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Derivatives of Spectral Function and Sobolev Norms of Eigenfunctions on a Closed Riemannian Manifold

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Abstract

Let $e(x,y,\lambda)$ be the spectral function and χ_λ the unit spectral projection operator, with respect to the Laplace-Beltrami operator on a closed Riemannian manifold M. We firstly review their history, including the asymptotic property of $e(x,x,\lambda)$, the story of the birth of χ_λ and the $L^2(M)\to L^p(M)$ $(p\geq 2)$ mapping properties of χ_λ. Then we give a generalization of the asymptotic formula of $e(x,x,\lambda)$ to $\partial_x^\alpha\partial_y^\beta e(x,y,\lambda)|_{x=y}$ for any multi-indices α,β in a sufficiently small geodesic normal coordinate chart of M. Finally, we apply this to the $(L^2$, Sobolev $L^p)$ $(p\geq 2)$ mapping properties of χ_λ.

1 Examples

Before giving the definitions of the spectral function $e(x,y,\lambda)$ and the unit spectral projection operator χ_λ on a general closed Riemannian manifold, let us see three examples, the first of which is concerned with the explicit computation of $e(x,y,\lambda)$ on a n-dimensional flat torus, the second of which are related with spherical harmonics and belongs to the prelude of the birth of χ_λ, and the third of which is on the $L^2(\mathbf{R}^n)\to L^p(\mathbf{R}^n)$ mapping property of an Euclidean space analogy of χ_λ.

Example 1.1. Let $T^n=\mathbf{R}^n/(2\pi\mathbf{Z})^n$ be the standard n-dimensional torus with the flat metric and Lesbesgue measure induced from $\mathbf{R}^n$. Let $\mathbf{k}=(k_1,\cdots,k_n)$ denote a lattice point in

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$\mathbf{Z}^n$ and $|\mathbf{k}|^2 := \sum_1^n k_j^2$. Let $\theta = (\theta_1, \cdots, \theta_n)$ denote a point in $[0, 2\pi)^n$ and $\mathbf{k} \cdot \theta := \sum_1^n k_j \theta_j$. Then $d\theta = d\theta_1 \cdots d\theta_n$ gives the Lebesgue measure on T^n. The functions $\frac{\exp(i\mathbf{k} \cdot \theta)}{(2\pi)^{n/2}}$, $\mathbf{k} \in \mathbf{Z}^n$, are L^2-normalized eigenfunctions of the the positive Laplacian $-\sum_{j=1}^n \frac{\partial^2}{\partial \theta_j^2}$ on T^n and their eigenvalues are $|\mathbf{k}|^2$. Moreover they exhaust all the eigenfunctions of the positive Laplacian since they form a completed orthonormal basis of $L^2(T^n, d\theta)$. The spectral function of T^n is defined by

$$\begin{aligned} e(\theta, \theta', \lambda) &:= \sum_{|\mathbf{k}| \leq \lambda} \frac{\exp(i\mathbf{k} \cdot \theta)}{(2\pi)^{n/2}} \cdot \overline{\frac{\exp(i\mathbf{k} \cdot \theta')}{(2\pi)^{n/2}}} \\ &= (2\pi)^{-n} \sum_{|\mathbf{k}| \leq \lambda} \exp(i\mathbf{k}(\theta - \theta')) . \end{aligned}$$

Restricting $e(\theta, \theta', \lambda)$ on the diagonal, we obtain $e(\theta, \theta, \lambda)$ equals the number of the lattice points in the Euclidean ball centered at 0 and having radius λ. Moreover, we have the asymptotic formula

$$e(\theta, \theta, \lambda) = (2\pi)^{-n} |B_n| \lambda^n + O(\lambda^{n-1}), \ \lambda \to +\infty,$$

where $|B_n|$ is the volume of the unit ball $B_n = \{x \in \mathbf{R}^n : |x| \leq 1\}$.

We remark that on the open subset $(-\pi/2, \pi/2)^n$ of T^n $\theta = (\theta_1, \cdots, \theta_n)$ gives the geodesic normal coordinates, whose definition will be given in Section 3. On the other hand, $\partial/\partial\theta_1, \cdots, \partial/\partial\theta_n$ are in fact global vector fields on T^n so that for any multi-index α, ∂_θ^α becomes a differential operator on T^n. For a positive integer m, we set the following notations:

$$(2m-1)!! := (2m-1)(2m-3) \cdots 3 \cdot 1, \ (-1)!! := 1 .$$

We say $\alpha \equiv \beta \pmod 2$ for two multi-indices $\alpha, \beta \in \mathbf{Z}_+^n$ if and only if $\alpha_j \equiv \beta_j \pmod 2$ for $1 \leq j \leq n$. By simple computation, we obtain the generalization of $e(\theta, \theta, \lambda)$,

$$\partial_\theta^\alpha \partial_{\theta'}^\beta e(\theta, \theta', \lambda)|_{\theta = \theta'} = \begin{cases} (2\pi)^{-n} (-1)^{(|\alpha| - |\beta|)/2} \sum_{|\mathbf{k}| \leq \lambda} \mathbf{k}^{\alpha + \beta} & \text{if } \alpha \equiv \beta \pmod 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

It is not difficult to prove (cf Theorem 1.1.7 in [8]) the asymptotic formula

$$\sum_{|\mathbf{k}| \leq \lambda} \mathbf{k}^{\alpha+\beta} = \lambda^{n + |\alpha + \beta|} \int_{B_n} x^{\alpha+\beta} \, dx + O(\lambda^{n + |\alpha+\beta| - 1}), \ \lambda \to \infty, \tag{2}$$

where for $\alpha \equiv \beta \pmod 2$

$$\int_{B_n} x^{\alpha+\beta} \, dx = \frac{\pi^{n/2}}{2^{|\alpha+\beta|/2} \Gamma(\frac{|\alpha+\beta|+n}{2} + 1)} \prod_{j=1}^n (\alpha_j + \beta_j - 1)!! . \tag{3}$$

The remainder term on the left hand side of (2) can be improved further. For example, if $\alpha = \beta = 0$, the remainder can be refined to be $O(\lambda^{n-2+\frac{2}{n+1}})$ (cf Theorem 11 in [2]) by using the stationary phase method, however it is an open problem even in dimension two to determine the precise reminder term. For general case, please see Ben Lichtin [9], in which an excellent survey is given on this interesting and difficult problem.

