha'}|\xi'|^{-|\alpha'|}e^{-\frac{1}{2}|\xi'|x_n}.$$

In fact, we have

$$\begin{aligned} |\partial_{\xi'}^{\alpha'} e^{-|\xi'|x_n}| &= \sum_{\sigma=1}^{|\alpha'|} x_n^\sigma e^{-|\xi'|x_n} \sum_{|\alpha_1'|+|\alpha_2'|+\cdots+|\alpha_\sigma'|=|\alpha'|,\ |\alpha_j|\ge 1} \left(\partial_{\xi'}^{\alpha_1'}|\xi'|\right)\cdots\left(\partial_{\xi'}^{\alpha_\sigma'}|\xi'|\right) \\ &\le C\sum_{\sigma=1}^{|\alpha'|} |\xi'|^{\sigma-|\alpha'|}x_n^\sigma e^{-|\xi'|x_n} < C|\xi'|^{-|\alpha'|}e^{-\frac{1}{2}|\xi'|x_n}. \end{aligned}$$

Therefore

$$\left|\partial_{\xi'}^{\alpha'}\xi_j x_n y_n e^{-|\xi'|(x_n+y_n)}\right| \le |x_n y_n||\xi'|^{1-|\alpha'|}e^{-\frac{1}{2}|\xi'|(x_n+y_n)} \le |\xi'|^{-1-|\alpha'|}e^{-\frac{1}{4}|\xi'|(x_n+y_n)},$$

And

$$\left|\partial_{\xi'}^{\alpha'}[\xi_j(x_n y_n)e^{-|\xi'|(x_n+y_n)}\widetilde{f}_j(\xi',y_n)]\right| \le C_{\alpha'}|\xi'|^{-1-|\alpha'|}\sum_{|\gamma'|\le|\alpha'|}|\partial_{\xi'}^{\gamma'}\widetilde{f}_j(\xi',y_n)|.$$

Applying Lemma 4.2 with $N = n-3,\ \sigma = 1$,

$$|g(x)| \le C|x'|^{-(n-2)}\int_{\mathbb{R}_+^n}\int_0^\infty \sum_{|\gamma'|\le n-1}|f_j(x',y_n)(x')^{\gamma'}|dx'dy_n \le C|x'|^{-(n-2)}\int_{\mathbb{R}_+^n}|f(x)|dx.$$

On the other hand, taking $(y_n|\xi'|)e^{-|\xi'||y_n|} \le 1$ into consideration, there holds the estimate:

$$|g(x)| \le \left(\frac{1}{2\pi}\right)^{n-1}\int_{\mathbb{R}^{n-1}} x_n e^{-|\xi'|x_n}d\xi'\int_{\mathbb{R}_+^n}|f(x)|dx = C_n x_n^{-(n-2)}\int_{\mathbb{R}_+^n}|f(x)|dx.$$

Hence

$$|g(x)| \le \begin{cases} C_n|x'|^{-(n-2)}\int_{\mathbb{R}_+^n}|f(x)|dx \\ C_n x_n^{-(n-2)}\int_{\mathbb{R}_+^n}|f(x)|dx \end{cases} \le C_n|x|^{-(n-2)}\int_{\mathbb{R}_+^n}|f(x)|dx.$$

Similarly applying Lemma 4.2 with $N = n-2, \sigma = 1$, we obtain

$$|\nabla g(x)| \le C_n|x|^{-(n-1)}\int_{\mathbb{R}_+^n}|f(x)|dx.$$

Next we consider

$$g_0(x) = \left(\frac{1}{2\pi}\right)^{n-1}\int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\phi_0(\xi')\frac{\xi_j}{|\xi'|}x_n e^{-|\xi'|x_n}\int_0^\infty y_n|\xi'|e^{-|\xi'|y_n}\widehat{f}_j(\xi',y_n)dy_nd\xi'.$$

then the following estmate holds:

$$|\partial_x^\alpha g_0(x)| \le C_\alpha\int_{|\xi'|\le 1}\frac{d\xi'}{|\xi'|}\int_{\mathbb{R}_+^n}|f_j(y)|dy \le C_\alpha\int_{\mathbb{R}_+^n}|f_j(y)|dy.$$

We choose $\phi_\infty(\xi') = 1 - \phi_0(\xi') \in C^\infty(\mathbb{R}^n)$ so that $\phi_\infty(\xi') = 1$ for $|\xi'| \geq 2$ and $\phi_\infty(\xi') = 0$ for $|\xi'| \leq 1$. Set

$$U_\infty(x) = \mathcal{F}_\xi^{-1}\left[\frac{\phi_\infty(\xi)}{|\xi|^2}\left(\widehat{F}(\xi) - \frac{\xi}{|\xi'|^2}(\xi\cdot\widehat{F}(\xi))\right)\right],$$

And then for $1 < p < \infty$

$$U_\infty(x) \in W^{2,p}(\mathbb{R}^n) \quad ; ||U_\infty||_{W^{2,p}(\mathbb{R}^n)} \leq C||f||_{L^p(\mathbb{R}^n_+)}. \tag{4.4}$$

In fact, noticing that

$$\partial_{\xi_n}^{\alpha_n}|\xi|^{-2} = \sum_{\sigma=1}^{\alpha_n} C_\alpha|\xi|^{-2(\sigma+1)} \sum_{a_1+\ldots+a_\sigma=\alpha_n} \partial_{\xi_n}^{a_1}\xi_n^2\cdot\cdots\cdot\partial_{\xi_n}^{a_\sigma}\xi_n^2 = \sum_{\sigma\geq m/2}^{\alpha_n} C_\alpha|\xi|^{-2(\sigma+1)}\sum_{p=2\sigma-\alpha_n}\xi_n^p,$$

and

$$|\partial_{\xi'}^{\alpha'}|\xi|^{-2(\sigma+1)}| \leq \sum_{\ell=1}^{|\alpha'|}|\xi|^{-2(\sigma+\ell+1)}|\xi'|^{2\ell-|\alpha'|} \leq C_\alpha|\xi|^{-2(\sigma+1)}|\xi'|^{-|\alpha'|},$$

we have

$$|\partial_{\xi'}^{\alpha'}\partial_\xi^{\alpha_n}[\phi_\infty(\xi')|\xi|^{-2}]| \quad \leq \quad \left|\sum_{\sigma\geq\alpha_n/2}^{\alpha_n} C|\xi|^{-2(\sigma+1)}\xi_n^{2\sigma-\alpha_n}|\xi'|^{-|\alpha'|}\right| \leq \begin{cases} C_\alpha|\xi|^{-2}|\xi'|^{-|\alpha'|}|\xi_n|^{-\alpha_n} & |\xi'|\geq 2 \\ 0 & |\xi'|\leq 1\end{cases}.$$

thus there holds the estimate for any β with $|\beta| \leq 2$ and α,

$$\left|\xi^\alpha\partial_{\xi'}^{\alpha'}\partial_{\xi_n}^{\alpha_n}[\phi_\infty(\xi')|\xi|^{-2}\xi^\beta]\right| \leq C_\alpha.$$

