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Indeterminacy of Buying and Selling Price Spread in Monetary Equilibrium without Transactions Costs

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Abstract

In this paper we report an existence result in a general monetary equilibrium model with buying and selling price spread but without transactions costs. Under the assumption that an initial endowment allocation is not Pareto optimal it is proved that an equilibrium with a positive value of money exists if traders take buying and selling prices of commodities as given even if transactions costs are not explicitly required in the buying and selling activities of traders in commodity markets. This result points to an indeterminacy problem arising from existing general equilibrium models with transaction technologies.

1 Introduction

Our purpose of the present paper is to point out an indeterminacy problem in a general monetary equilibrium model with buying and selling price spread and with or without transaction technologies. Under the assumption that an initial endowments allocation is not Pareto optimal it is shown in Yamazaki (2004) that an equilibrium with a positive value of money exists if traders take buying and selling prices of commodities as given *even if transactions costs are not explicitly required* in the buying and selling activities of transactions in commodity markets. This result gives an affirmative answer to a conjecture of Professor Duffie (1990) that his monetary equilibrium existence theorem does not seem to rely on the requirement of transactions costs.

When one introduces pure outside fiat money into a standard general equilibrium model of Arrow-Debreu-McKenzie-Nikaido (ADMN), the fact that an equilibrium exists in the original ADMN model implies that there always exists an equilibrium in which the value of money is zero. But, of course, unless money receives positive valuation in markets, it is impossible to play a basic role as a means of payment. This is the well-known "Hahn problem" in a general monetary equilibrium model pointed out by Hahn (1965). In a series of efforts to solve the problem in 1970's, traders are either simply forced to hold the positive amount of money or motivated to such holdings by required tax payments in the terminal period. (See, e.g., Starr (1974), Kurz (1974a, 1974b), Heller (1974), Okuno (1973), etc.) The reason for such a requirement is that fiat money has no apparent purchasing power in the last period so that traders have no incentives to hold money.

Duffie (1990) in his contribution to this problem has specifically taken up the problem of terminal value of money. In the setting of an ADMN model with outside fiat money and transactions costs expressed by individual transactions possibility sets as in Kurz (1974b) and Heller (1974), he showed that an equilibrium with the positive terminal value of money exists if traders take buying and selling prices in markets as given provided that they have incentives to trade (that is, the initial endowments allocation is not Pareto optimal). Traders are thought to face distinct buying and selling prices due to transactions costs as in a market transactions costs' model of Foley (1970). Duffie's basic idea is that accounting identity forces the total value of purchases to exceed the total value of sales by the amount of outside money if traders are to use the money balances in their transactions at the terminal period.

An equilibrium is composed of buying prices, selling prices, and preference-maximizing transactions in traders' budgets given these prices such that the total amount of each commodity (or money) bought does not exceed the total amount of each commodity (or money) sold. This equilibrium concept was introduced by Foley (1970) and Hahn (1971), and others such as Kurz (1974a, 1974b), Starrett (1973), Hayashi (1974), Okuno (1974), Duffie (1990), etc. worked with this definition. We employ an exchange economy version of a standard ADMN model as in Duffie (1990) but without explicitly introducing transactions costs in the form of individual transactions possibility sets as is done in his paper as well as in the works of Kurz (1974b) and Heller (1974).

The model describes the terminal period of a finite sequence economy. We work with the above equilibrium concept.

How should we interpret the implication of the result reported in this paper? One might wonder whether something went wrong in the setting of transactions costs models treating the above mentioned Hahn problem. There seem to be two possible interpretations. One favorable interpretation is to regard bid-ask spreads as representing endogenous indirect taxation. This interpretation may be of interest as writers such as Lerner (1947), Starr (1974), Kurz (1974b) and others thought that it is necessary to require that terminal money be used for tax purposes to force traders to demand the terminal money. The formulation of taxes by distinct buying and selling prices without transactions costs do endogenize this taxation. The second interpretation that seems more appropriate to us is that the equilibrium concept needs to be strengthened so that bid-ask spreads of buying and selling prices do satisfy an arbitrage-free condition defined in an appropriate way vis-à-vis explicitly introduced transactions costs. Or, it points to a need for reformulation of monetary models with transactions costs to explicitly take account of market price-making technology. It would resolve the indeterminacy problem of bid-ask spreads in our setup.