Example 1.2. Consider the Euclidean unit sphere S^n in $\mathbf{R}^{n+1}$, and the standard positive Laplace-Beltrami operator Δ on S^n. The distinct eigenvalues of Δ are $j(j+n-1)$, $(j = 0, 1, \cdots)$, and the corresponding eigenfunctions are the restrictions to S^n of the harmonic homogeneous polynomials of degree j in $\mathbf{R}^{n+1}$, which are called the spherical harmonics of degree j. Moreover, the multiplicity of the eigenvalue $j(j+n-1)$ $(j \geq 2)$ equals $\begin{pmatrix} n+j \\ n \end{pmatrix} - \begin{pmatrix} n+j-2 \\ n \end{pmatrix}$, which can be comparable to j^{n-1} as $j \to +\infty$. The proof of the above facts can be founded, for example, in §17.5 of [7]. Let H_j denote the projection operator with respect to the space of the spherical harmonics of degree j. Then H_j takes an L^2 function f on S^n, which can be written as $f = \sum_{k=0}^{\infty} H_k f$, to $H_j f$. Let $\delta(r)$ be the critical exponent $\max(n \cdot |1/r - 1/2| - 1/2, 0)$ for Bochner Riesz means on $L^r(\mathbf{R}^n)$. Sogge [11] obtained the sharp estimates:

$$||H_j f||_{L^r(S^n)} \leq C j^{\varepsilon(r)} ||f||_{L^2(S^n)}, \; j \geq 1, \tag{4}$$

where the constant C does not depend on j and

$$\varepsilon(r) = \begin{cases} \frac{(n-1)(r-2)}{4r} & \text{if } 2 \leq r \leq \frac{2(n+1)}{n-1}, \\ \delta(r) & \text{if } \frac{2(n+1)}{n-1} \leq r \leq \infty. \end{cases}$$

The sharpness of (4) means that the bounds can not be replaced by $o(j^{\varepsilon(r)})$. Sharpness of the bounds of estimates in what follows will always have this meaning.

Example 1.3. Tomas and Stein [16] showed that for $n \geq 2$ and $1 \leq p \leq 2(n+1)/(n+3)$ then Fourier transform of an $L^p(\mathbf{R}^n)$ function restricts to the unit sphere as an element of $L^2(S^{n-1})$. That is, if $d\sigma$ denotes the induced Lebesgue measure on S^{n-1}, then the following inequality holds:

$$\left(\int_{S^{n-1}} |\hat{f}(\xi)|^2 d\sigma(\xi) \right)^{1/2} \leq C ||f||_{L^p(\mathbf{R}^n)} \tag{5}$$

for f belonging to the Schwarz function space $\mathscr{S}(\mathbf{R}^n)$. A straightforward calculation involving Plancherel's theorem for $\mathbf{R}^n$ shows that if we define projection operators P_λ as follows

$$P_\lambda f(x) = \int_{|\xi| \in (\lambda, \lambda+1]} \hat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi,$$

then (5) is equivalent to a uniform inequality of the following form:

$$||P_\lambda f||_{L^2(\mathbf{R}^n)} \le C\lambda^{\delta(p)}||f||_{L^p(\mathbf{R}^n)},\ 1 \le p \le 2(n+1)/(n+3),\ \lambda \ge 1.$$

By dual argument and interpolation with $||P_\lambda f||_2 \le ||f||_2$, we obtain from the above inequality that

$$||P_\lambda f||_{L^r(\mathbf{R}^n)} \le C\lambda^{\varepsilon(r)}||f||_{L^2(\mathbf{R}^n)},\ 2 \le r \le \infty,\ \lambda \ge 1, \tag{6}$$

which can be comparable to (4).

Consider the Laplace-Beltrami operator $\Delta = -\sum_{j=1}^n \partial^2/\partial x_j^2$, which is a self-adjoint operator with domain $H^2(\mathbf{R}^n)$ on $L^2(\mathbf{R}^n)$. Let E_λ be its spectral family. Then by Theorem 10.17 in [1], up to a constant

$$P_\lambda = E_{(\lambda+1)^2} - E_{\lambda^2}\ .$$

Next section we will define the unit spectral operator χ_λ on a closed Riemannina manifold as an analogy of P_λ.

2 Spectral function and Fourier restriction theorems on manifold

In this section we shall give the definitions of the spectral function $e(x,y,\lambda)$ and the unit spectral projection operator χ_λ, recall the asymptotic formula of $e(x,x,\lambda)$ as $\lambda \longrightarrow +\infty$ by Hörmander and the (L^2, L^r) $(r \ge 2)$ mapping properties of χ_λ by Sogge and explain their interrelation.

Let M be a closed (compact and boundaryless) smooth manifold of dimension $n \ge 2$ and let P be an essentially self-adjoint and positive elliptic differential operator with smooth coefficients in $L^2(M, d\mu)$, where $d\mu$ is a positive smooth density. Let $\{E_\lambda\}$ be the spectral family of P, and let $e(x,y,\lambda)$ $(\lambda \ge 0)$ be the kernel of E_{λ^2}. This is an element of $C^\infty(M \times M)$ called the spectral function of P. Let p be the principal symbol of P, which is a real homogeneous polynomial of degree m on the cotangent bundle T^*M. The density $d\mu$ defines a Lebesgue measure $d\xi$ in each fiber of T^*M, which is a vector space of dimension n. Hörmander [5] proved the following uniform estimate by using the Fourier integral operator:

$$e(x,x,\lambda) = (2\pi)^{-n} \times \int_{B_x} d\xi \times \lambda^{2n/m} + \mathrm{O}(\lambda^{2(n-1)/m}),\ \lambda \longrightarrow +\infty \tag{7}$$

where $B_x = \{\xi \in T_x^*M | p(\xi) \le 1\}$.

Let g be a Riemannian metric on M, which is a $(2,0)$ tensor field such that for any x in M, $g(x)$ is a scalar product on the tangent space $T_x(M)$ at x of M. Let $|\cdot|_g$ be the norm on $T_x(M)$ with respect to $g(x)$. We define a distance s on M and a positive Radon measure $f \mapsto \int_M f\,dv(g)$ as follows. The distance $s(x,y)$ of x and y in M is defined to be the infimum of the lengths $L(\gamma)$ of all piecewise C^1 curves $\gamma : [a,\, b] \to M$ from x to y, where

$$L(\gamma) = \int_a^b |\frac{d\gamma}{dt}|_g\, dt$$

While the Riemannian volume element is given in any chart by

$$dv(g) = \sqrt{\det\,(g_{ij}(x))}\,dx =: \sqrt{\mathbf{g}(x)}\,dx,$$

where the g_{ij}'s are the components of g in the chart, and dx is the Lebesgue's volume element of $\mathbf{R}^n$. One can also define the Levi-Civita connection ∇ of g as the unique linear connection on M which is torsion free and which is such that the covariant derivative of g is zero. The Christofell symbols of the Levi-Civita connection are then given in any chart by

