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Robustness of Rational Expectations in Dynamic General Equilibrium Model

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Abstract

We consider a one sector dynamic general equilibrium model with possibility that a consumer does not know about a future economy. If a consumer updates his forecast by learning, we show that even a rough expectation can maintain stability of a steady state though a learning process does not necessarily leads to rational expectations.

1 Introduction

A one sector optimal growth model has been treated by Ramsey(1928), Koopmans(1965), Cass(1966) etc. They claimed that the movement of the capital accumulation path is monotone and the steady state is globally stable. That model is originally a descriptive model. The otimal paths of the model are, however, interpreted as paths of a dynamic general equilibrium model with many consumers and produsers(Becker(1980), Bewley(1982)). In such a model, economic agents maximize their objective functions over an infinite time horizen. It is usually assumed that they know the equilibrium prices of future markets infinitetly ahead. That concept is called the rational expectation. That has been often exposed to the criticizm that to know equilibrium prices before markets open means to know the shape of a demand function and a supply function, namely, economic agents know the other agents' preferences and production functions. In many cases we cannot, however, necessarily know the real economic model but know the economies that the various kinds of data depicit. Economic agents construct models based on the observed data. We expect that the more data we obtain the more precise approximate model we can get.

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In this paper, we demonstrate that even if we donot assume the rational expectation, the equiliprium capital path can converge to the steady state which is indentical to a rational expectation model and study economic validity that an optimization problem of infinite time horizon is solved by the Bellman Principle..

If we do not assume the rational expectation, there are two questions. One is what is available information for economic agents. In this paper we assume that agents only know their private information or history. Consemers can observe only past equilibrium prices. The second is how to get information about the economy. We assume that consumers update their forecast by learning based on all past equilibrium prices. We consider a capital path as a solution of learning process.

There are many literatures which point out that the dynamic behavior of macro-economic models depends crucially on the way the public is assumed to form expectations of future economic variables. They say that *myopic* perfect foresight generally misleads the public away from the long-run equilibrium of the model if the system is not initially on the stable manifold (Tobin(1965), Nagatani(1970), Ohyama(1989)), but truly rational public always discovers and follows the path leading to the long-run equilibrium under *long-run* perfect foresight (Sargent and Wallace(1973)). But these papers do not consider the dynamic optimization of a consumer or a firm. In the literature of optimal economic growth, Easley and Kiefer(1988) formulate a Bayesian learning process in stochastic economic growth model. In their setting, the social planner knows the shape of the reduced form utility function though he does not know the true probability measure about an exogenous stochastic process. The reduced form utility function includes information about consumers or firms. So we regard such a model as the rational expectaions model even if the probability measure is unknown. There seems to be no literature which studies stability of a steady state without the assumption of rational expectations in dynamic general equilibrium model considering the learnig process.

This paper organized as follows. Section 2 presents our model and defines an equilibium capital path. Section 3 shows our main results. In Section 4 we studies counterexamples in which the stability of the steady state failes.

2 The model

We consider the following problem that f is unknown.

$$\max_{c_t} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$

$$\text{subject to} \qquad c_t + k_{t+1} = f(k_t) \quad t \in \mathbb{N} \quad \text{given} \quad k_1$$

The above setting is equivalent to the following dynamic general equilibrium model. We consider a representative firm which produces single perishable good, and identical consumers(workers, capital stock holders).

Assumption1. A production function $F(K, L)$ is in $C^2(\mathbb{R}^2_+, \mathbb{R}_+)$ and homogenous of degree one with $f'(k) > 0, f''(k) < 0$, $f(0) = 0$, $\lim_{k \downarrow 0} f'(k) > 1/\beta$ and $\lim_{k \uparrow \infty} f'(k) = 0$ where $f(k) \equiv F(K/L, 1)$, $k \equiv K/L$, $\beta \in (0, 1)$, K and L are a discount factor, a capital stock and a labor respectively. A utility function u is in $C^2(\mathbb{R}_+, \mathbb{R}_+)$ and $u' > 0$, $u'' < 0$ and $\lim_{x \downarrow 0} u'(x) = \infty$.

A firm maximizes the following problem at t,

$$\Pi(\frac{w_t}{p_t}, \frac{r_t}{p_t}) \equiv \max_{K_t, L_t}[F(K_t, L_t) - \frac{w_t}{p_t} L_t - \frac{r_t}{p_t} K_t]$$

where p_t, w_t and r_t mean a price of a good, a wage rate and a nominal rental price at t.

For simplicity, a labor is supplied at $\bar{L}$ inelastically. A demand of K_t and L_t, denote K_t^d and L_t^d, are determined by

$$\frac{w_t}{p_t} = F_L(K_t^d, L_t^d), \qquad \frac{r_t}{p_t} = F_K(K_t^d, L_t^d)$$

for all t. We assume w_t/p_t is determined so that $L_t^d = \bar{L}$ for all t.

A consumer solves a following problem. Definitions of W and $\mathcal{V}_t$ are introduced later.

$$\max_{c_t, k_{t+1}^s} [u(c_t) + \beta \mathcal{V}_t(k_{t+1}^s)]$$

$$\text{subject to} \qquad c_t + k_{t+1}^s \leq W(k_t^s)$$

We assume r_t/p_t is determined as $K_t^d = K_t^s (= \bar{L} k_t^s)$.