2 Model of a Monetary Economy with Buying and Selling Price Spread

$\mathbb{R}^\ell$ is the commodity space. Consumption sets are taken to be $\mathbb{R}^\ell_+$ for simplicity. Preference relations $\succ \subset \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$ are relatively open, irreflexive, locally nonsatiated, and convex (i.e., the set $\{z \in \mathbb{R}^\ell_+ \mid z \succ x\}$ is convex for each $x \in \mathbb{R}^\ell_+$).

There are m agents. Each agent $i \in \{1, \dots, m\}$ is characterized by a preference relation $\succ_i$, initial endowments $e_i \in \mathbb{R}^\ell_+$, and an endowment of money $M_i \geq 0$.

Buying and selling are separate transaction activities which command different price systems. All the prices are expressed in terms of the unit of account. $p^B \in \mathbb{R}^\ell$ and $p^S \in \mathbb{R}^\ell$ denote buying and selling prices respectively. Some of the prices may be negative. We express a pair composed of buying prices p^B and "reverse-signed" selling prices $-p^S$ by $p := (p^B, -p^S)$. p represents a vector of prices to be *paid* for a unit of purchases or sales of commodities in the market. A basic assumption is that each agent takes both buying and selling prices in markets as given and determines its transaction of commodities.

Means of payment are fiat money the amount $M = \sum_{i=1}^{m} M_i > 0$ of which is fixed exogenously. It is implicitly assumed that a public authority is set up for coordinating market transactions and collecting money for its service provided in the markets. The monetary unit is used as the unit of account.

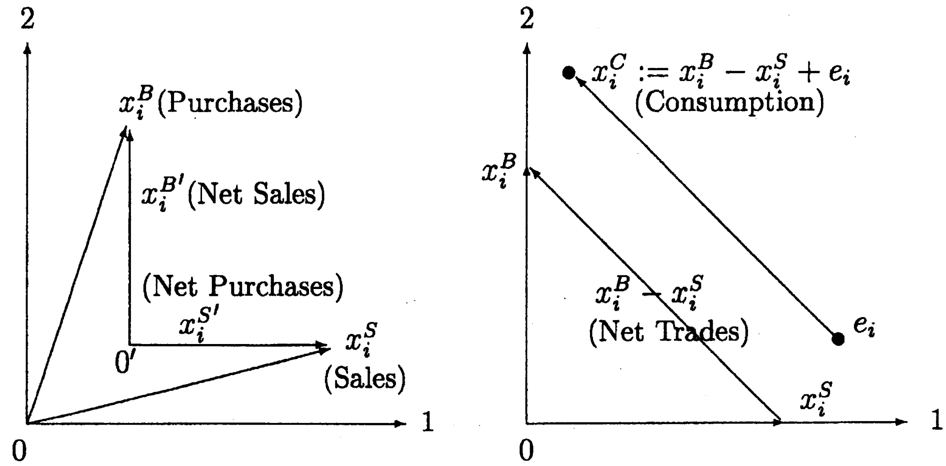

Figure 1: Purchases, Sales, Net Trades, and Consumption

Let x_i^B, x_i^S ($\in \mathbb{R}_+^\ell$) represent purchases and sales of ℓ commodities by agent i in the markets. Write $x_i := (x_i^B, x_i^S) \in \mathbb{R}_+^\ell \times \mathbb{R}_+^\ell$. The transaction of x_i by agent i leaves the consumption vector $x_i^C := x_i^B - x_i^S + e_i$ for i. (See Figure 1.) It is *budget feasible* for i at buying and selling prices $p = (p^B, -p^S) \in \mathbb{R}^{2\ell}$ if $x_i^C \geq 0$ and $p \cdot x_i \leq M_i$. When agent i engages in market transactions x_i, he receives $p^S \cdot x_i^S$ in money and spends $p^B \cdot x_i^B$ that should not exceed $p^S \cdot x_i^S + M_i$. Or, one can think of the above budget inequality as allowing netting between obligations and claims before the payment of money is made to the market authority; that is, $p \cdot x_i = p^B \cdot x_i^B - p^S \cdot x_i^S$ may be interpreted to represent netted obligations and claims. Figure 2 illustrates the budget feasibility in the transaction space[1] whereas Figure 3 illustrates the same budget feasibility in the commodity space which is more familiar. One may

[1]If u_i is a utility function for $\succ_i$, then the induced preferences of i in the transaction space is defined by the utility function U_i given by $U_i(x_i) = u_i(x_i^B - x_i^S + e_i)$ for $x_i = (x_i^B, x_i^S)$.