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \Gamma_{ij}^k \partial_k,\ \Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^n g^{lk}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})\,,$$

where (g^{ij}) denotes the inverse matrix of g_{ij}. Let Δ be the positive Laplace-Beltrami operator associated to g acting on functions. In any chart,

$$\Delta = -\frac{1}{\sqrt{\mathbf{g}}}\sum_{j=1}^n \partial_j(\sqrt{\mathbf{g}}\sum_{k=1}^n g^{jk}\partial_k)$$

Let $L^2(M)$ be the space of square integrable functions on M with respect to the positive Radon measure $dv(g)$. Let P be the self-adjoint extension of the positive Laplace-Beltrami operator Δ on $L^2(M)$. Then applying the above result (7) to this P, we have

$$e(x,x,\lambda) = (2\pi)^{-n}|B_n|\lambda^n + \mathrm{O}(\lambda^{n-1}),\ \lambda \to +\infty. \tag{8}$$

Let $e_1(x), e_2(x), \cdots$ be a complete orthonormal basis in $L^2(M)$ for the real-valued eigenfunctions of Δ such that $0 \le \lambda_1^2 \le \lambda_2^2 \le \cdots$ for the corresponding eigenvalues, where λ_j are nonnegative real numbers. Let $\mathbf{e}_j$ denote the projection onto the 1-dimensional space $\mathbf{C}e_j$. Thus, an L^2 function f can be written as $f = \sum_{j=1}^{\infty} \mathbf{e}_j(f)$, where the partial sum converges in the L^2 norm. It follows from the spectral resolution of the Laplace-Beltrami operator Δ that

$$e(x,y,\lambda) = \sum_{\lambda_j \le \lambda} e_j(x)e_j(y),$$

by which and (8) we have the uniform estimate for $x \in M$ of the following form:

$$\sum_{\lambda_j \in (\lambda,\lambda+1]} |e_j(x)|^2 \le C\lambda^{n-1},\ \lambda \ge 1, \tag{9}$$

which is also sharp.

Recalling this model case S^n in Example 1.2, the eigenvalues $j(j+n-1)$ repeat with a high frequency comparable to j^{n-1} as $j \longrightarrow +\infty$. For a general compact Riemannian manifold M, by the integral of (9) on M, we obtain the number of λ_j in $(\lambda, \lambda+1]$ is always comparable to λ^{n-1} as $\lambda \longrightarrow +\infty$. With the support of Examples 1.2 and 1.3, Sogge defined

$$\chi_\lambda : f \mapsto \sum_{\lambda_j \in (\lambda, \lambda+1]} \mathbf{e}_j f$$

as the appropriate generalizations of H_j and P_λ in [12] [13], where he also proved the corresponding projection theorem of the form:

$$||\chi_\lambda f||_r \leq C\lambda^{\varepsilon(r)}||f||_2,\ 2 \leq r \leq \infty, \tag{10}$$

where $||\cdot||_r$ is the L^r norm of the function on M. Moreover, Sogge proved in [13] that this estimate is sharp. We call χ_λ the unit spectral projection operator of Δ. We may consider (10) as the Fourier restriction theorems on compact Riemannian manifold because it has the same expression with (6). The following lemma gives the relationship between the uniform estimate (9) of eigenfunctions and the (L^2, L^∞) mapping property of χ_λ.

Lemma 2.1. *The uniform estimate* (9) *is equivalent to the* (L^2, L^∞) *estimate of* χ_λ:

$$||\chi_\lambda f||_\infty \leq C\lambda^{(n-1)/2}||f||_2,\ \lambda \geq 1. \tag{11}$$

Proof. The idea of the proof is due to Sogge [11]. Let the estimate (9) hold. Without loss of generality, we assume that f is a real-valued function on M in what follows. Since

$$\chi_\lambda f(x) = \int_M \sum_{\lambda_j \in (\lambda, \lambda+1]} e_j(x)e_j(y)f(y)dv(M)\,,$$

for any $x \in X$, by the Cauchy-Schwarz inequality and (9) we have

$$\begin{aligned} |\chi_\lambda f(x)|^2 &\leq \sum_{\lambda_j \in (\lambda, \lambda+1]} |e_j(x)|^2 \sum_{\lambda_j \in (\lambda, \lambda+1]} \left(\int_M e_j(y)f(y)dv(M) \right)^2 \\ &\leq C\lambda^{n-1}||f||_2^2\ (\lambda \geq 1)\,. \end{aligned}$$

Let the estimate (11) hold. Taking a point $x \in M$ and substituting

$$f(\cdot) = \sum_{\lambda_j \in (\lambda, \lambda+1]} e_j(x)\, e_j(\cdot)$$

into (11), we obtain the inequality (9). q.e.d.

The main ideas of Sogge's proof to (10) is as follows. With the help of the oscillatory integral theorems of Carleson-Sjölin [3] and Stein [15], Sogge showed in [12] and [13]

$$||\chi_\lambda f||_q \leq C\lambda^{\delta(q)}||f||_2,\ q = 2(n+1)/(n-1) \tag{12}$$

by using the Hadamard parametrix for $\Delta - (\lambda + i)^2$ and the wave operator $(\partial/\partial t)^2 + \Delta$ respectively. Interpolating (12) with (11) and the trivial inequality $||\chi_\lambda f||_2 \leq ||f||_2$, Sogge proved (10).

3 Derivatives of spectral function

In this section we give the definitions of Sobolev spaces and geodesic normal coordinates on M and state our generalizations of Hörmander's results (8), (9) to the derivatives of the spectral function and eigenfunctions.

Let (g^{ij}) denote the inverse matrix of (g_{ij}). For k a nonnegative integer and $u \in C^\infty(M)$, $\nabla^k u$ denotes the kth covariant derivative of u (with the convention $\nabla^0 u = u$). As an example, the components of ∇u in local coordinates are given by $(\nabla u)_i = \partial_i u$, while the components of $\nabla^2 u$ in local coordinates are given by

$$(\nabla^2 u)_{ij} = \partial_{ij}^2 u - \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k u\,. \tag{13}$$

We define the length $|\nabla^k u|$ of $\nabla^k u$ by

$$|\nabla^k u|^2 := \sum g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k}\,.$$

where the sum is taken for $1 \leq i_1, \cdots, i_k, j_1, \cdots, j_k \leq n$.