Hence by Fourier multiplier theory, we have

$$||U_\infty||_{W^{2,p}(\mathbb{R}^n)} \leq C_p||f||_{L^p(\mathbb{R}^n)}.$$

Furthermore, we choose $\psi_\infty(\xi') \in C^\infty(\mathbb{R}^n)$ so that $\psi = 1$ for $|\xi'| \geq 1$ and $\psi = 0$ for $|\xi'| \leq \frac{1}{2}$. Setting $\widetilde{v_{n,\infty}}(\xi', x_n) = \psi_\infty(\xi')x_n e^{-|\xi'|x_n} i\xi'\cdot\widetilde{U_\infty}(\xi', 0)$, we consider the following estimates for $|\beta'| \leq 2$:

$$(\xi')^{\beta'}\widetilde{v_{n,\infty}}(\xi', x_n) = x_n|\xi'|e^{-\frac{1}{2}|\xi'|x_n}\frac{\psi_\infty(\xi')}{|\xi'|}(\xi')^\beta i\xi\cdot\widetilde{U_\infty}(\xi', 0)e^{-\frac{1}{2}|\xi'|x_n}.$$

In the case where $|\beta'| \leq 1$, we set $\mathcal{F}_{\xi'}^{-1}\left[i\xi'\cdot\widetilde{U_\infty}(\xi', 0)e^{-|\xi'|x_n}\right](x') = W_\infty(x)$. We have

$$\left|\partial_{\xi'}^{\alpha'}[x_n|\xi'|e^{-\frac{1}{2}|\xi'|x_n}\frac{\psi_0(\xi')}{|\xi'|}(\xi')^{\beta'}]\right| \leq C_\alpha|\xi'|^{-|\alpha'|},$$

so that by Fourier multiplier theory, we obtain the following estimate:

$$||\partial_{x'}^{\beta'}v_{n,\infty}||_{L^p(\mathbb{R}^n_+)} \leq C||W_\infty||_{L^p(\mathbb{R}^n_+)} \leq C||\nabla U_\infty||_{L^p(\mathbb{R}^n_+)}$$

In the case where $|\beta'| = 2$, we set $\beta' = \beta_1' + \beta_2'$ with $|\beta_1'| = |\beta_2'| = 1$ then we have

$$||\partial_x^{\beta_1}W_\infty||_{L^p(\mathbb{R}^n_+)} = \left\|\mathcal{F}_{\xi'}^{-1}\left[(\xi')^{\beta_1'}i\xi'\cdot\widetilde{U_\infty}(\xi', 0)e^{-|\xi'|x_n}\right](x')\right\| \leq C||\nabla^2 U_\infty||_{L^p(\mathbb{R}^n_+)}.$$

Thus we obtatin the following estimate:

$$\|\partial_x^{\beta_1} v_{n,\infty}\|_{L^p(\mathbb{R}^n_+)} \leq C\|\partial_{x'}^{\beta'} W_\infty\|_{L^p(\mathbb{R}^n_+)} \leq C\|\nabla^2 U_\infty\|_{L^p(\mathbb{R}^n_+)}.$$

Moreover noticing that

$$\partial_n \widetilde{v_{n,\infty}}(\xi', x_n) = (1 - x_n|\xi'|)e^{-\frac{1}{2}|\xi'|x_n} e^{-\frac{1}{2}|\xi'|x_n} i\xi' \cdot \widetilde{U_\infty}(\xi', 0),$$

we have the following estimates :

$$\|\partial_{x'}^{\beta'} \partial_n \widetilde{v_{n,\infty}}\|_{L^p(\mathbb{R}^n_+)} \leq C\|\partial_{x'}^{\beta'} W_\infty\|_{L^p(\mathbb{R}^n_+)} \leq C\|\nabla^2 U_\infty\|_{L^p(\mathbb{R}^n_+)},$$

and

$$\partial_n^2 \widetilde{v_{n,\infty}}(\xi', x_n) = \sum_{j=1}^{n-1} (2 - x_n|\xi'|)\frac{\xi_j}{|\xi'|} e^{-\frac{1}{2}|\xi'|x_n} e^{-\frac{1}{2}|\xi'|x_n} \xi_j i\xi' \cdot \widetilde{U_\infty}(\xi', 0).$$

Therefore we obtain

$$\|\partial_n^2 \widetilde{v_{n,\infty}}(\xi', x_n)\|_{L^p(\mathbb{R}^n_+)} \leq C\|\nabla^2 U_\infty\|_{L^p(\mathbb{R}^n_+)}.$$

We have

$$\|v_{n,\infty}\|_{W^{2,p}(\mathbb{R}^n_+)} \leq C\|U_\infty\|_{W^{2,p}(\mathbb{R}^n_+)} \leq C\|f\|_{L^p(\mathbb{R}^n_+)}.$$

The other term of I_n enjoys the same estimate. As a result we see $I_n \in W^{2,p}_{loc}(\mathbb{R}^n_+)$ and

$$\|I_n\|_{W^{2,p}(|x|\leq R)} \leq C_R\|f\|_{L^p(\mathbb{R}^n_+)},$$
$$|I_n(x)| \leq C|x|^{-(n-2)}\|f\|_{L^p(\mathbb{R}^n_+)}, \qquad |\nabla I_n(x)| \leq C|x|^{-(n-1)}\|f\|_{L^p(\mathbb{R}^n_+)}.$$

In the same way, we can obtain the decay of velocity term and pressure term:

$$\|v_n\|_{W^{2,p}(|x|\leq R)} \leq C_R\|f\|_{L^p(\mathbb{R}^n_+)},$$
$$|v_n(x)| \leq C|x|^{-(n-2)}\|f\|_{L^p(\mathbb{R}^n_+)}, \qquad |\nabla v_n(x)| \leq C|x|^{-(n-1)}\|f\|_{L^p(\mathbb{R}^n_+)}.$$
$$\theta_n \in W^{1,p}(\mathbb{R}^n_+) \quad \text{and} \quad \|\theta_n\|_{W^{1,p}(|x|<R, x\in\mathbb{R}^n_+)} \leq C_R\|f\|_{L^p(\mathbb{R}^n_+)}.$$

Lemma 4.3 *Let $n \geq 3$ and $1 < p < \infty$. Let $\Omega = \mathbb{R}^n_+$ or a purturbed half-space. Let $u \in W^{2,p}_{loc}(\Omega)$ and $\pi \in W^{1,p}_{loc}(\Omega)$ enjoy*

$$-\Delta u + \nabla\pi = 0, \ \nabla\cdot u = 0 \quad in \quad \Omega, \qquad u|_{\partial\Omega} = 0. \tag{4.5}$$

Moreover

$$\sup_{x\in B^+_{R+3}} \left[|x|^{n-2}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|\pi(x)|\right] < \infty.$$

Then $u = 0$ and $\pi = 0$.