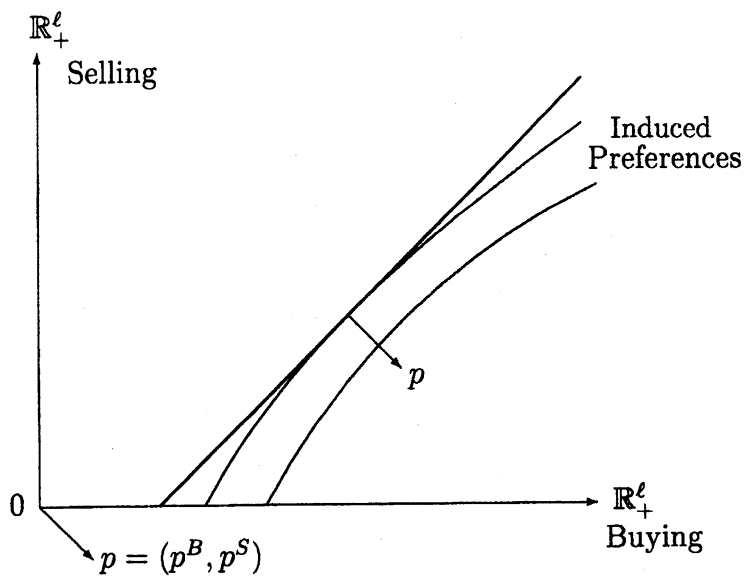

Figure 2: Budget Feasibility in the Transaction Space

note that initial endowments vector e_i lies in the interior of the budget feasible set which would force the value of money to zero in an equilibrium in a standard ADMN model.

A budget feasible transaction x_i is a *preference maximizer* for i provided $y_i^C \not\succ_i x_i^C$ for any budget feasible transaction y_i for i.

Allocations and Monetary Equilibrium

A m-tuple of transaction vectors $(x_1, \ldots, x_m)$ is a *transaction allocation* if $x_i \in \mathbb{R}^{2\ell}_{+}$ and $x_i^C \in \mathbb{R}^{\ell}_{+}$ for every $i = 1, \ldots, m$. $(x_1^C, \ldots, x_m^C)$ is called the associated *consumption allocation.* A transaction allocation is *feasible* if $\sum_{i=1}^m x_i^B = \sum_{i=1}^m x_i^S$. Note that the feasibility of a transaction allocation is equivalent to the feasibility condition $\sum_{i=1}^m x_i^C = \sum_{i=1}^m e_i$ of the associated consumption allocation $(x_1^C, \ldots, x_m^C)$, which is more familiar.

Given two transaction allocations $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$, $(x_1, \ldots, x_m)$ is said to *Pareto improve* $(y_1, \ldots, y_m)$ if $x_i^C \succ_i y_i^C$ for every agent i. (This requirement of improvement is stronger than the usual one unless preferences satisfy monotonicity.) A feasible transaction allocation $(x_1, \ldots, x_m)$ is *Pareto optimal* if no other feasible transaction allocations can Pareto improve it.

A *monetary equilibrium* for the economy $(\succ_i, e_i, M_i)_{i=1,\ldots,m}$ is a collection $((x_1, \ldots, x_m), p) \in (\mathbb{R}^{2\ell})^m \times \mathbb{R}^{2\ell}$ such that, given prices p^B, p^S, the transaction