Definition 3.1. The Sobolev space $H_k^r(M)$ is the completion of $C^\infty(M)$ with respect to the norm

$$||u||_{H_k^r} := \left(\sum_{j=0}^{k} \int_M |\nabla^j u|^r dv(g)\right)^{1/r},\ 1 \leq r < \infty\,,$$

$$||u||_{H_k^r} := \sum_{j=0}^{k} \sup_{x \in M} |\nabla^j u(x)|, \qquad r = \infty\,.$$

Sometimes we also write C^k, H^k instead of H_k^∞, H_k^2.

The following result is well known.

Proposition 3.1. *$H_k^r(M)$ does not depend on the Riemannian metric. And $H^k(M)$ is a Hilbert space.*

We also need some preliminary knowledge about the geodesic normal coordinates on the Riemannian manifold (M,g). A smooth curve γ is said to be a geodesic iff $\nabla_{\frac{d\gamma}{dt}}\frac{d\gamma}{dt}=0$. In local coordinate, this means that for any $k=1,\cdots,n$,

$$(\gamma^k)''(t)+\sum_{1\le i,j\le n}\Gamma^k_{ij}(\gamma(t))(\gamma^i)'(t)(\gamma^j)'(t)=0\,,$$

which is a second order nonlinear ordinary differential system. For a point p in M and a tangent vector V in the tangent space $T_p(M)$ of M at p, there always a positive number $a>0$ such that the above system has a solution $\gamma_V(t)$ for t in $(-a,a)$ with $\gamma_V(0)=p$ and $\frac{d\gamma_V}{dt}(0)=V$. p and V are called the initial point and the initial velocity of the solution geodesic $\gamma_V(t)$ respectively. On the other hand, since M is closed and then is complete with respect to the distance s, by the Hopf-Rinow's theorem any geodesic on M can be defined on the whole of $\mathbf{R}$.

A geodesic $\gamma(t)$ is minimizing locally, i.e. the length $L(\gamma|_{[t_1,t_2]})$ of the geodesic arc $\gamma|_{[t_1,t_2]}$ equals $s(\gamma(t_1),\gamma(t_2))$ if $|t_1-t_2|$ is sufficiently small. The injectivity radius $inj_M(p)$ at p is defined as the largest $r>0$ for which any geodesic γ of length less than r and having p as the initial point is minimizing. The injectivity radius inj_M of (M,g) is then defined as the infimum of $inj_M(p)$, $p\in M$. It is a positive number by the compactness of M. The exponential map $\exp_p$ at p in M is the map from $T_p(M)$ to M defined by the map $\exp_p(V)=\gamma_V(1)$. Up to the identification of $T_p(M)$ with $\mathbf{R}^n$, it is smooth and it defines geodesic normal coordinates at p on

$$B_p(inj_M(p))=\{q\in M: s(p,q)<inj_M(p)\},$$

in which a point q has the coordinates $V\in T_p(M)$ with

$$\exp_p V=q\,.$$

The preimage of $B_p(inj_M(p))$ by $\exp_p$ is a neighborhood of 0 in $T_p(M)$. Let

$$\mathscr{W}=\{(q,p)\in M\times M: s(q,p)<inj_M\}\,.$$

Globally there is a neighborhood $\mathscr{V}$ of the zero section $\{0\}\times M$ in the tangent bundle TM and a well-defined diffeomorphism

$$\mathscr{V}\ni(V,p)\mapsto(\exp_p V,p)\in\mathscr{W}\,.$$

Taking a point p in M and fixing it, we can see $B_p(\frac{1}{4}inj_M)\times B_p(\frac{1}{4}inj_M)\subset\mathscr{W}$. In what follows, let $(X,x)=(B_p(\frac{1}{4}inj_M)),\exp_p^{-1})$ be the geodesic normal coordinates on $B_p(\frac{1}{4}inj_M)$. In particular, $x(p)=0$. We will generalize Hörmander's result (8) by considering the derivatives of $e(x,y,\lambda)$ on the diagonal of $X\times X$.

Theorem 3.1. *In the geodesic normal coordinate chart* (X,x) *of* M, *for multi-indices* $\alpha,\beta \in \mathbf{Z}_+^n$ *the following estimates hold uniformly for* $x \in X$ *as* $\lambda \longrightarrow \infty$:

$$\partial_x^\alpha \partial_y^\beta e(x,y,\lambda)|_{x=y} = \begin{cases} C_{n,\alpha,\beta}\,\lambda^{n+|\alpha+\beta|} + O(\lambda^{n+|\alpha+\beta|-1}) & \text{if } \alpha \equiv \beta \ (\mathrm{mod}\ 2), \\ O(\lambda^{n+|\alpha+\beta|-1}) & \text{otherwise,} \end{cases} \tag{14}$$

where for multi-indices α,β *such that* $\alpha \equiv \beta$ (mod 2),

$$\begin{aligned} C_{n,\alpha,\beta} &= (2\pi)^{-n}(-1)^{(|\alpha|-|\beta|)/2}\int_{B_n} x^{\alpha+\beta}\,dx \\ &= (-1)^{(|\alpha|-|\beta|)/2}\frac{\prod_{j=1}^n(\alpha_j+\beta_j-1)!!}{\pi^{n/2}2^{n+|\alpha+\beta|/2}\Gamma(\frac{|\alpha+\beta|+n}{2}+1)} \end{aligned}$$

In particular, if $\alpha = \beta$, *then the following estimate holds uniformly for* $x \in X$ *as* $\lambda \longrightarrow \infty$:

$$\sum_{\lambda_j \le \lambda} |\partial^\alpha e_j(x)|^2 = C_{n,\alpha}\lambda^{n+2|\alpha|} + O(\lambda^{n+2|\alpha|-1}), \tag{15}$$

where $C_{n,\alpha} = C_{n,\alpha,\alpha} > 0$.

Remark 3.1. Since $e(x,y,\lambda) = \sum_{\lambda_j \le \lambda} e_j(x)e_j(y)$, an immediate and interesting consequence of Theorem 3.1 says that if λ is sufficiently large, then in the geodesic normal coordinate (X,x) the function $\sum_{\lambda_j \le \lambda} \partial^\alpha e_j(x)\partial^\beta e_j(x)$ with $\alpha \equiv \beta$ (mod 2) is positive (negative) iff $|\alpha| - |\beta|$ can (not) be divided by 4.