Since K_t^s is determined at $t - 1$, it is an exogenous variable at t. Then w_t/p_t and r_t/p_t are determined by k_t^s. W represents an income or a wealth a consumer has at t. Define

$$W(k_t^s) = \frac{w_t}{p_t}(k_t^s) + \frac{r_t}{p_t}(k_t^s) k_t^s + \pi(\frac{w_t}{p_t}(k_t^s), \frac{r_t}{p_t}(k_t^s))$$

where

$$\pi(\frac{w_t}{p_t}(k_t^s), \frac{r_t}{p_t}(k_t^s)) = \Pi(\frac{w_t}{p_t}(k_t^s), \frac{r_t}{p_t}(k_t^s))/\bar{L}.$$

We call the W a wealth function. We have to remark that $W(k_t^s) = f(k_t^s)$ by a homogenity of F. But $\frac{w_t}{p_t}(k_t^s)$ and $\frac{r_t}{p_t}(k_t^s)$ are functions such that

$$k_t^s \mapsto K_t^s \mapsto (K^*, \frac{w_t}{p_t}, \frac{r_t}{p_t}).$$

The first map includes information about consumers and the second map infomation about firms. So those functions are never known to a consumer. A consumer expects a shape of $W(\cdot)$ based on the observed data. He only knows at t that $k_t^s \mapsto W(k_t^s)$, namely ,that one point of a real shape of the function $W(\cdot)$ (here the real shape function is $f(\cdot)$). So he constructs a function $W(\cdot)$ which passes $(k_\tau^s, f(k_\tau^s))$ $\tau = 1, 2, \cdots, t$ in his own way.
At $t+1$, a point $(k_{t+1}^s, W(k_{t+1}^s))$ is determined. He learns $t+1$ points of real shape of $W(\cdot)$. At this procedure, he gets more precise information about $f(\cdot)$ as time goes by. Let $k^* > 0$ and $k_H > 0$ be

$$f'(k^*) = \frac{1}{\beta}, \quad f(k_H) = k_H.$$

From assumption 1, the existence and uniqeness of such points is clear. In addition, we can see that $0 < k^* < k_H$.

The set of wealth functions that a consumer expects is

$$\Phi = \{W \in C^2([0,\xi],[0,\xi]) \mid W(0) = 0, W' \geq 0, -a \leq W'' \leq 0\}$$

where $a > 0$ is a uniform bound on this set. We assume $\xi > k_*$. Existence and uniqueness of this point is guaranteed by assumption1. $(\Phi, \|\cdot\|_{C^2})$ is a closed set of a seperable Banach space where $\|W\|_{C^2} = \max_{x\in[0,\xi]} |W(x)| + \max_{x\in[0,\xi]} |W'(x)| + \max_{x\in[0,\xi]} |W''(x)|$ which is called C^2 norm topology. We suppose that Φ is endowed with Borel σ-algebra $\mathcal{B}(\Phi)$, i,e., the σ-algebra generated by all open subsets of Φ. We define $V : \Phi \times [0,\xi] \to \mathbb{R}$ such as

$$V(W,x) = \max_{\{y_1,y_2,\cdots,\}} \sum_{t=0}^{\infty} \beta^t u(W(y_t) - y_{t+1})$$

where $y_0 = x$.

Definition. A path $\{k_t\}_{t=1}^{\infty}$ is feasible if there is a path $c_t \geq 0$ $t \in \mathbb{N}$ which satisfies $c_t + k_{t+1} \leq f(k_t)$ $t \in \mathbb{N}$.

Remark1. Let $\{k_t\}_{t=1}^{\infty}$ be a feasible path. If $k_1 \in [0, k_H]$, then $k_t \in [0, k_H]$ for all $t \in \mathbb{N}$.

Proposition1.
existence:There exists the unique optimal solution in $[0,\xi]^{\infty}$ with product topology of the problem,

$$\max_{\{k_2,k_3,\cdots\}} \sum_{t=1}^{\infty} \beta^{t-1} u(W(k_t) - k_{t+1})$$

for each initial condition $k_1 \in [0, \xi]$.
pointwise continuity:

$$\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$

is continuous on $(c_1, c_2, \cdots,) \in [0,\xi]^{\infty}$ with product topology.

Lemma 1. $V(\cdot, x)$ is $(\mathcal{B}(\Phi), \mathcal{B}(\mathbb{R}))$-measurable for each $x \in [0, \xi]$.

Proof; It suffices to prove the continuity of $V(\cdot, x)$ about $W \in \Phi$.
Let W_n, $W \in \Phi$ such that $W_n \longrightarrow W$ as $n \to \infty$ in $\|\cdot\|_{C^2}$. Take a $x \in [0,\xi]$ arbitrary and fix. Put

$$(y_1^n(x), y_2^n(x), \cdots y_i^n(x), \cdots)$$
$$= \arg\max_{y_1,y_2,\cdots}[u(W_n(x) - y_1) + \beta u(W_n(y_1) - y_2) + \beta^2 u(W_n(y_2) - y_3) + \cdots],$$

$$(y_1(x), y_2(x), \cdots y_i(x), \cdots)$$
$$= \arg\max_{y_1,y_2,\cdots}[u(W(x) - y_1) + \beta u(W(y_1) - y_2) + \beta^2 u(W(y_2) - y_3) + \cdots].$$

Since $y_1^n(x) \in [0,\xi]$ for all n, we may choose a subsequence of n, call it n_1, such that $y_1^{n_1}(x) \to y_1^*(x)$ as $n_1 \to \infty$. Since $y_2^{n_1}(x) \in [0,\xi]$ for all n_1, we may choose a subsequence of n_1, call it n_2, such that $y_2^{n_2}(x) \to y_2^*(x)$ as $n_2 \to \infty$. Then, for each $i \geq 3$, choose inductively a subsequence of n_{i-1}, call it n_i, such that $y_i^{n_i}(x) \to y_i^*(x)$ as $n_i \to \infty$. Choose ith number of n_i, denote n'. Then, by construction, $y_i^{n'}(x) \to y_i^*(x)$ as $n' \to \infty$ for all i. Namely $(y_1^{n'}(x), y_2^{n'}(x), \cdots, y_i^{n'}(x), \cdots) \longrightarrow (y_1^*(x), y_2^*(x), \cdots, y_i^*(x), \cdots)$ pointwise as $n' \to \infty$.