[proof of Lemma 4.3] Taking local regularity into account, we may assume that $u \in W^{2,q}_{loc}(\Omega)$, $\pi \in W^{1,p}_{loc}(\Omega)$ for $1 < q < \infty$. In particularly we may assume that $u \in W^{2,2}_{loc}(\Omega)$, $\pi \in W^{1,2}_{loc}(\Omega)$. Now we choose $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| < 1$ and $\psi(x) = 0$ for $|x| > 2$. Set $\psi_L(x) = \psi(x/L)$. Since $u(x), \pi(x)$ satisfy (4.5), we have

$$\begin{aligned} 0 &= (-\Delta u + \nabla\pi, \psi_L u) \\ &= (\nabla u, (\nabla\psi_L)u) + (\nabla u, (\psi_L)\nabla u) - (\pi, (\nabla\psi_L)u) - (\pi, \psi_L(\nabla\cdot u)) \\ &= (\nabla u, (\nabla\psi_L)u) + (\nabla u, (\psi_L)\nabla u) - (\pi, (\nabla\psi_L)u). \end{aligned}$$

Then the first and the third terms of right hand side tend to 0 as $L \to \infty$, and therefore we have

$$0 = ||\nabla u||_{L^2}^2,$$

which implies that $\nabla u = 0$. Therefore $u =$ constant. Since $u(x)|_{\partial\Omega} = 0$, we obtain $u = 0$. By the equation $\nabla\pi = 0$, which implies $\pi =$ constant. Since $\pi(x) = O(|x|^{-(n-1)})$ as $|x| \to \infty$, $\pi = 0$. This completes the proof. ■

5 Analysis in Ω by constructing a parametrix

First of all, to construct a parametrix in Ω , we intoroduce the Bogovskiĭ lemma [3] which plays an important role in this paper. To introduce the Bogovskiĭ lemma for any domain D in $\mathbb{R}^n$, we define the spaces $W_0^{N,p}(D)$ and $\dot{W}^{N,p}(D_R)$ as follows.

$$\begin{aligned} W_0^{N,p}(D) &= \{f \in W^{N,p}(D); \partial_x^\alpha f|_{\partial D} = 0 \quad \text{for} \quad |\alpha| \le N-1\} \qquad N \ge 1, \\ \dot{W}^{N,p}(D) &= \{f \in W_0^{N,p}(D); \int_{D_R} f dx = 0\} \qquad N \ge 1, \\ \dot{W}^{0,p}(D) &= \{f \in L^p(D); \int_{D_R} f dx = 0\}. \end{aligned}$$

Proposition 5.1 (Bogovskiĭ lemma) *Let* $1 < p < \infty$. *For* $D_R = \{x \in \mathbb{R}_+^n; R < |x| < R+1\}$ *and any integer* $N \ge 0$, *there is a linear operator* $\mathbb{B}_R$ *from* $\dot{W}_0^{N,p}(D_R)$ *into* $W_0^{N+1,p}(D_R)$ *such that*

$$\nabla \cdot \mathbb{B}_R f = f, \quad ||\mathbb{B}_R f||_{W^{N+1,p}(D_R)} \le C_{N,p,R}||f||_{W^{N,p}(D_R)},$$

for any $f \in \dot{W}_0^{N,p}(D_R)$.

Next we shall introduce the notation in order to constructing a parametrix. Fix $R > 0$ such that $B_+^R \cap \Omega = B_+^R = \{x \in \mathbb{R}_+^n; |x| > R\}$. Let E_R be a bounded domain with smooth boundary ∂E_R such that $E_R \cap B_{R+4} = \Omega \cap B_{R+4}$. And let D_R be a bounded domain such that $D_R = \{x \in \overline{\mathbb{R}_+^n}; R+1 \le |x| \le R+2\}$. In particular $D_R \subset \overline{\Omega} \cap B_{R+3} \subset E_R$.

Let $P_\lambda f$ and $\pi_\lambda f$ be operators given in section 2. Recall that w and θ satisfy the equation:

$$(\lambda - \Delta)w + \nabla\theta = f_0, \ \nabla \cdot w = 0 \quad \text{in} \quad \mathbb{R}_+^n, \qquad w|_{x_n=0} = 0,$$

and the estimate:

$$||P_\lambda f||_{\mathbb{W}^{2,p}(D_R)} + ||\pi f||_{W^{1,p}(D_R)} \le C_R||f||_{\mathbb{L}^p(\Omega)}.$$

Given $f \in \mathbb{L}_{R+3}^p$, we set $Af = w$, $\Phi f = \pi$ where w and π are the solution to the equation;

$$-\Delta w + \nabla\pi = f, \qquad \nabla \cdot w = 0 \quad \text{in } E_R, \qquad w|_{\partial E_R} = 0. \tag{5.1}$$

We know the unique existence of $Af \in \mathbb{W}^{2,p}(E_R)$ and $\Phi f \in W^{1,p}(E_R)$ satisying the estimate:

$$||Af||_{\mathbb{W}^{2,p}(E_R)} + ||\nabla\Phi f||_{L^p(E_R)} \le C_{R,p}||f||_{L^p(E_R)}.$$

(cf. Farwig and Sohr [7]). By addition some constant to Φf, we may assume that

$$\int_{D_R} (\Phi f - \pi f)\, dx = 0.$$

Given function f defined on Ω, let $f_0(x)$ be defined by the formula; $f_0(x) = f(x)$ for $|x| > R$ and $f_0(x) = 0$ for $|x| \leq R$. We choose $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ so that $\phi(x) = 1$ for $|x| \leq R+1$ and $\phi(x) = 0$ for $|x| \geq R+2$. We set

$$R_\lambda f \quad = \quad (1-\phi)P_\lambda f + \phi A f + \mathbb{B}[(\nabla\phi)\cdot(P_\lambda - A_0 f)].$$

Then $R_\lambda f|_{\partial\Omega} = 0$. Since $\mathrm{supp}(\nabla\phi) \subset D_R$ and

$$\int_\Omega (\nabla\phi)\cdot(P_\lambda f - Af)dx = 0,$$

we see $(\nabla\phi)\cdot(P_\lambda f - Af) \in \dot{W}^{1,p}(D_R)$ and $\mathbb{B}[(\nabla\phi)\cdot(P_\lambda f - Af)] \in \mathbb{W}^{2,p}(D_R)$. Thus $R_\lambda f \in \mathbb{W}^{2,p}_{loc}(\Omega)$. Now set $\Pi_\lambda f = (1-\phi)\pi f + \phi\Phi_0 f$, then

$$(\lambda - \Delta)R_\lambda f + \nabla\Pi_\lambda f = f + S_\lambda f, \qquad \nabla\cdot R_\lambda f = 0 \quad \text{in} \quad \Omega, \qquad R_\lambda f|_{\partial\Omega} = 0,$$

where

$$\begin{aligned} S_\lambda f \quad = \quad & 2(\nabla\phi) : (\nabla P_\lambda f) + (\Delta\phi)P_\lambda f + (\lambda-\Delta)\mathbb{B}[(\nabla\phi)\cdot(P_\lambda f - Af)] \\ & +\lambda\phi Af - 2(\nabla\phi) : (\nabla Af) - (\Delta\phi)(Af) - (\nabla\varphi)\pi f + (\nabla\varphi)\Phi f. \end{aligned}$$

Then we know $S_\lambda : \mathbb{L}^p_{R+3}(\Omega) \to \mathbb{L}^p_{R+3}(\Omega)$ is compact operator (see [8]). What is more we have the two following lemmas.