Remark 3.2. Let $(\tilde{Y},x)$ be an arbitrary coordinate chart in M and Y be a relatively compact subset of $\tilde{Y}$. Then Theorem 17.5.3 of [7] claims that the following uniform estimate holds for $(x,y) \in Y \times Y$:

$$|\partial_{x,y}^\gamma e(x,y,\lambda)| \le C\lambda^{n+|\gamma|},\ \lambda \ge 1. \tag{16}$$

Theorem 3.1 refines this rough estimate on the diagonal of $X \times X$ for the geodesic normal coordinate chart X.

Remark 3.3. Since M is compact, considering a finite covering of geodesic coordinate charts on M, we obtain from (15) that

$$\sum_{\lambda_j \in (\lambda,\lambda+1]} ||e_j||_{C^k} = O(\lambda^{n+2k-1}),\ \lambda \longrightarrow +\infty.$$

Using the same idea in the proof of Lemma 2.1, by the above estimate we can prove the (L^2, C^k) mapping properties of χ_λ of the following form:

$$||\chi_\lambda f||_{C^k} \le C\lambda^{n+2k-1}||f||_2,\ \lambda \ge 1. \tag{17}$$

4 Outline of proof of the $\alpha = \beta$ case of Theorem 3.1

4.1 The Hadamard parametrix

Let $\mathscr{P}$ be the self-adjoint extension of $1+\Delta$ in $L^2(M)$ with $\mathscr{D}_{\mathscr{P}} = H^2(M)$. Let $\cos(t\sqrt{\mathscr{P}})$ be the wave operator associated with $\mathscr{P}$ defined by

$$\cos(t\sqrt{\mathscr{P}}) = \int_0^\infty \cos(t\sqrt{\mu})dE_\mu\,,$$

where E_μ is the spectral family of $\mathscr{P}$. Since the spectral function $\tilde{e}(x,y,\lambda)$ of $\mathscr{P}$ has the relation with $e(x,y,\lambda)$ of Δ as $\tilde{e}(x,y,\lambda) = \begin{cases} 0 & \text{if } \lambda \in [0,1) \\ e(x,y,\sqrt{\lambda^2-1}) & \text{if } \lambda \geq 1 \end{cases}$, for the proof of Theorem 3.1 we only need to consider $\tilde{e}(x,y,\lambda)$ instead of $e(x,y,\lambda)$. For simplicity of notations, in the following of this section we still write $\tilde{e}(x,y,\lambda)$ to be $e(x,y,\lambda)$. By the standard computations (cf Section 17.5 of [7]), the wave kernel $K(t,x,y) \in \mathscr{D}'(\mathbf{R}\times M \times M)$ of $\cos(t\sqrt{\mathscr{P}})$ is the Fourier transformation with respect to τ of the temperate measure $dm(x,y,\tau)$,

$$m(x,y,\tau) = \sqrt{\mathbf{g}(y)}\,(\operatorname{sgn}\tau)\,e(x,y,|\tau|)/2\,. \tag{18}$$

We remark that $K(t,x,y)$ is even with respect to t.

In the following we shall review a remarkably simple and precise construction due to J. Hadamard, which gives the singularities of the wave kernel $K(t,x,y)$ with any desired precision. All the details of this construction can be found in § 17.4-5 of [7].

Let the distribution χ_+^a ($a \in \mathbf{C}$) on $\mathbf{R}$ is defined to be $x_+^a/\Gamma(a+1)$ for $\operatorname{Re} a > -1$ and is defined on the other values of a in $\mathbf{C}$ by analytic continuation so that $d\chi_+^a/dx = \chi_+^{a-1}$ (cf (3.2.17) in [6]). In particular, χ_+^0 is the Heaviside function and $\chi_+^{-k} = \delta_0^{(k-1)}$ for $k = 1,2,\cdots$. In $\mathbf{R}_t \times \mathbf{R}_x^n$ we define the homogeneous distributions E_ν ($\nu \in \mathbf{Z}$) of degree $2\nu+1-n$ with support in the forward light cone $\{(t,x) : t \geq |x|\}$ by

$$E_\nu = 2^{-2\nu-1}\pi^{(1-n)/2}\chi_+^{\nu+(1-n)/2}(t^2-|x|^2),\ t > 0\,. \tag{19}$$

We have

$$(\partial^2/\partial t^2 - \textstyle\sum \partial^2/\partial x_j^2)E_\nu = \nu E_{\nu-1}, \nu \neq 0;\ (\partial^2/\partial t^2 - \textstyle\sum \partial^2/\partial x_j^2)E_0 = \delta_{0,0}; \tag{20}$$

$$-2\partial E_\nu/\partial x = xE_{\nu-1}. \tag{21}$$

With some abuse of the notation we shall write $E_\nu(t,|x|)$ instead of $E_\nu(t,x)$ in what follows; when $t = 0$ this should be interpreted as the limit when $t \to +0$. Moreover it follows

from the proof of Lemma 17.4.2 in [7] with the notation (3.2.10)$'$ of [6] that
$F_\nu(t) := \partial_t(E_\nu(t,0) - \check{E}_\nu(t,0))$

$$= \begin{cases} 2^{-2\nu}\pi^{(1-n)/2}t^{2\nu-n}/\Gamma(\nu+(1-n)/2), & \text{if n is even} \\ 2^{-2\nu-1}\pi^{(1-n)/2}|t|^{2\nu-n}/\Gamma(\nu+(1-n)/2), & \text{if } n \text{ is odd and } 2\nu > n \\ (-1)^k 2^{-2\nu-k}\pi^{(1-n)/2}\delta^{(2k)}/(2k-1)!!, & \text{if } n = \text{odd and } n-1-2\nu = 2k \geq 0, \end{cases} \tag{22}$$

where $\check{E}_\nu$ is the reflection of E_ν with respect to the origin of $\mathbf{R}_t$.

Recall the notations X and $\mathscr{W}$ appearing in the above section. Let $X^c = \{q \in M : \inf_{p\in X} s(p,q) < c\}$ and put $c = \frac{1}{4}inj_M$ in what follows. Then $X^c \times X \subset \mathscr{W}$ and the geodesic coordinates on X can be extended onto X^c. By the Hadamard construction (cf §17.4 in [7]), there exists a sequence of smooth functions $U_\nu(x,y)$ $(\nu = 0,1,\cdots)$ in $\mathscr{W}$ with $U_0(x,x) = 1$ such that for

$$\mathscr{E}(t,x,y) = \sum_0^N U_\nu(x,y)E_\nu(t,s(x,y))$$

with the positive integer N sufficently large, the followings hold:
(i) For $(t,x,y) \in (-c,c) \times \mathscr{W}$,

$$K(t,x,y) - \partial_t(\mathscr{E}(t,x,y) - \check{\mathscr{E}}(t,x,y))\sqrt{\mathbf{g}(y)} \in C^{N-n-3}. \tag{23}$$