Because $W_{n'}$ converges to W uniformly and W is continuous,
$$\left|\{W_{n'}(y_i^{n'}(x)) - y_{i+1}^{n'}(x)\} - \{W(y_i^*(x)) - y_{i+1}^*(x)\}\right|$$
$$=\left|W_{n'}(y_i^{n'}(x) - W(y_i^*(x))\right| + \left|y_{i+1}^{n'}(x) - y_{i+1}^*(x)\right|$$

$\leq \left|W_{n'}(y_i^{n'}(x)) - W(y_i^{n'}(x))\right| + \left|W(y_i^{n'}(x)) - W(y_i^*(x))\right| + \left|y_{i+1}^{n'}(x) - y_{i+1}^*(x)\right|$
$\leq \left\|W_{n'} - W\right\|_{C^2} + \left|W(y_i^{n'}(x)) - W(y_i^*(x))\right| + \left|y_{i+1}^{n'}(x) - y_{i+1}^*(x)\right|$
$\longrightarrow 0$ as $n' \to \infty$ for all i. Since $\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$ is pointwise continuous and $\{W_{n'}(y_i^{n'}(x)) - y_{i+1}^{n'}(x)\} \longrightarrow \{W(y_i^*(x)) - y_{i+1}^*(x)\}$ from the above discussion, we obtain

$$u(W_{n'}(x)-y_1^{n'}(x))+\beta u(W_{n'}(y_1^{n'}(x))-y_2^{n'}(x))+\beta^2 u(W_{n'}(y_2^{n'}(x))-y_3^{n'}(x))+\cdots$$
$$\longrightarrow u(W(x)-y_1^*(x))+\beta u(W(y_1^*(x))-y_2^*(x))+\beta^2 u(W(y_2^*(x))-y_3^*(x))+\cdots$$

as $n' \to \infty$. $(*)$

In order to complete the proof, it suffices to demonstrate that $(y_1(x), y_2(x), \cdots) = (y_1^*(x), y_2^*(x), \cdots)$, namely,

$$u(W(x) - y_1^*(x)) + \beta u(W(y_1^*(x)) - y_2^*(x)) + \beta^2 u(W(y_2^*(x)) - y_3^*(x)) + \cdots$$
$$\geq u(W(x) - y_1) + \beta u(W(y_1) - y_2) + \beta^2 u(W(y_2) - y_3) + \cdots$$

for any feasible path $\{y_1, y_2, y_3 \cdots\}$.

Claim:For any feasible path $\{y_1, y_2, y_3, \cdots\}$, we can take a sequence of feasible path $\{y_1^n, y_2^n, y_3^n, \cdots\}$ such that $\{y_1^n y_2^n, y_3^n, \cdots\} \longrightarrow \{y_1, y_2, y_3, \cdots\}$ pointwise.

proof of Claim; Let $\{y_1, y_2, y_3, \cdots\}$ be a feasible path. Since $y_1 \leq W(x)$, we have

$$[0, W(x)] \cap (y_1 - 1, y_1 + 1) \neq \emptyset.$$

Because $W_n(x) \to W(x)$, there exists n_1 shch that

$$[0, W_n(x)] \cap (y_1 - 1, y_1 + 1) \neq \emptyset \qquad \text{for any } n \geq n_1.$$

Take $y_{n_1} \in [0, W_{n_1}(x)] \cap (y_1 - 1, y_1 + 1)$ arbitrary.
Since

$$[0, W(x)] \cap (y_1 - \frac{1}{2}, y_1 + \frac{1}{2}) \neq \emptyset,$$

there exists $n_2 > n_1$ such that

$$[0, W_n(x)] \cap (y_1 - \frac{1}{2}, y_1 + \frac{1}{2}) \neq \emptyset \qquad \text{for any } n \geq n_2.$$

Take $y_{n_2} \in [0, W_{n_2}(x)] \cap (y_1 - \frac{1}{2}, y_1 + \frac{1}{2})$ arbitrarily. Similarly we take $y_{n_k} \in [0, W_{n_k}(x)] \cap (y_1 - \frac{1}{k}, y_1 + \frac{1}{k})$ arbitrary for $n_{k+1} > n_k$ where $k \in \mathbb{N}$.
For $n < n_1$, take $y_n \in [0, W_n(x)]$ arbitrary. For $n_k < n < n_{k+1}$, choose $y_n \in [0, W_n(x)] \cap (y_1 - \frac{1}{k}, y_1 + \frac{1}{k})$ arbitrary. So by construction $\{y_1^n\}$ satisfies that $y_1^n \to y_1$ and $y_1^n \leq W_n(x)$.
Note (y_1, y_2) satisfies $y_2 \leq W(y_1)$. Because $W_n \to W$ uniformly and $y_1^n \to y_1$,

$$|W_n(y_1^n) - W(y_1)| \leq |W_n(y_1^n) - W(y_1^n)| + |W(y_1^n) - W(y_1)|$$
$$\leq \|W_n - W\|_\infty + |W(y_1^n) - W(y_1)| \to 0 \text{ as } n \to \infty.$$

Since $W_n(y_1^n) \to W(y_1)$, by the same discussion, we can take $\{y_2^n\}$ such that $y_2^n \to y_2$ and $y_2 \leq W_n(y_1^n)$. Similarly, there exists $\{y_i^n\}$ such that $y_i^n \to y_i$ and $y_i \leq W_n(y_{i-1}^n)$ for all i. We complete the proof of claim.