Lemma 5.2 *For $1 < p < \infty$, the following relation holds*

$$\lim_{\lambda\to 0,\ \lambda\in\Sigma_\epsilon} \|S_\lambda - S_0\|_{\mathcal{L}(\mathbb{L}_{p,R+3}(\Omega))} = 0.$$

[proof of lemma 5.2] Lemma 5.2 follows immediately from lemma 4.1 . ■

Lemma 5.3 *It holds that*

$$(1+S_0)^{-1} \in \mathcal{L}(\mathbb{L}_{p,R+3}(\Omega)).$$

[proof of Lemma 5.3] Since

$$S_0 f \quad = \quad 2(\nabla\phi) : (\nabla P_0 f - Af) + (\Delta\phi)(P_0 f - Af) - \Delta\mathbb{B}[(\nabla\phi)\cdot(P_0 f - Af)]$$

is the compact operator in $\mathbb{L}^p_{R+3}(\Omega)$, we will show that $1 + S_0$ is injective in $\mathbb{L}^p_{R+3}$. Let $f \in \mathbb{L}^p_{R+3}(\Omega)$ satisfy $(1+S_0)f = 0$. Set $u = R_0 f$ and $\pi = \Pi f$, and by uniqueness we see $u(x) = 0$, $\pi(x) = 0$. Therefore

$$\begin{cases} (1-\phi)P_0 f + \phi Af + \mathbb{B}[(\nabla\phi)\cdot(P_0 - A_0 f)] = 0 & \text{in } \Omega, \\ (1-\phi)\pi f + \phi\Phi f = 0 & \text{in } \Omega. \end{cases} \tag{5.2}$$

Since $\phi(x) = 0$ for $|x| \geq R+2$, $P_0 f(x) = 0$, $\pi f(x) = 0$. And since $\phi(x) = 1$ for $|x| \leq R+1$, we have $Af(x) = 0$, $\Phi f(x) = 0$. Put $\widetilde{E_R} = \{x \in E_R; |x| \geq R\} \cup \{x \in \mathbb{R}^n_+; |x| < R\}$. If we put

$$w(x) = \begin{cases} Af(x) & |x| \geq R, x \in E_R, \\ 0 & |x| < R, \end{cases} \qquad p(x) = \begin{cases} \Phi f(x) & |x| \geq R, x \in E_R, \\ 0 & |x| < R, \end{cases}$$

then $w \in \mathbb{W}^{2,p}(\widetilde{E_R})$ and $p \in W^{1,p}(\widetilde{E_R})$ and (w,p) satisfies the equation;

$$-\Delta w + \nabla p = f_0, \ \nabla \cdot w = 0 \quad \text{in } \widetilde{E_R}, \qquad w|_{\partial \widetilde{E_R}} = 0.$$

On the other hand, we have

$$\begin{cases} -\Delta P_0 f + \nabla \pi f = f_0, \ \nabla \cdot (P_0 f) = 0 \quad \text{in } \widetilde{E_R}, \\ P_0 f|_{\partial \widetilde{E_R}} = 0. \end{cases}$$

By uniqueness, we obtain $P_0 f = w$ in $\widetilde{E_R}$. Notice that $\nabla(\pi f - p) = 0$ and

$$\int_{D_R} (\pi f - p) dx = \int_{D_R} (\pi f - \Phi_0 f) dx = 0,$$

we see $\pi f - p = \text{constant} = c$ in $\widetilde{E_R}$. Since

$$0 = \int_{D_R} (\pi f - p) dx = \int_{D_R} c dx = c|D_R|,$$

we obtain $c = 0$. Therefore $\pi f = p$ in $\widetilde{E_R}$. As a result, we have

$$P_0 f = w = A_0 f, \ \pi f = p = \Phi_0 f \quad \text{in } \widetilde{E_R}.$$

In particular it holds $(\nabla \phi) \cdot [P_0 f - A_0 f] = 0$ in Ω. By (5.2), we have

$$\begin{aligned} 0 = P_0 f + \phi(A_0 f - P_0 f) = P_0 f \qquad & \text{for } |x| \geq R+1, \ x \in \Omega, \\ 0 = \pi f + \phi(\Phi_0 f - \pi f) = \pi f \qquad & \text{for } |x| \geq R+1, \ x \in \Omega. \end{aligned}$$

Since $A_0 f = 0$ and $\Phi_0 f(x) = 0$ for $x \in \Omega, \ |x| \leq R+1$,

$$\begin{aligned} 0 &= -\Delta A_0 f(x) + \nabla \Phi_0 f(x) = f \qquad \text{for } |x| \leq R+1, \ x \in \Omega, \\ 0 &= -\Delta P_0 f + \nabla \pi f = f \qquad \text{for } |x| \geq R+1, \ x \in \Omega. \end{aligned}$$

Consequently we obtain $f = 0$. ■

We get the following lemma from lemma 5.2 and 5.3.

Lemma 5.4 *There exists $\lambda_0 > 0$ such that for $\lambda \in \Sigma_\varepsilon \cup \{0\}$, $|\lambda| \leq \lambda_0$, the following relations holds:*

$$(1 + S_\lambda)^{-1} \in \mathcal{L}(\mathbb{L}^p_{R+3}(\Omega)), \qquad \|(1 + S_\lambda)^{-1}\|_{\mathcal{L}(\mathbb{L}^p_{R+3}(\Omega))} \leq C$$

By lemma 5.4, we can denote the solution (u, π) as follows:

$$\begin{aligned} u(x) &= R_\lambda (1 + S_\lambda)^{-1} f \\ &= (1 - \phi) P_\lambda (1 + S_\lambda)^{-1} f + \phi A_0 (1 + S_\lambda)^{-1} f + \mathbb{B}[(\nabla \phi) \cdot (P_\lambda (1 + S_\lambda)^{-1} f - A_\lambda (1 + S_\lambda)^{-1} f)], \\ \pi(x) &= \Pi_\lambda (1 + S_\lambda)^{-1} f = (1 - \phi) \pi (1 + S_\lambda)^{-1} f + \phi \Phi_0 (1 + S_\lambda)^{-1} f, \end{aligned}$$

where

$$\begin{aligned} (1 + S_\lambda)^{-1} f &= [1 + S_0 + (S_\lambda - S_0)]^{-1} f \\ &= (1 + S_0)^{-1} [1 + (1 + S_0)^{-1} (S_\lambda - S_0)]^{-1} f \\ &= (1 + S_0)^{-1} \sum_{j=0}^{\infty} [(1 + S_0)^{-1} (S_\lambda - S_0)]^j f, \\ S_\lambda - S_0 &= 2(\nabla \phi) : \nabla (P_\lambda - P_0) + (\Delta \phi)(P_\lambda - P_0) + \lambda \phi A_0 f \\ &\quad + \lambda \mathbb{B}[(\nabla \phi) \cdot (P_\lambda - P_0)] - \Delta \mathbb{B}[(\nabla \phi)(P_\lambda - P_0)]. \end{aligned}$$

6 Proofs of Theorem 1.1 and Theorem 1.2

We study coerciveness estimates for A^m and $(A-\lambda)^{-1}$ when $\lambda \in \Sigma = \{\lambda \in \mathbb{C}\backslash\{0\}; |\arg\lambda| < \delta\}$ with $|\lambda| \geq 1$.