(ii) For $(t,x,y) \in (-c,c) \times X^c \times X$,

$$\begin{aligned} \left|\partial^\alpha_{t,x,y}\left(K(t,x,y) - \partial_t(\mathscr{E}(t,x,y) - \check{\mathscr{E}}(t,x,y))\sqrt{\mathbf{g}(y)}\right)\right| &\leq C|t|^{2N-n-|\alpha|}, \\ |\alpha| &\leq N-n-3. \end{aligned} \tag{24}$$

4.2 The derivatives of the wave kernel

Let $\alpha = (\alpha_1,\cdots,\alpha_n), \beta = (\beta_1,\cdots,\beta_n) \in \mathbf{Z}^n_+$ be two multi-indices. In the coordinate chart $(X \times X, (x,y))$ of $M \times M$, we shall consider the singularities of the distribution $\partial^\alpha_x\partial^\beta_y K(t,x,y)|_{x=y}$ with respect to t at the origin of $\mathbf{R}_t$. By (23), we know

$$\begin{aligned} \partial^\alpha_x\partial^\beta_y K(t,x,y)|_{x=y} &= \partial^\alpha_x\partial^\beta_y(\partial_t(\mathscr{E}(t,x,y) - \check{\mathscr{E}}(t,x,y))\sqrt{\mathbf{g}(y)})|_{x=y} \\ &+ C^{N-n-|\alpha+\beta|-3}\text{term}. \end{aligned} \tag{25}$$

By the above equality we know that $\partial^\alpha_x\partial^\beta_y K(t,x,y)|_{x=y}$ is the sum of a continuous function of $(t,x) \in (-c,c) \times X$ and finite homogeneous distributions of t with coefficients of smooth functions of $x \in X$. We call the distribution summand of $\partial^\alpha_x\partial^\beta_y K(t,x,y)|_{x=y}$ with the lowest homogeneous degree the *principal singular term* of $\partial^\alpha_x\partial^\beta_y K(t,x,y)|_{x=y}$. Thoroughly analyzing the derivatives

$$\partial^\alpha_x\partial^\beta_y\left(\partial_t(E_\nu(t,s(x,y)) - \check{E}_\nu(t,s(x,y)))\right)_{x=y}, \; x \in X,$$

we obtain the following

Lemma 4.1. *Let α, β be two multi-indices such that $\alpha \equiv \beta$ (mod 2) and let (t,x) be in $(-c,c)\times X$. Then the principal singular term of $\partial_x^\alpha \partial_y^\beta K(t,x,y)|_{x=y}$ is $q_{\alpha,\beta}\sqrt{g(x)}F_{-|\alpha+\beta|/2}(t)$, where $q_{\alpha,\beta}$ is a constant only depending on n, α, β and is positive (negative) iff $|\alpha|-|\beta|$ can (not) be divided by 4. Moreover, if n is even, $\partial_x^\alpha \partial_y^\beta K(t,x,y)|_{x=y}$ equals the principal singular term plus*

$$\sum_{1-|\alpha+\beta|/2}^{(n-2)/2} F_\nu(t) \times \text{ a smooth function of } x + \text{ a smooth function;}$$

if n is odd, $\partial_x^\alpha \partial_y^\beta K(t,x,y)|_{x=y}$ equals the principal singular term plus

$$\sum_{1-|\alpha+\beta|/2}^{(n-1)/2} F_\nu(t) \times \text{a smooth function of } x + |t| \times \text{a smooth function}\,.$$

Remark 4.1. Suppose that $\alpha \equiv \beta$ (mod 2) does not hold. We also can determine the principal singular term of $\partial_x^\alpha \partial_y^\beta K(t,x,y)|_{x=y}$. Precisely speaking, if $|\alpha+\beta|$ is even, then it is $F_{1-|\alpha+\beta|/2}(t)$ times a smooth function of x; if $|\alpha+\beta|$ is odd, then it is $F_{-r(\alpha,\beta)}(t)$ times a smooth function of x, where $r(\alpha,\beta)$ equals either $(|\alpha+\beta|-1)/2$ or $(|\alpha+\beta|-3)/2$.

4.3 The Tauberian method

In this subsection we shall prove the $\alpha=\beta$ case of Theorem 3.1. Firstly we need a Tauberian lemma.

It is well known that there exists an even positive function ϕ in $\mathscr{S}(\mathbf{R})$ such that

$$\int_{\mathbf{R}} \phi(\tau)d\tau = 1, \quad \operatorname{supp}\hat{\phi} \subset (-1,1)\,.$$

For a positive number ε, let $\phi_\varepsilon(\tau) := \phi(\tau/\varepsilon)/\varepsilon$.

Lemma 4.2. (Tauberian lemma, cf Lemma 17.5.6 in [7]) *Let ι be a nonnegative number and κ in $[0,\iota]$. Let a be a positive number and a_0, a_1 be two real numbers $\geq a$. Let v be a function of locally bounded variation such that $v(0)=0$ and $|dv(\tau)| \leq M_0(|\tau|+a_0)^\iota d\tau$. Let u be an increasing temperate function with $u(0)=0$ such that*

$$|(du-dv)*\phi_a(\tau)| \leq M_1(|\tau|+a_1)^\kappa, \ \tau \in \mathbf{R}\,. \tag{26}$$

Then

$$|u(\tau)-v(\tau)| \leq C\Big(M_0 a(|\tau|+a_0)^\iota + M_1(|\tau|+a)(|\tau|+a_1)^\kappa\Big) \tag{27}$$

where C only depends on ι and κ.

PROOF OF THE $\alpha = \beta$ CASE OF THEOREM 3.1

Step 1 We shall show that there exists a positive number $C_{n,\alpha}$ only dependent on n and α such that (15) holds in the follows.

By the equality (22) and Example 7.1.17 of [6], there exists a positive constant $D_{n,\nu}$ such that $F_\nu(t,0)$ with $2\nu < n$ is the Fourier transform of

$$\frac{d}{d\tau}\left(D_{n,\nu}(\operatorname{sgn}\tau)|\tau|^{n-2\nu}\right). \tag{28}$$

Let $C_{n,\alpha} = 2q_\alpha \times D_{n,-|\alpha|}$. We shall apply Lemma 4.2 with $a = 1/c = 4/inj_M$ and

$$\begin{aligned} u(\tau) &= (1/2)\sqrt{\mathbf{g}(x)}\,(\operatorname{sgn}\tau)\sum_{\lambda_j \le |\tau|} |\partial^\alpha e_j(x)|^2 = (1/2)\sqrt{\mathbf{g}(x)}\,(\operatorname{sgn}\tau)\,\partial_x^\alpha\partial_y^\alpha e(x,y,|\tau|)|_{x=y} \\ v(\tau) &= C_{n,\alpha}\sqrt{\mathbf{g}(x)}\operatorname{sgn}\tau\,|\tau|^{n+2|\alpha|}/2\,. \end{aligned}$$

It is clear that (T1) holds with $\iota = n+2|\alpha|-1$. By (16), $u(\tau)$ is an increasing temperate function with $u(0) = 0$. We connect $u(\tau)$ with the wave kernel $K(t,x,y)$ by the following claim.