Let $\{y_1, y_2, y_3, \cdots\}$ be a feasible path. By the claim we can take a feasible path $\{y_1^n, y_2^n, y_3^n, \cdots\}$ such that $\{y_1^n, y_2^n, y_3^n, \cdots\} \to \{y_1, y_2, y_3, \cdots\}$. Since we have

$$u(W_{n'}(x)-y_1^{n'}(x))+\beta u(W_{n'}(y_1^{n'}(x))-y_2^{n'}(x))+\beta^2 u(W_{n'}(y_2^{n'}(x))-y_3^{n'}(x))+\cdots$$
$$\geq u(W_{n'}(x)-y_1^{n'})+\beta u(W_{n'}(y_1^{n'})-y_2^{n'})+\beta^2 u(W_{n'}(y_2^{n'})-y_3^{n'})+\cdots,$$

then from pointwise continuity of $\sum_{t=1}^{\infty}\beta^{t-1}u(c_t)$,

$$u(W(x)-y_1^*(x))+\beta u(W(y_1^*(x))-y_2^*(x))+\beta^2 u(W(y_2^*(x))-y_3^*(x))+\cdots$$
$$\geq u(W(x)-y_1)+\beta u(W(y_1)-y_2)+\beta^2 u(W(y_2)-y_3)+\cdots.$$

Because $\{y_1, y_2, y_3, \cdots\}$ is arbitrary, we obtain $(y_1(x), y_2(x), \cdots) = (y_1^*(x), y_2^*(x), \cdots)$. Therefore we get from $(*)$, $V(W_{n'}, x) \longrightarrow V(W, x)$ as $n' \to \infty$.

Suppose $V(W_n, x) \not\to V(W, x)$. Then there is a subsequence $\tilde{n}$ such that $\left|V(W_{\tilde{n}}, x) - V(W, x)\right| \geq \varepsilon$ for some $\varepsilon > 0$. In the same way, we can take a subsequence of $\tilde{n}$, call it $\tilde{\tilde{n}}$, such that $V(W_{\tilde{\tilde{n}}}, x) \to V(W, x)$. That is a contradiction. Then $V(W_n, x) \longrightarrow V(W, x)$. Since x is arbitrary, the proof is complete. Q.E.D.

Now we define $\mathcal{V}_t$. Let k_1 be an initial stock per capita. A consumer gets $f(k_1)$ at $t = 1$. Define

$$F_1 = \left\{W \in \Phi \;\middle|\; W \text{ wihch passes through} (k_1, f(k_1))\right\}.$$

Note that F_1 is a closed set. Let μ_1 be a subjective probability measure on $\mathcal{B}(\Phi)$ with $\mu_1(F_1) = 1$. We assume that this probability is commonly shared by all consumers. Then we caluculate a value which is generated by a stock for next period in a following way,

$$\mathcal{V}_1(y) = \int_{\Phi} V(W, y)\mu_1(dW).$$

Then a consumer solves a following problem at $t = 1$,

$$\max_{c_1, k_2^s}[u(c_1) + \beta\mathcal{V}_1(k_2^s)]$$

$$\text{subject to} \qquad c_1 + k_2^s \leq f(k_1^s).$$

So k_2 is determined by the above problem and a consumer gets $f(k_2)$ at $t = 2$. Define

$$F_2 = \left\{W \in \Phi \;\middle|\; W \text{which passes through} (k_1, f(k_1)), (k_2, f(k_2))\right\}.$$

Note that F_2 is a closed set. Let μ_2 be a subjective probability measure on

$\mathcal{B}(\Phi)$ with $\mu_2(F_2) = 1$. This probability measrure is commonly shared by all consumers.

So we caluculate a value which is generated by a stock for next period in a following way,

$$\mathcal{V}_2(y) = \int_{\Phi} V(W, y)\mu_2(dW).$$

Then a consumer solves a following problem at $t = 2$,

$$\max_{c_2, k_3^s}[u(c_2) + \beta\mathcal{V}_2(k_3^s)]$$

$$\text{subject to} \qquad c_2 + k_3^s \leq f(k_2^s).$$

So k_3 is determined by the above problem and a consumer gets $f(k_3)$ at $t = 3$. Define F_3 and μ_3 in same way and k_4 is determined by $\mathcal{V}_3$. In the same way, we define F_t, μ_t and $\mathcal{V}_t$ for $t \geq 4$. Let $F_\infty = \cap_{t=1}^{\infty} F_t$ and be μ_∞ a probability measure with $\mu_\infty(F_\infty) = 1$.

Definition. Let

$$g^t(x) = \arg\max_y [u(f(x) - y) + \beta\mathcal{V}_t(y)].$$

For $k_1 \in (0, k_H]$, define $k_t = g^{t-1}(g^{t-2}(\cdots(g^1(k_1))\cdots))$. We call the $\{k_t\}_{t=1}^{\infty}$ *an equilibrium capital path.*

3 Main Results

Lemma 2. $\mathcal{V}_t$ is differentiable for all $t \in \mathbb{N}$.