Proposition 6.1 *Let $1 < q < \infty$ and let A be the Stokes operator in $J_q(\Omega)$.*
(i) Assume that $u \in \mathfrak{D}_q(A)$ and $Au \in W_q^m(\Omega)$ for a nonnegative integer m. Then $u \in W_q^{m+2}(\Omega)$ and for some constant $C_m > 0$,

$$\|u\|_{W^{m+2,q}(\Omega)} \leq C_m \left(\|Au\|_{W^{m,q}(\Omega)} + \|u\|_{L^q(\Omega)}\right). \tag{6.1}$$

(ii) $u \in \mathfrak{D}_q(A^m)$, $m \geq 0$, then $u \in W_q^{2m}(\Omega)$ and

$$\|u\|_{W^{2m,q}(\Omega)} \leq C_m \left(\|A^m u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}\right). \tag{6.2}$$

[proof of Proposition 6.1] The proof of (6.2) is carried out by applying (6.1) and the following estimate: for any $\varepsilon > 0$ and integer $\ell \geq 1$,

$$\|A^{\ell-1}u\|_{L^q(\Omega)} \leq \varepsilon\|A^{\ell}u\|_{L^q(\Omega)} + C_\varepsilon\|u\|_{L^q(\Omega)}, \qquad u \in \mathfrak{D}_q(A^{\ell})$$

with some $C_\varepsilon > 0$. ∎

Proposition 6.2 *Let $1 < q < \infty$ and let A be the Stokes operator in $J_q(\Omega)$.*
(i) For a nonnegative integer m,

$$\|A^m u\|_{L^q(\Omega)} \leq C_m\|u\|_{W^{2m,q}(\Omega)}, \qquad u \in \mathfrak{D}_q(A^m). \tag{6.3}$$

(ii) Let $0 < \delta < \pi$ and let m be a nonnegative integer. If $f \in \mathfrak{D}_q(A^m)$, then

$$\|(A+\lambda)^{-1}f\|_{W^{2m+2,q}(\Omega)} \leq C_m\|f\|_{W^{2m,q}(\Omega)}, \tag{6.4}$$

for any $\lambda \in \Sigma(\delta)$.

[proof of Proposition 6.2]
(i) Since $Au = -P\Delta u$ for $u \in \mathfrak{D}_q(A)$ and P is bounded in $W_q^{\ell}(\Omega)$ for any nonnegative integer ℓ, we have

$$\|A^m u\|_{L^q(\Omega)} = \|P\Delta(A^{m-1}u)\|_{L^q(\Omega)} \leq C\|A^{m-1}u\|_{W^{2,q}(\Omega)} \leq C\|A^{m-2}u\|_{W^{4,q}(\Omega)}.$$

Repeating this manipulation leads us to (6.3).
(ii) The estimate (6.4) is an immediate consequence of Proposition 6.1 ,(6.2) and (6.3). In fact,

$$\begin{aligned}\|(A+\lambda)^{-1}f\|_{W^{2m+2,q}(\Omega)} &\leq C_m \left(\|A^{m+1}(A+\lambda)^{-1}f\|_{L^q(\Omega)} + \|(A+\lambda)^{-1}f\|_{L^q(\Omega)}\right) \\ &\leq C_m \left(\|A^m f\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}\right) \leq C_m\|f\|_{W^{2m,q}(\Omega)}\end{aligned}$$

∎

We shall prove Theorem 1.1 and 1.2 from what we showed in section 5 in the same way as in Iwashita [9]. First, we shall show Theorem 1.1. For this purpose, it suffices to prove the following theorem:

Theorem 6.3 *Let $n \geq 3$ and $1 < q < \infty$. Then there exists a positive constant $C = C(q)$ such that the inequality*

$$\|e^{-tA}f\|_{L^q(\Omega)} \leq C(1+t)^{-\frac{n+1}{2}}\|f\|_{L^q(\Omega)} \qquad t \geq 0 \tag{6.5}$$

is valid for any $f \in \mathbb{J}^q(\Omega)$.

[proof of Theorem 6.3] Since the semigroup e^{-tA} is bounded in $J_q(\Omega)$, it suffices to show (6.5) for large $t > 0$. Then let $\frac{\pi}{2} < \delta_0 < \delta < 2\pi$ and $0 < \varepsilon < \varepsilon_0$. Let Γ be a contour as follows :$\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 = \{\lambda \in \mathbb{C}; 0 < |\lambda| < \varepsilon,\ \arg\lambda = \pm\delta\}, \quad \Gamma_2 = \{\lambda \in \mathbb{C}; |\lambda| > \varepsilon,\ \arg\lambda = \pm\delta\}.$$

The semigroup is described as follows :

$$e^{-tA} = \frac{-1}{2\pi i}\int_{\Gamma_1} e^{-t\lambda}R(\lambda)d\lambda + \frac{-1}{2\pi i}\int_{\Gamma_2} e^{-t\lambda}(A+\lambda)^{-1}d\lambda. \tag{6.6}$$

The second term of the right hand side of (6.6) is estimated as

$$\left\|\frac{-1}{2\pi i}\int_{\Gamma_2} e^{-t\lambda}(A+\lambda)^{-1}d\lambda\right\|_{\mathcal{L}(J_q(\Omega))} \leq C\int_{\varepsilon}^{\infty} e^{-tr}dr \leq Ce^{-ct}. \tag{6.7}$$

for some $c, C > 0$. In order to estimate the first term of the right hand side of (6.6), we need the following lemma that is a direct consequence of the formula of the gamma function $\Gamma(\sigma)$.

Lemma 6.4 *(i) For $\sigma > 0$ and $t > 0$, it holds that*

$$\frac{-1}{2\pi i}\int_{\Gamma} e^{-tz}z^{\sigma-1}dz = -\frac{\sin\sigma\pi}{\pi}\Gamma(\sigma)e^{i\pi\sigma}t^{-\sigma}.$$

(ii) For a nonnegative integer j and any $t > 0$,

$$\frac{-1}{2\pi i}\int_{\Gamma} e^{-tz}z^j\log z dz = -\frac{d}{dz}\left[\frac{\sin\sigma\pi}{\pi}\Gamma(\sigma)e^{i\pi\sigma}\right]\Bigg|_{\sigma=j+1} t^{-j-1}.$$

Since $R(\lambda)$ is described as

$$R(\lambda) = \begin{cases} G_1(\lambda)\lambda^{\frac{n-1}{2}} + G_2(\lambda)\lambda^{\frac{n}{2}}\log\lambda + G_3(\lambda) & n \text{ is even,} \\ G_1(\lambda)\lambda^{\frac{n}{2}} + G_2(\lambda)\lambda^{\frac{n-1}{2}}\log\lambda + G_3(\lambda) & n \text{ is odd,} \end{cases} \tag{6.8}$$

we can apply Lemma 6.4 to obtain

$$\left\|\frac{-1}{2\pi i}\int_{\gamma_1} e^{-t\lambda}G_1(\lambda)\lambda^{\frac{n-1}{2}}fd\lambda\right\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}-\frac{1}{2}}\|f\|_{L^q(\Omega)}, \tag{6.9}$$

$$\left\|\frac{-1}{2\pi i}\int_{\gamma_1} e^{-t\lambda}G_2(\lambda)\lambda^{\frac{n}{2}}(\log\lambda)fd\lambda\right\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}-1}\|f\|_{L^q(\Omega)}. \tag{6.10}$$