Claim 1 The Fourier transform of

$$\frac{d}{d\tau}\left(\sqrt{\mathbf{g}(y)}\,(\operatorname{sgn}\tau)\,\partial_x^\alpha\partial_y^\beta e(x,y,|\tau|)/2\right)$$

with respect to τ can be written by

$$\partial_x^\alpha\partial_y^\beta K(t,x,y) + \sum_{\gamma<\beta} P_\gamma(y)\partial_x^\gamma\partial_y^\beta K(t,x,y),$$

where $P_\gamma(y)$ $(\gamma < \beta)$ are smooth functions of x depending on the metric g of M. In particular, $\widehat{du}(t)$ equals

$$\left(\partial_x^\alpha\partial_y^\alpha K(t,x,y) + \sum_{\gamma<\alpha} P_\gamma(y)\partial_x^\gamma\partial_y^\alpha K(t,x,y)\right)_{x=y}.$$

Proof of Claim 1: We argue by induction with respect to the nonnegative integer $|\alpha+\beta|$. The case of $\alpha = \beta = 0$ follows from (18). We denote the Fourier transform of $w(\tau)$ by $\mathbf{F}[w](t)$. Since

$$\begin{aligned} & \mathbf{F}[(d/d\tau)\sqrt{\mathbf{g}(y)}\,(\operatorname{sgn}\tau)\,\partial_{y_j}\partial_x^\alpha\partial_y^\beta e(x,y,|\tau|)/2](t) \\ = \;& \partial_{y_j}\mathbf{F}[(d/d\tau)\sqrt{\mathbf{g}(y)}\,(\operatorname{sgn}\tau)\,\partial_x^\alpha\partial_y^\beta e(x,y,|\tau|)/2](t) \\ - \;& \partial_{y_j}\log(\sqrt{\mathbf{g}(y)})\,\mathbf{F}[(d/d\tau)\sqrt{\mathbf{g}(y)}\,(\operatorname{sgn}\tau)\,\partial_x^\alpha\partial_y^\beta e(x,y,|\tau|)/2](t)\,, \end{aligned}$$

the left part of the induction argument can be completed by direct computation.

By Claim 1, (28), Lemma 4.1 and Remark 4.1, when t in $(-c, c)$, the principal singular term of $\widehat{du}$ equals that of $\partial_x^\alpha \partial_y^\alpha K(t,x,y)|_{x=y}$, which is the Fourier transform of dv; the other singular terms are Fourier transforms of $|t|^{n+2|\alpha|-2j-1}$ times smooth functions of x for $1 = j \leq |\alpha| + (n-1)/2$. Hence $(du - dv) * \phi_a$ is the sum of the regularizations of these functions and a bounded function. Then we use the idea in the proof of Theorem 17.5.7 in [7] to show that (26) holds with $\kappa = \max(n+2|\alpha|-3, 0)$ as follows.

By the choice of $a = 1/c$ and

$$(du - dv) * \phi_a(\tau) = \mathbf{F}^{-1}[(\widehat{du} - \widehat{dv})\,\widehat{\phi_a}](\tau), \ \ \operatorname{supp} \widehat{\phi_a} \subset (-c, c)\,,$$

we have

$$\begin{aligned} |(du - dv) * \phi_a(\tau)| &\leq C \sum_{j=1}^{|\alpha|+(n-1)/2} \phi_a * |t|^{n+2|\alpha|-2j-1}(\tau) \\ &\leq C\phi_a * (1+|t|)^\kappa(\tau) \\ &\leq C \int_{\mathbf{R}} (1+|t|)^{-\kappa-2}(1+|t-\tau|)^\kappa \, dt \\ &\leq C \int_{\mathbf{R}} (1+|t|)^{-\kappa-2}(1+|t|)^\kappa(1+|\tau|)^\kappa \, dt \\ &\leq C(1+\tau)^\kappa \,. \end{aligned}$$

Therefore by Lemma 4.2, we obtain

$$|u(\lambda) - v(\lambda)| \leq C\lambda^{n+2|\alpha|-1}, \ \lambda \geq 1. \tag{29}$$

Step 2 Since the constant $C_{n,\alpha}$ does not depend on the Riemannian manifold M, we only need to consider its computation on a particular closed Riemannian manifold. In fact, we have done it on a flat torus in Example 1.1 and obtained its value in (1), (2) and (3).

5 Sobolev norms of eigenfunctions

In this section we generalize Sogge's result (10) on the (L^2, L^r) mapping properties of χ_λ to its $(L^2$, Sobolev $L^r)$ ones. Moreover, we give an example of the Sobolev norms of certain spherical harmonics.

Theorem 5.1. *Let k be a nonnegative integer and $2 \leq r \leq \infty$. Then the following estimate*

$$||\chi_\lambda f||_{H_k^r} \leq C\lambda^{\varepsilon(r)+k}||f||_2, \ \lambda \geq 1, \tag{30}$$

holds and it is sharp. In particular, for a single eigenfunction $e_j(x)$ the following holds:

$$||e_j||_{H_k^r} \leq C\lambda_j^{\varepsilon(r)+k}, \ \lambda \geq 1,$$

which in general can not be improved in the sense of the following example.

Example 5.1. Let M^n be the unit n-sphere S^n of the Euclidean space $\mathbf{R}^{n+1}$. Let Z_m be the zonal harmonic function of degree m with respect to the north pole and Q_m the spherical harmonic defined by

$$Q_m(x) = (x_2 + ix_1)^m .$$

Then there exists a positive constant C independent of m such that the following inequalities hold:

$$||Z_m||_{H_k^r}/||Z_m||_2 \geq Cm^{\varepsilon(r)+k},\ 2(n+1)/(n-1) \leq r \leq \infty;$$

$$||Q_m||_{H_k^r}/||Q_m||_2 \geq Cm^{\varepsilon(r)+k},\ 2 \leq r \leq 2(n+1)/(n-1).$$

For the proof of Theorem 5.1, we cite a well known elliptic estimates as following

Proposition 5.1. *Let u be a smooth function on M, $1 \leq r < \infty$ and k a positive integer. Then the followings hold:*

$$||u||_{H_{2k}^r} \leq C\sum_{j=0}^{k} ||\Delta^j u||_r,\ ||u||_{H_{2k+1}^r} \leq C\sum_{j=0}^{k} ||\Delta^j u||_{H_1^r}, \tag{31}$$

where the constant C only depends on the metric g of M and k.