Choose $k \in (0, \xi)$ and $t \in \mathbb{N}$ arbitrarily.

$$(*) \lim_{h\to 0} \frac{\mathcal{V}_t(k+h) - \mathcal{V}_t(k)}{h}$$

$$= \lim_{h\to 0} \int_{\Phi} \frac{V(W, k+h) - V(W, k)}{h} \mu_t(dW).$$

Let $\bar{V} = u(\xi) + \beta u(\xi) + \beta^2 u(\xi) + \cdots = u(\xi)/(1-\beta)$. Because $V(W, \cdot)$ is nondecreasing and concave,

$$\left|\frac{V(W, k+h) - V(W, k)}{h}\right| \leq \frac{\bar{V}}{k} \quad \text{for all } h > 0, W \in \Phi.$$

Becouse μ_t is a finte measure, by the bounded convergence theorem,

$$(*) = \int_{\Phi} \lim_{h\to 0} \frac{V(W,k+h) - V(W,k)}{h} \mu_t(dW) \tag{1}$$

$$= \int_{\Phi} V_k(W,k)\mu_t(dW) \tag{2}$$

$$= \int_{\Phi} u'(W(k) - h(W)(k))W'(k)\mu_t(dW) \tag{3}$$

where $h(W)(x) = \arg\max_y[u(W(x) - y) + \beta V(W,y)]$. On the last equality, see Benveniste and Scheinkman(1979), Araujo(1991), Stokey and Lucas (1989)etc. Q.E.D.

Lemma 3. g^t is nondecreasing for all $t \in \mathbb{N}$.

Proof; The proof is essentially the same as Dechert and Nishimura(1983, Theorem 1). Q.E.D.

Lemma 4. Let $x \in (0,\xi]$ and $t \in \mathbb{N}$ satisfy $\text{ess.inf}_{W\in F_t} h(W)(x) < ess.\sup_{W\in F_t} h(W)(x)$. Then,

$$g^t(x) \in (\text{ess.}\inf_{W\in F_t} h(W)(x), \text{ess.}\sup_{W\in F_t} h(W)(x)).$$

Proof; Select $x \in (0,\xi]$ and $t \in (0,\xi]$ arbitrarily such that $\text{ess.inf}_{W\in F_t} h(W)(x) < ess.\sup_{W\in F_t} h(W)(x)$. We assume that $h(W)(x) \le g^t(x)$ for any $E \subset F_t$ such that $mu_t(E) > 0$. Because $\text{ess.inf}_{W\in F_t} h(W)(x) < ess.\sup_{W\in F_t} h(W)(x)$, for some $E' \subset F_t$ such that $\mu_t(E') > 0$ and some $W \in E'$, we have $h(W)(x) < g^t(x)$. By the assumption $\lim_{x\downarrow 0} u'(x) = \infty$, we see $h(W) \in (0,\xi)$ for all $W \in F_t$ (then $0 < g^t(x)$). Because $u(f(x) - \cdot) + \beta\mathcal{V}_t(\cdot)$ is differentiable,

$$u'(f(x) - g^t(x)) \begin{cases} = \beta \int_\Phi V_k(W, g^t(x))\mu_t(dW) & g^t(x) < \xi \\ \ge \beta \int_\Phi V_k(W, g^t(x))\mu_t(dW) & g^t(x) = \xi. \end{cases} \tag{4}$$

$u(f(x) - \cdot) + \beta V(W,\cdot)$, $W \in F_t$ is differentiable and strctly concave,

$$u'(f(x) - h(W)(x)) = \beta V_k(W, h(W)(x)), \quad W \in F_t$$

and

$$u'(f(x) - g^t(x)) \begin{cases} < \beta V_k(W, g^t(x)) & h(W)(x) < g^t(x) \\ = \beta V_k(W, g^t(x)) & h(W)(x) = g^t(x) < \xi, \quad W \in F_t. \end{cases} \tag{5}$$

Then

$$u'(f(x) - g^t(x)) < \beta \int_\Phi V_k(W, g^t(x))\mu_t(dW).$$

But this can not occur. If we assume that $h(W)(x) \geq g^t(x)$ for all $E \subset F_t$ such that $\mu_t(E) > 0$ and all $W \in E$, a contradction occurs in the same way. Q.E.D.

Lemma 5. Let $\{k_t\}_{t=1}^\infty$ be an equilibrium capital path. If there exists $\bar{t}$ such that $k_{\bar{t}} = k_{\bar{t}+1}$, then $k_t = k_{\bar{t}} > 0$ for all $t \geq \bar{t}$.

Proof; If there exists $\bar{t}$ such that $k_{\bar{t}} = k_{\bar{t}+1}$, consumers have same information at $\bar{t}$ and $\bar{t}+1$. Then $\mathcal{V}_{\bar{t}} = \mathcal{V}_{\bar{t}+1}$. So $k_{\bar{t}+1} = k_{\bar{t}+2}$. Then $\mathcal{V}_{\bar{t}+1} = \mathcal{V}_{\bar{t}+2}$. So $k_{\bar{t}+2} = k_{\bar{t}+3}$. By the same way, $k_t = k_{\bar{t}}$ for all $t \geq \bar{t}$. Because $W(x) > 0 (x > 0)$ for $W \in F_t$, $t \geq 1$, we see that $k_t > 0$ for $t \geq 1$.
Q.E.D.