Finally the operator $G_3(\lambda)$ is bounded so that we have

$$\left\|\frac{-1}{2\pi i}\int_{\gamma_1} e^{-t\lambda}G_3(\lambda)fd\lambda\right\|_{L^q(\Omega)} \leq Ce^{-ct}\|f\|_{L^q(\Omega)}. \tag{6.11}$$

Combining (6.7) and (6.9)-(6.11) completes the proof of Theorem 6.3. ■

We can show Theorem 1.2 from Theorem 1.1 and cut-off technique in the same way as in Iwashita [9]. In order to show Theorem 1.2, we introduce the Ukai operator $E(t)f(x)$ (cf. Ukai [14]) which solves the equation:

$$\begin{cases} u_t - \Delta u + \nabla p = 0, & \nabla \cdot u = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^n_+, \\ u|_{x_n=0} = 0, & u|_{t=0} = f, \end{cases}$$

with $u = E(t)f(x)$ and some pressure term $p(t,x)$. We know the following fact (cf. Ukai [14], Borchres and Miyakawa [2]) .

Lemma 6.5 *Let $1 \le q \le r \le \infty$ and put $\sigma = (\frac{n}{q} - \frac{n}{r})/2$. Then, we have*

$$\begin{aligned} \|E(t)f\|_{L^r(\mathbb{R}^n_+)} &\le C_\alpha t^{-\sigma} \|f\|_{L^q(\mathbb{R}^n_+)}, \\ \|\nabla E(t)f\|_r &\le C_\alpha t^{-\sigma-\frac{1}{2}} \|f\|_{L^q(\mathbb{R}^n_+)}, \end{aligned} \tag{6.12}$$

for $t > 0$.

The next lemma is concerned with the estimate of derivatives of $e^{-tA}f$.

Lemma 6.6 *Let $n \ge 3, 1 < q < \infty, d > R_0$, and let m be a nonnegative integer.*

(i) There exists a constant $C = C(d,m) > 0$ such that

$$\|e^{-tA}f\|_{\Omega_d,q,2m} \le C(1+t)^{-\frac{n}{2}-\frac{1}{2}} \|f\|_{\Omega,q,2m}, \tag{6.13}$$

for any $f \in \mathfrak{D}_q(A^m)$ with $f = 0$ for $|x| > d$.

(ii) If $f \in \mathfrak{D}_q(A^{m+1})$ and $f = 0$ for $|x| > d$, then

$$\|\partial_t e^{-tA}f\|_{\Omega_d,q,2m} \le C(1+t)^{-\frac{n}{2}-\frac{3}{2}} \|f\|_{\Omega,q,2m+2}, \tag{6.14}$$

where the constant $C > 0$ is independent of f.

[proof of Lemma 6.6] By proposition 6.1 and proposition 6.2, it suffies to verify the assertions for t large. The proof of (i) can be carried out similarly as in proof of theorem 6.3 with the aid of (6.4) and expansion of $R(\lambda)$.
(ii) For $t \gg 1$ and the contour γ introduced in proof of theorem 6.3, the identity

$$\partial_t e^{-tA} f = \frac{-1}{2\pi i} \int_{\gamma_1} e^{-t\lambda}(-\lambda) R(\lambda) f d\lambda + \frac{-1}{2\pi i} \int_{\gamma_2} e^{-t\lambda}(-\lambda)(A-\lambda)^{-1} f d\lambda \tag{6.15}$$

is valid so that (6.14) is obtained by using the expansion and (6.4) in (6.15). ■

Put $u_0 = e^{-Af}$ for $f \in J_q(\Omega)$. Since $u_0 \in \mathfrak{D}_q(A^N)$ for any integer $N \ge 0$, it follows from proposition 6.1 that $u_0 \in W^{2N}_q(\Omega)$ and

$$\|u_0\|_{\Omega,q,2N} \le C_N \|f\|_{\Omega,q}. \tag{6.16}$$

Put $u(t) = e^{-tA}u_0$ and $u(t)$ is satisfied with the following:

$$\begin{cases} \partial_t u(t) - \Delta u(t) + \nabla p(t) = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ \nabla \cdot u(t) = 0 & \text{in} \quad (0,\infty) \times \Omega, \\ u(t)|_{\partial\Omega} = 0 & \text{on} \quad (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases}$$

In the same manner as in Iwashita [9], by Lemma 6.5 and Lemma 6.6, we have the two lemmas.

Lemma 6.7 *Let $u(t)$ be as above and $d \geq R_0 + 5$. For a nonnegative integer m, the inequalities*

$$\begin{aligned} \|u(t)\|_{\Omega_d,q,2m} &\leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+2m+2}, & (6.17)\\ \|\partial_t u(t)\|_{\Omega_d,q,2m} &\leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+2m+4}, & (6.18)\end{aligned}$$

are valid for any $t \geq 0$, where the constant C depends only on d, m, and q.

Lemma 6.8 *Let $p(t)$ be a certain pressure assosiated with $u(t)$. Then,*

$$\|p(t)\|_{\Omega_d,q,2m} \leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+2m+4}. \qquad (6.19)$$

We choose $\psi \in C^\infty(\mathbb{R}^n_+)$ so that $\psi(x) = 1$ for $|x| \geq d$ and $= 0$ for $|x| \leq d-1$. By proposition 5.1, Lemma 6.7, we can find $v_3(t)$ to satisfy the following relation:

$$\nabla \cdot v_3(t) = \nabla \cdot [\psi u(t)], \qquad \mathrm{supp} v_3(t) \subset \{d-1 \leq |x| \leq d\} \cap \mathbb{R}^n_+.$$

Then $v_3(t)$ satisfies the two following estimates:

$$\begin{aligned} &\|v_3(t)\|_{q,m} \leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+m+2}, \\ &\|\partial_t v_3(t)\|_{q,m} \leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+m+4}, \end{aligned} \qquad (6.20)$$

In fact

$$\begin{aligned} \|v_3(t)\|_{q,m} &\leq C\|\nabla \cdot [\psi u(t)]\|_{q,m-1} = C\left(\|\nabla\psi \cdot u(t)\|_{q,m-1} + \|\psi\nabla \cdot u(t)\|_{q,m-1}\right) \\ &\leq C\|u(t)\|_{\Omega_d,q,m-1} \leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+m+2}. \end{aligned}$$

Set $v_4(t) = \psi u(t) - v_3(t)$.