Let u be a real valued smooth function on the Riemannian manifold M. The gradient $\mathbf{grad}\, u$ of u is defined to be the dual vector field of one form $du = \nabla u$ by

$$g(\mathbf{grad}\, u, V) = du(V)$$

for arbitrary smooth vector field V on M. In the coordinate chart (X,x)

$$|\mathbf{grad}\, u| = |\nabla u| = \sum g^{jk}\partial_j u \partial_k u\ , \tag{32}$$

we define the L^p $(1 \leq p < \infty)$ norm of $\mathbf{grad}\, u$ as

$$||\mathbf{grad}\, u||_p = \left(\int_M |\mathbf{grad}\, u(x)|^p\, dv(M)\, dx\right)^{1/p} .$$

Then

$$||u||_{H_1^p} \approx ||u||_p + ||\mathbf{grad}\, u||_p\ ,$$

where $f \approx g$ means that there exists a positive constant C depending only on the metric g of M such that $g/C \leq f \leq Cg$. By Proposition 5.1, we have the following

Corollary 5.1. *Let u be a smooth function on M, $1 \leq r < \infty$ and k a positive integer. Then the following relations hold:*

$$||u||_{H_{2k}^r} \approx \sum_{j=0}^{k} ||\Delta^j u||_r,\ ||u||_{H_{2k+1}^r} \approx \sum_{j=0}^{k} (||\Delta^j u||_r + ||\mathbf{grad}\, \Delta^j u||_r), \tag{33}$$

where $f \approx g$ means that there exists a positive constant C depending on k, r and the metric g of M such that $g/C \leq f \leq Cg$.

PROOF OF THEOREM 5.1: By (17) we can let $2 \leq r < \infty$. By Corollary 5.1, we have only to prove the following estimates hold for $j = 0, 1, \cdots$:

$$||\Delta^j \chi_\lambda f||_r \leq C\lambda^{2j+\varepsilon(r)}||f||_2, \ ||\mathbf{grad}\,\Delta^j \chi_\lambda f||_r \leq C\lambda^{2j+1+\varepsilon(r)}||f||_2,$$

and they are sharp. By the duality, we need only to prove the estimates

$$||\Delta^j \chi_\lambda f||_2 \leq C\lambda^{2j+\varepsilon(r)}||f||_{r'}, \ ||\mathbf{grad}\,\Delta^j \chi_\lambda f||_2 \leq C\lambda^{2j+1+\varepsilon(r)}||f||_{r'} \tag{34}$$

hold for $r' = r/(r-1)$ and they are sharp. The dual version of Proposition ?? says that the following estimate holds and it is sharp:

$$||\chi_\lambda f||_2 \leq C\lambda^{\varepsilon(r)}||f||_{r'} . \tag{35}$$

The proof is completed by the following relations:

$$||\Delta^j \chi_\lambda f||_2 \approx \lambda^{2j}||\chi_\lambda f||_2, \ ||\mathbf{grad}\,\Delta^j \chi_\lambda f||_2 \approx \lambda^{2j+1}||\chi_\lambda f||_2 . \tag{36}$$

The first relations follows from

$$\Delta \chi_\lambda f = \sum_{\lambda_j \in (\lambda, \lambda+1]} \lambda_j^2 \mathrm{e_j}(f) .$$

The second one can be deduced from the equality

$$\int_M \mathbf{grad}\, e_j(x) \cdot \mathbf{grad}\, e_k(x) dv(M) = \delta_{jk}\lambda_j^2$$

derived by the Green's formula.

6 A remark on Dirichlet boundary value problem

Let N be a compact Riemannian manifold with smooth boundary ∂N. On N we consider the Dirichlet Laplacian Δ_N with respect to the Dirichlet boundary value problem

$$\Delta_N u = f, \ \ x \in N^\circ; \ u(x) = 0, \ \ x \in \partial N.$$

Let $\{e_j^N(x)\}_{j=1}^\infty$ be the real normalized eigenfunctions of Δ_N such that

$$\Delta_N e_j^N(x) = \mu_j^2 e_j^N(x), \ x \in N^\circ; \ e_j^N(x) = 0, \ x \in \partial N;$$

where $0 < \mu_1^2 \leq \mu_2^2 \leq \cdots$ are the eigenvalues of Δ_N. Similarly we can also define the unit spectral projection operator $\chi_{N,\lambda}$ associated to Δ_N. In particular, when N is a bounded region in $\mathbf{R}^n$, by studying the heat kernel of Δ_N, Ozawa [10] proved

$$\sum_{\mu_j \leq \lambda} \left| \frac{\partial e_j^N(x)}{\partial v} \right|^2 = \frac{\lambda^{n+2}}{(4\pi)^{n/2}\Gamma(2+\frac{n}{2})} + \mathrm{o}(\lambda^{n+2}), \text{ as } \lambda \to \infty, \tag{37}$$

for every $x \in \partial N$, where ν is the unit outward normal derivative at $x \in \partial N$. For the general Riemannian manifold N with boundary ∂N, Grieser [4] and Sogge [14] proved that the estimate (11) holds for $\chi_{N,\lambda}$, by which Xiangjin Xu [17] used a clever maximum principle argument to show the estimate

$$||\chi_{N,\lambda} f||_{C^1(N)} \leq C\lambda^{(n+1)/2}||f||_{L^2(N)} \,. \tag{38}$$

The results of Ozawa and Xiangjin Xu stimulated me to think of Theorem 5.1. We conclude the note with a problem on the spectral function of Dirichlet Laplacian.

Problem Can we show the analogy of Theorem 3.1 in the geodesic coordinate chart with respect to the submanifold ∂N in N? In particular, for integer $k \geq 0$ do there exist the corresponding nonnegative constants $C_{n,k}$ such that the following equalities

$$\sum_{\mu_j \leq \lambda} \left| \frac{\partial^k e_j^N(x)}{\partial \nu^k} \right|^2 = C_{n,k}\lambda^{n+2k} + O(\lambda^{n+2k-1}), \ \lambda \to \infty, \ x \in \partial N,$$

hold ?

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