Lemma 6. Let $\{k_t\}_{t=1}^\infty$ be an equilibrium capital path. If there is not $\bar{t}$ such that $k_{\bar{t}} = k_{\bar{t}+1}$, then $\{k_t\}_{t=1}^\infty$ consists of infinite different points.

Proof; Assume, on the contrary, $\{k_t\}_{t=1}^\infty$ consists of finite points. Let that number be N and write $\{k1, k2, \cdots, kN\}$. Let T be the first time such that $\{k1, k2, \cdots, kN\} \subset \{k_1, k_2, \cdots, k_T\}$. For $t \geq T$, consumers have same infomation. So $\mathcal{V}_t = \mathcal{V}_T$ for all $t \geq T$. Let $(t', t")$ be the first time after T such that $t' > t" \geq T$ and $k_{t'} = k_{t"}$. Since $\mathcal{V}_{t'} = \mathcal{V}_{t"}$, then $k_{t'+1} = k_{t"+1}$. Because of $\mathcal{V}_{t'+1} = \mathcal{V}_{t"+1}$, we have $k_{t'+2} = k_{t"+2}$. In the same way, $k_{t'+(t'-t")} = k_{t'}$. Therefore, after $t"$, the capital path describes a $t' - t"$ cycle. But because $g^t = g$ for all $t \geq t"$ and g is nondeceasing, it is impossible the cycles occur except a stationary point. Since we consider only the case that there is not $\bar{t}$ such that $k_{\bar{t}} = k_{\bar{t}+1}$, the capital path is not a stationary point at any time. Then $\{k_t\}_{t=1}^\infty$ consists of infinite different points. Q.E.D.

We put a following assumption about the μ_t. For an equilibrium capital path $\{k_t\}_{t=1}^\infty$, let there exists $t \in \mathbb{N}$ and $W \in F_t$ such that $W'(k_t) = 1/\beta$. if $\mu_t(\{W\}) = 1$, then $k_t = k_{t+1}$. We eliminate such a case.

Assumption 2. Let $\{k_t\}_{t=1}^\infty$ be an equilibrium capital path. Then, there is not $t \in \mathbb{N}$ such that $k_t = k_{t+1}$.

Lemma 7. Let k be an accumulation point of an equilibrium capital path

$\{k_t\}_{t=1}^{\infty}$[1]. Then $f'(k) = W'(k)$ for all $W \in F_\infty$. ($W \in F_\infty$ means that $f(k_t) = W(k_t)$ for all t.)

Proof; Let $\{t'\}$ be a subsequence of $\{t\}$ such that $k_{t'} \to k^*$ and define $h_{t'} = k_{t'} - k$. Because f and W are differentiable,

$$f'(k) = \lim_{h\to 0} \frac{f(k+h) - f(k)}{h} \tag{6}$$

$$= \lim_{t'\to\infty} \frac{f(k+h_{t'}) - f(k)}{h_{t'}} \tag{7}$$

$$= \lim_{t'\to\infty} \frac{W(k+h_{t'}) - W(k)}{h_{t'}} \tag{8}$$

$$= \lim_{h\to 0} \frac{W(k+h) - W(k)}{h} = W'(k) \tag{9}$$

Because $f(k_{t'}) = W(k_{t'})$ for all t', $f(k_{t'}) \to f(k)$ and $W(k_{t'}) \to W(k)$, then $f(k) = W(k)$. The third equality from that fact. Q.E.D.

Proposition 2. (Sokey and Lucas(1989) etc.) Let $t \in \mathbb{N}$ and $W \in F_t$. If $W'(x) < (>) 1/\beta$, then $h(W)(x) < (>) x$.

Let $\{k_t\}_{t=1}^{\infty}$ be an equilibrium path. Let π be a permutation such that $k_{\pi(1)} \leq k_{\pi(2)} \leq \cdots \leq k_{\pi(t)} \leq \cdots$ and define $k_{\pi(t)} = x_t$ for all $t \in \mathbb{N}$. Assume $x_1 < x_2 \leq k^*$ at some $T > 1$. Since f is strictly concave, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > \frac{1}{\beta}.$$

Let $t \geq T$ and $W \in F_t$. Because $W(x_1) = f(x_1)$、$W(x_2) = f(x_2)$, by concavity of $W \in F_t$ for $h(\neq 0)$ such that $x_1 + h < x_2$,

$$\frac{W(x_1+h) - W(x_1)}{h} \geq \frac{W(x_2) - W(x_1)}{x_2 - x_1} \tag{10}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{11}$$

Then $W'(x_1) > 1/\beta$. From Proposition 2, $h(W)(x_1) > x_1$ for all $W \in F_t$、$t \geq T$. So by Lemma 4, $g^t(x_1) > x_1$ for $t \geq T$. Therefore, because g^t is nondecreasing, x_1 is a lower bound of the equilibrium path. Note that because $W(x) > x (x > 0)$ for $W \in F_{T-1}$ and $\lim_{x\downarrow 0} u'(x) = \infty$, then $x_1 > 0$.

[1] k is an accumulation point of $\{k_t\}_{t=1}^{\infty}$ *if* for any $\varepsilon > 0$ and all t there exists $\bar{t} \geq t$ such that $k_{\bar{t}} \in (k - \varepsilon, k + \varepsilon) \setminus \{k\}$.