Lemma 6.9 *Let $q < r < \infty$ and $v_4(t)$ be as above. Then*

$$\|v_4(t)\|_r \leq C(1+t)^{-(\frac{n}{q}-\frac{n}{r})/2}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+7}, \qquad (6.21)$$

[proof of Lemma 6.9] 1st step. We set

$$h(t) = -\{2(\nabla\psi)\cdot\nabla + (\Delta\psi)\}u(t) - (\partial_t - \Delta)v_3(t) + p(t)\nabla\psi,$$

and then see supp $h(t) \subset \{d-1 \leq |x| \leq d\} \cap \mathbb{R}^n_+$. Moreover, $h(t)$ satisfies the estimate:

$$\|h(t)\|_{q,m} \leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+m+4}. \qquad (6.22)$$

Here we set $v_4 = \psi u_0 - v_3(0)$ and see $\nabla \cdot v_4 = 0$ in $\mathbb{R}^n_+$ and

$$\|v_4\|_{q,m} \leq C\|u_0\|_{\Omega,q,m}. \qquad (6.23)$$

$v_4(t)$ satisfies the following problem:

$$\begin{cases} \partial_t v_4(t) - \Delta v_4(t) + \nabla(\psi p(t)) = h(t) & \text{in } (0,\infty)\times\mathbb{R}^n_+, \\ \nabla \cdot v_4(t) = 0 & \text{in } (0,\infty)\times\mathbb{R}^n_+, \\ v_4(0) = v_4 & \text{in } \mathbb{R}^n_+. \end{cases} \qquad (6.24)$$

and is hence desrcibed as

$$v_4(t) = E(t)v_4 + v_5(t), \quad v_5(t) = \int_0^t E(t-s)P_0 h(s)ds.$$

By (6.12),(6.23), we see

$$\|E(t)v_4\|_r \leq C(1+t)^{-\sigma}\|v_4\|_{q,[2\sigma]+1} \leq C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[2\sigma]+1}.$$

Next we will estimate $v_5(t)$ in the second step.
2nd step: Case 1. We consider the case of $\frac{n}{2q} > 1$. We can estimate $v_5(t)$ as follows ;

$$\begin{aligned}
\|v_5(t)\|_r &\leq C\int_0^t (1+(t-s))^{-\sigma}\|P_0 h(s)\|_{q,[2\sigma]+1}ds \\
&\leq C\int_0^t (1+(t-s))^{-\sigma}\|h(s)\|_{q,[2\sigma]+1}ds \\
&= C\int_0^t (1+(t-s))^{-\sigma}(1+s)^{-\frac{n}{2q}}ds\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+5},
\end{aligned}$$

where

$$\begin{aligned}
\int_{\frac{t}{2}}^t (1+(t-s))^{-\sigma}(1+s)^{-\frac{n}{2q}}ds &\leq C(1+t)^{-\frac{n}{2q}} \leq C(1+t)^{-\sigma}, \\
\int_0^{\frac{t}{2}} (1+(t-s))^{-\sigma}(1+s)^{-\frac{n}{2q}}ds &\leq C(1+t)^{-\sigma}\int_0^{\frac{t}{2}}(1+s)^{-\frac{n}{2q}}ds \leq C(1+t)^{-\sigma}.
\end{aligned}$$

We see

$$\|v_5(t)\|_r \leq C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+5}.$$

Case 2: We consider the case of $\frac{n}{2q} \leq 1$. Taking $\sigma = (\frac{n}{q} - \frac{n}{r})/2 < 1$ and $n(1-\frac{1}{r}) > 1$ into account, we see $r > \frac{n}{n-2}$. When $n \geq 4$, $r > \frac{n}{n-2}$ follows from $r > q > \frac{n}{2}$. Then We can choose $\rho > 1$ so that $(\frac{n}{\rho} - \frac{n}{r})/2 = 1+\kappa \quad (0<\kappa<\frac{1}{2})$. By (6.22), we can estimate $v_5(t)$ as follows:

$$\begin{aligned}
\|v_5(t)\|_r &\leq C\int_0^t (1+(t-s))^{-(1+\kappa)}\|P_0 h(s)\|_{\rho,3}ds \\
&\leq C\int_0^t (1+(t-s))^{-(1+\kappa)}\|h(s)\|_{\rho,3}ds \\
&\leq C\int_0^t (1+(t-s))^{-(1+\kappa)}(1+s)^{-\frac{n}{2q}}ds\|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \\
&\leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \leq C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[\frac{n}{q}]+7}.
\end{aligned}$$

Case 3 :We will show $\frac{3}{2} \leq q < r \leq 3$ for $n=3$. We choose $\rho > 1$ so that $\frac{3}{2\rho} > 1$ and $3(\frac{1}{\rho} - \frac{1}{r})/2 < 1$. We have

$$\begin{aligned}
\|v_5(t)\|_r &\leq C\int_0^t (1+(t-s))^{-3(\frac{1}{\rho}-\frac{1}{r})/2}\|P_0 h\|_{\rho,2}ds \\
&\leq C\int_0^t (1+(t-s))^{-3(\frac{1}{\rho}-\frac{1}{r})/2}(1+s)^{-\frac{n}{2q}}ds\|u_0\|_{\Omega,q,[\frac{n}{q}]+6}.
\end{aligned}$$

When $q > \frac{3}{2}$, taking account of $-3(\frac{1}{\rho} - \frac{1}{r})/2 - \frac{3}{2q} + 1 < -\sigma$ we see

$$\|v_5(t)\|_r \le C(1+t)^{-3(\frac{1}{\rho}-\frac{1}{r})/2-\frac{3}{2q}+1}\|u_0\|_{\Omega,q,[\frac{n}{q}]+6},$$

and

$$\|v_5(t)\|_r \le C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[\frac{n}{q}]+6}.$$

For $q = \frac{3}{2}$, noticing that $\rho < \frac{3}{2} = q$ we have

$$\|v_5(t)\|_r \le C(1+t)^{-3(\frac{1}{\rho}-\frac{1}{r})/2}\log(1+t)\|u_0\|_{\Omega,q,[\frac{n}{q}]+6} \le C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[\frac{n}{q}]+6},$$

since $-\frac{3}{2\rho} + \frac{3}{2r} + \varepsilon < -\sigma$. Summing up, we obtain the assertion of lemma. ■

[proof of Thorem 1.2 (1)] At first we shall show (1.3) for $t \ge 1$. Since (6.16),(6.17)and Sobolev's embedding theorem, we have

$$\begin{aligned} \|u(t)\|_{\Omega_d,r} &\le C\|u(t)\|_{\Omega_d,q,[2\sigma]+1} \\ &\le C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+3} \le C(1+t)^{-\frac{n}{2q}}\|f\|_{\Omega,q}, \end{aligned} \tag{6.25}$$

for $d \ge R_0 + 5$. On the other hand, by (6.16),(6.20) and (6.21),we see

$$\begin{aligned} \|u(t)\|_{\{x\ge d\},r} &\le \|v_4(t)\|_r + \|v_3(t)\|_r \\ &\le C(1+t)^{-\sigma}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+7} + C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+3} \\ &\le C(1+t)^{-\sigma}\|f\|_{\Omega,q}. \end{aligned} \tag{6.26}$$

Since $u(t) = e^{-tA}u_0 = e^{-(1+t)A}f$, we obtain (1.3) for $t \ge 1$.
Secondly we will show (1.3) for $0 < t < 1$. Set $N = [2\sigma]$. When N is even, by proposion 6.1 we see

$$\|e^{-tA}f\|_{\Omega,q,N} \le C\left(\|A^{\frac{N}{2}}e^{-tA}f\|_{\Omega,q} + \|e^{-tA}f\|_{\Omega,q}\right) \le Ct^{-\frac{N}{2}}\|f\|_{\Omega,q}.$$