If $x_1 < x_2 < x_3 \leq k^*$, by the same discussion, x_2 is a lower bound of the equilibrium path. If $x_1 < x_2 < \cdots x_{n-1} < x_n \leq k^*$, then x_{n-1} is a lower bound. In the case of $k^* \leq x_{T-n} < x_{T-n-1} < \cdots < x_T$ for some $T > 1$, in the same way, x_{T-n-1} is an upper bound of the equilibrium path. Let $\{k_s\}$ be a sequence of lower bounds and $\{k_u\}$ be a sequence of upper bounds where $\{s\}$ and $\{u\}$ are supsequence of $\{t\}$. By the above discussion, $\{k_s\}$ is nondecreasing, and $\{k_u\}$ is nonincreasing. Put $\lim_{s\uparrow\infty} k_s = \underline{k}$, $\lim_{u\uparrow\infty} k_u = \bar{k}$. By assumption 2 and Lemma 6, $\{k_s\}$ or $\{k_u\}$ consisits of infinite different points. Then $\underline{k}(\leq k^*)$ or $\bar{k}(\geq k^*)$ is an accumulation point. Without loss of generality, let $\bar{k}$ be an accumulation point. Therefore the following lemma can be proved.

Lemma 8. $\bar{k} = k^*$.

Proof; Assume, on the contrary, $\bar{k} > k^*$. First we prove the following claim; For any $\varepsilon > 0$ there exists $u_0 \in \mathbb{N}$ such that;

$$|W_u'(\bar{k}) - \frac{W_u(k_u) - W_u(\bar{k})}{k_u - \bar{k}}| < \varepsilon, \quad u \geq u_0, W_u \in F_u. \qquad (\dagger)$$

Assume that there are some $\varepsilon > 0$, the subsequence of $\{u\}$(without loss of generality we write $\{u\}$) and $W_u \in F_u$,

$$|W_u'(\bar{k}) - \frac{W_u(k_u) - W_u(\bar{k})}{k_u - \bar{k}}| \geq \varepsilon.$$

By the concavity of W_u,

$$W_u'(\bar{k}) \geq \frac{W_u(k_u) - W_u(\bar{k})}{k_u - \bar{k}} \geq W_u'(k_u).$$

Then,

$$W_u'(\bar{k}) - W_u'(k_u) \qquad (12)$$

$$= W_u'(\bar{k}) - \frac{W_u(k_u) - W_u(\bar{k})}{k_u - \bar{k}} + \frac{W_u(k_u) - W_u(\bar{k})}{k_u - \bar{k}} - W_u'(k_u) \qquad (13)$$

$$\geq \varepsilon, \qquad u \geq u_0 \qquad (14)$$

Because $k_u \downarrow \bar{k}$,

$$\frac{W_u'(\bar{k}) - W_u'(k_u)}{\bar{k} - k_u} \downarrow -\infty \quad (u \uparrow \infty).$$

That is a contradiction to the definition of Φ. Then the claim($\dagger$) is proved. From Lemma 7, for any $\varepsilon > 0$ there exists $u_1 \in \mathbb{N}$ such that;

$$|f'(\bar{k}) - \frac{W_u(k_u) - W_u(\bar{k})}{k_u - \bar{k}}| < \varepsilon \quad u \geq u_1, W_u \in F_u. \tag{$\dagger\dagger$}$$

Therefore from ($\dagger$) and ($\dagger\dagger$) for any $\varepsilon > 0$ there exists $\bar{u} \in \mathbb{N}$ such that;

$$|W_u'(\bar{k}) - f'(\bar{k})| < \varepsilon, \quad u \geq \bar{u}, W_u \in F_u.$$

By $\bar{k} > k^*$, $f'(\bar{k}) < 1/\beta$. So there is $\bar{u}_0$ such that;

$$\sup_{u \geq \bar{u}_0} \sup_{W_u \in F_u} W_u'(\bar{k}) < \frac{1}{\beta}.$$

Since $F_t \subset F_{\bar{u}_0}$ for $t \geq \bar{u}_0$,

$$\sup_{t \geq \bar{u}_0} \sup_{W_t \in F_t} W_t'(\bar{k}) < \frac{1}{\beta}.$$

Then,

$$\sup_{t \geq \bar{u}_0} \sup_{W_t \in F_t} h(W_t)(\bar{k}) < \bar{k}.$$

By Lemma 4,

$$\sup_{t \geq \bar{u}_0} g^t(\bar{k}) < \bar{k}.$$

Since g^t is nondecreasing,

$$\sup_{t \geq \bar{u}_0} g^t(x) \leq \sup_{t \geq \bar{u}_0} g^t(\bar{k}) < \bar{k}, \quad x \leq \bar{k}.$$

If there is $t_1 \geq \bar{u}_0$ such that $k_{t_1} \leq \bar{k}$, then $g^{t_1}(k_{t_1}) < \bar{k}$, and $g^{t_1+1}(g^{t_1}(k_{t_1})) < \bar{k} \cdots$. So $k_t < \bar{k}$ for all $t \geq t_1$. But this is a contradiction to that $\bar{k}$ is an accumulation point of $\{k_u\}$. Then $\bar{k} < k_t$ for all $t \geq \bar{u}_0$. Therefore,

$$k_t \downarrow \bar{k} \ (t \uparrow \infty). \tag{$**$}$$

Note that for any $t \geq \bar{u}_0$ there exists $W_t \in F_t$ such that $\bar{k} \leq h(W_t)(k_t)$ and $h(W_t)(0) = 0$. the continuity Since $h(W_t)$ is continuous(The Berge maximum theorem) and nondecreasing, there exists $0 \leq y_t \leq k_t$ such that $\bar{k} = h(W_t)(y_t)$. Then $y_t \downarrow \bar{k}$. By the first condition, we see

$$u'(W_t(y_t) - h(W_t)(y_t)) = \beta u'(W_t(h(W_t)(y_t)) - h(W_t)(h(W_t)(y_t)))W_t'(h(W_t)(y_t)).$$

For sufficiently large $T \geq \bar{u}_0$, we have $W_t(\bar{k}) = W_t(h(W_t)(y_t)) < 1/\beta$ for $t \geq T$. Then for $t \geq T$,

$$u'(W_t(y_t) - h(W_t)(y_t)) < u'(W_t(h(W_t)(y_t)) - h(W_t)(h(W_t)(y_t)).$$

By the concavity of u, for $t \geq T$,

$$W_t(y_t) - h(W_t)(y_t) > W_t(\bar{k}) - h(W_t)(\bar{k}).$$

Put $\bar{k} - \sup_{t \geq \bar{u}_0} \sup_{W_t \in F_t} h(W_t)(\bar{k}) = B > 0$. Then,

$$W_t(y_t) - h(W_t)(y_t) \tag{15}$$
$$> W_t(\bar{k}) - h(W_t)(\bar{k}) \tag{16}$$
$$> W_t(\bar{k}) - \bar{k} + B \tag{17}$$

Since $W_t(y_t)$ and $W_t(\bar{k})$ converge to $f(\bar{k})$, $f(\bar{k}) - \bar{k} \geq f(\bar{k}) - \bar{k} + B$. That is a contradiction. So the proof is complete. Q.E.D.

From (**), $\bar{k}$ is not only the limit point of the subsequence but also of the equilibrium capital path itself. In the case that $\underline{k}$ is an accumulation point, by the same discussion, we can say $\underline{k} = k^*$. So we get the following.

Theorem 1. Let $\{k_t\}_{t=1}^{\infty}$ be an equilibrium capital path. Then $\lim_{t \uparrow \infty} k_t = k^*$.

This result states the relationship between the way of expectations and the stability of a steady state of a perfect foresight model. The quantity of information which consumers get plays an essential role for determination of the property of the dynamics.
Because μ_∞ is not equal to δ_f, the Dirac Measure concentrating at f, the limits of expectations are not rational expectations. Even the rough expectations, however, an equilibrium capital path can reach the steady state which is identical to that of rational expectations model.

4 Nondifferentiable, nonconvex case

In this section, we consider the case in which the expected wealth function W is nondifferentiable or nonconvex. we construct examples which a capital path converges to a point x^* where $f'(x^*) \neq 1/\beta$ even if a consumer learns infinitly many points.

Nondifferentiable case: Let $x^* \in (0, \xi)$ satisfy $f'(x^*) < 1/\beta < f(x^*)/x^*$. Put $a = f(x^*)/x^*$. The expected wealth function is the following.

$$W(x) = \begin{cases} ax & 0 \leq x \leq x^* \\ f(x) & x^* \leq x \leq \xi. \end{cases} \tag{18}$$

Note $\lim_{h\uparrow 0}(W(x^* + h) - W(x^*))/h = a \neq f'(x^*) = \lim_{h\downarrow 0}(W(x^* + h) - W(x^*))/h$. Because $1/\beta \in \partial W(x^*)$ where ∂ means subdifferential, we have $x^* = \arg\max_x[\beta W(x) - x]$. So the x^* is the unique stationary state of the problem; $\max\sum_{t=1}^{\infty}\beta^{t-1}u(W(k_t) - k_{t+1})$. If $x^* < k_1$, the capital path $\{k_t\}_{t=1}^{\infty}$ is $W(k_t) = f(k_t)$, $k_t \in (x^*, \xi)$ $\forall t \in \mathbb{N}$ and $k_t \downarrow x^*$. Then $W \in F_{\infty}$. But $f'(x^*) < 1/\beta$, which means that x^* is not an optimal steady state in the original rational expectations model.

Nonconvex case: We construct two type expected wealth functions, W_R and W_L, which are differentiable in following way. Take some interval $[a, b]$ where $0 < a < b < k_H$ such that $1/\beta > f'(x)$ for $x \in [a, b]$. Put $k_R > b$ so that it is the unique stationary state of the problem; $\max\sum_{t=1}^{\infty}\beta^{t-1}u(W_R(k_t) - k_{t+1})$, namely, $f(k_R)$ is sufficiently large and $f'(k_R) = 1/\beta$. Note that if $k_1 < b$, a capital path $\{k_t\}_{t=1}^{\infty}$ of the solution of $\max\sum_{t=1}^{\infty}\beta^{t-1}u(W_R(k_t) - k_{t+1})$, $k_t \uparrow k_R$. (Any bounded optimal path converges to a steady state. See Kamihigashi and Roy(2003)). Let $k_L < a$ be the unique stationary point of the problem; $\max\sum_{t=1}^{\infty}\beta^{t-1}u(W_L(k_t) - k_{t+1})$. Note that if $k_1 > a$ a capital path $\{k_t\}_{t=1}^{\infty}$ of the solution of $\max\sum_{t=1}^{\infty}\beta^{t-1}u(W_L(k_t) - k_{t+1})$, $k_t \downarrow k_L$. Put μ so that $k_t \in \arg\max_y[u(f(k_{t-1}) - y) + \int_{\Phi} V(W_R, y)\mu_t(dW_R) + \int_{\Phi} V(W_L, y)\mu_t(dW_L)]$ is in $[a, b]$, intuitively, a consumer thinks two possibilities that the best choice is heading upward to k_R or downward to k_L. Because the accumulation point of the capital path k^* is in $[a, b]$, we have $f'(k^*) \neq 1/\beta$.

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