Similary

$$\|e^{-tA}f\|_{\Omega,q,N+2} \le Ct^{-\frac{N+2}{2}}\|f\|_{\Omega,q}.$$

By using Sobolev's embedding theorem and an interpolation method, we have

$$\begin{aligned} \|e^{-tA}f\|_{\Omega,r} &\le C\|e^{-tA}f\|_{\Omega,q,2\sigma} \\ &\le C\left(t^{-\frac{N}{2}}\right)^{-\sigma+\frac{N}{2}+1}\left(t^{-\frac{N+2}{2}}\right)^{1+\sigma-\frac{N}{2}-1}\|f\|_{\Omega,q} \\ &= Ct^{-\sigma}\|f\|_{\Omega,q}. \end{aligned}$$

When N is odd, we consider $N-1$ instead of N in the same way. ■

[proof of Theorem 1.2 (2)] We shall prove (1.4) for large t only. Since $v_4(t)$ satisfy (6.24), $\partial v_4(t)$ is described as the integral form :

$$\partial v_4(t) = \partial E(t)v_4 + v_6, \quad v_6(t) = \int_0^t \partial E(t-s)P_0 h(s)ds.$$

We can estimate $v_4(t)$ as follows:

$$\|\partial v_4(t)\|_r \le C(1+t)^{-\sigma-\frac{1}{2}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+7}. \tag{6.27}$$

In fact, since by (6.23) and (6.23) we have

$$\|\partial E(t)v_4\|_r \leq C(1+t)^{-\sigma-\frac{1}{2}}\|v_4\|_{q,[2\sigma]+2} \leq C(1+t)^{-\sigma-\frac{1}{2}}\|u_0\|_{q,[2\sigma]+2},$$

it is suficient that we consider the estimate of $v_6(t)$.

Case 1: We consider the estimate of $v_6(t)$ for $\sigma > \frac{1}{2}$. We have

$$\begin{aligned}\|v_6(t)\|_r &\leq C\int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}}\|P_0 h(s)\|_{q,[2\sigma]+2}ds \\ &\leq C\int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}}(1+s)^{-\frac{n}{2q}}ds\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+6}.\end{aligned}$$

Since $q \leq r \leq n$, it follows that $\frac{n}{2r} \geq \frac{1}{2}$, that is, $\sigma + \frac{1}{2} \leq \frac{n}{2q}$, we can see

$$\begin{aligned}\int_{\frac{t}{2}}^t (1+(t-s))^{-\sigma-\frac{1}{2}}(1+s)^{-\frac{n}{2q}}ds &\leq C(1+t)^{-\frac{n}{2q}} \leq C(1+t)^{-\sigma-\frac{1}{2}}, \\ \int_0^{\frac{t}{2}} (1+(t-s))^{-\sigma-\frac{1}{2}}(1+s)^{-\frac{n}{2q}}ds &\leq C(1+t)^{-\sigma-\frac{1}{2}-\frac{n}{2q}+1} \leq C(1+t)^{-\sigma-\frac{1}{2}}.\end{aligned}$$

Therefore we can show

$$\|v_6(t)\|_r \leq C(1+t)^{-\sigma-\frac{1}{2}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+6}.$$

Case 2: We consider the case that $\sigma \leq \frac{1}{2}$, $n(1-\frac{1}{r}) > 1$ ie. $r > \frac{n}{n-1}$. We can find $\rho > 1$ so that $\rho < q$ and $\frac{n}{\rho} - \frac{n}{r} = 1 + 2\kappa$ for $0 < \kappa < \frac{1}{2}$. We have

$$\begin{aligned}\|v_6(t)\|_r &\leq C\int_0^t (1+(t-s))^{-(1+\kappa)}\|P_0 h(s)\|_{\rho,3}ds \\ &\leq C\int_0^t (1+(t-s))^{-(1+\kappa)}\|h(s)\|_{\rho,3}ds \\ &\leq C\int_0^t (1+(t-s))^{-(1+\kappa)}(1+s)^{-\frac{n}{2q}}ds\|u_0\|_{\Omega,q,[\frac{n}{q}]+7}.\end{aligned}$$

If $\frac{n}{2q} \geq 1$, we have

$$\begin{aligned}\int_{\frac{t}{2}}^t (1+(t-s))^{-(1+\kappa)}(1+s)^{-\frac{n}{2q}}ds &\leq C(1+t)^{-\frac{n}{2q}} \leq C(1+t)^{-1}, \\ \int_0^{\frac{t}{2}} (1+(t-s))^{-(1+\kappa)}(1+s)^{-\frac{n}{2q}}ds &\leq C(1+t)^{-(1+\kappa)-\frac{n}{2q}+1} \leq C(1+t)^{-1}.\end{aligned}$$

Therefore we can obtatin the estimate of $v_6(t)$:

$$\|v_6(t)\|_r \leq C(1+t)^{-1}\|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \leq C(1+t)^{-\sigma-\frac{1}{2}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+7}.$$

If $\frac{n}{2q} < 1$, then $\sigma + \frac{1}{2} \leq \frac{n}{2q}$. So we have

$$\|v_6(t)\|_r \leq C(1+t)^{-\frac{n}{2q}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+7} \leq C(1+t)^{-\sigma-\frac{1}{2}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+7}.$$

Case 3 : We consider the case that $\sigma \leq \frac{1}{2}$ and $r \leq \frac{n}{n-1}$. Then $\frac{n}{2q} \geq \frac{n-1}{2}$. Provided that $\frac{n}{2q} > 1$, which is always valid when $n \geq 4$, we have

$$\begin{aligned}
\|v_6(t)\|_r &\leq C\int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}}\|P_0h(s)\|_{q,[2\sigma]+2}ds \\
&\leq C\int_0^t (1+(t-s))^{-\sigma-\frac{1}{2}}(1+s)^{-\frac{n}{2q}}ds\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+6} \\
&\leq C(1+t)^{-\sigma-\frac{1}{2}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+[2\sigma]+6}.
\end{aligned}$$

Case 4: Finally it remain to consider the case that $n = 3$ and $q = r = \frac{3}{2}$. We choose ρ so that $1 < \rho < \frac{3}{2}$. Then we have

$$\begin{aligned}
\|v_6\|_{\frac{3}{2}} &\leq C\int_0^t (1+(t-s))^{-\frac{3}{2\rho}+\frac{1}{2}}\|P_0h(s)\|_{\rho,2}ds \\
&\leq C\int_0^t (1+(t-s))^{-\frac{3}{2\rho}+\frac{1}{2}}\|h(s)\|_{\rho,2}ds \\
&\leq C\int_0^t (1+(t-s))^{-\frac{3}{2\rho}+\frac{1}{2}}(1+s)^{-1}ds\|u_0\|_{\Omega,q,[\frac{n}{q}]+6} \\
&\leq C(1+t)^{-\frac{3}{2\rho}+\frac{1}{2}}\log(1+t)\|u_0\|_{\Omega,q,[\frac{n}{q}]+6} \leq C(1+t)^{-\frac{1}{2}}\|u_0\|_{\Omega,q,[\frac{n}{q}]+6}.
\end{aligned}$$

Summing up, we have prove (6.27). ■

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