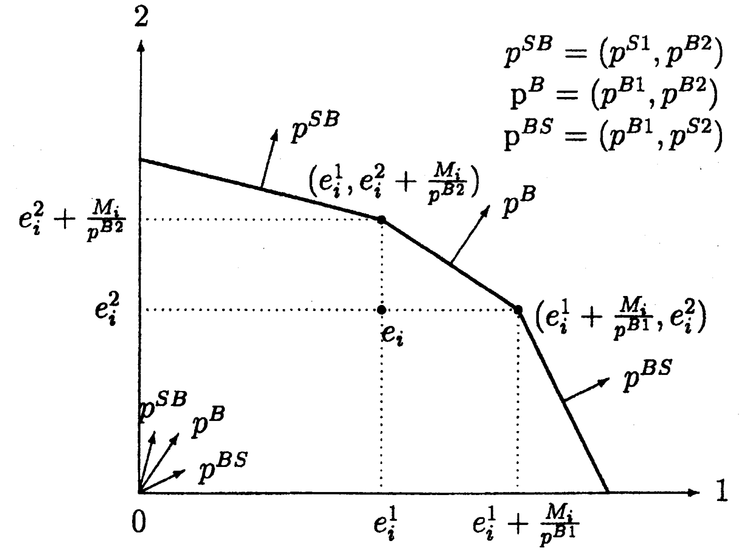

Figure 3: Budget Feasibility in the Commodity Space

x_i is budget feasible and a preference maximizer among budget feasible transactions for each $i \in \{1, \ldots, m\}$, and the transaction allocation $(x_1, \ldots, x_m)$ is feasible, i.e., $\sum_{i=1}^m x_i^B = \sum_{i=1}^m x_i^S$.

In this definition of a monetary equilibrium buying and selling prices are expressed in terms of the unit of account. Thus, whenever equilibrium prices $p \in \mathbb{R}^{2\ell}$ exist, the value of money must be positive. One may note that in seeking candidate equilibrium prices of commodities they cannot be constrained to lie in a compact subset of $\mathbb{R}^{2\ell}$ unless the value of money which might fall to zero is explicitly introduced. It is to be noted that the free disposability of commodities is not assumed so that some of the prices may be negative at equilibrium.

Let us give a statement of a property concerning the preference distribution of an economy which will be needed as a condition of the theorem below.

> [*Possibility of Individual Utility Enhancement for Feasible Allocations*] There is a positive number k^* such that for any given feasible consumption allocation $(x_1^C, \ldots, x_m^C)$, $\sum_{i=1}^m x_i^C = \sum_{i=1}^m e_i$, and for any commodity j, $(1 \leq j \leq \ell)$, there is some agent $i \in \{1, \ldots, m\}$ with a consumption vector $0 \leq y \leq k^* \sum_{i=1}^m e_i$ satisfying

(i) $y^h = x_i^{Ch}$ for all $h \neq j$,

(ii) $y \succ_i x_i^C$.

Take any one commodity j. If it is distributed freely among agents exhausting all of the resource, then, it should be the case that some one prefers to reduce or increase (within some uniform bound) his/her consumption of the commodity. If there is at least one agent who consumes and regards the commodity j to be desirable (i.e., a "good") or undesirable (i.e., a "bad") in all the range of feasible consumptions, this condition is automatically satisfied. Thus, it is immediate that if all the agents have monotone preferences, the above property is satisfied. Note that it does not rule out some of the commodities to be bads or some of the commodities to become undesirable beyond some levels of their consumption as long as all the agents do not reach their satiation level of a particular commodity simultaneously.

The only case of preference distribution which is ruled out by this condition is the following: There is a commodity j that is regarded desirable only up to some positive consumption levels by *all* agents, and the sum of the "satiation levels" of the commodity, beyond which an increase of its consumption by agents become undesirable just happens to be exactly equal to the total endowment of the commodity j in the economy. This condition is of a "generic" nature in the sense that even if a particular economy does not satisfy the condition, a slight perturbation of agents' preferences or total endowments will make the condition satisfied.

THEOREM (Yamazaki (2004)) *Let* $(\succ_i, e_i, M_i)_{i=1,\dots,m}$ *be an economy with* $\sum_{i=1}^m e_i > 0$ *and* $M = \sum_{i=1}^m M_i > 0$ *satisfying the property of the possibility of individual utility enhancement for feasible allocations. Then, there is a monetary equilibrium* $((x_1, \dots, x_m), p)$ *for the economy provided that the transaction allocation* $(0, \dots, 0)$ *inducing the initial endowments consumption allocation* $(e_1, \dots, e_m)$ *is not Pareto optimal.*

In Figure 4 an Edgeworth-Pareto box economy is illustrated and an equilibrium consumption allocation in a monetary equilibrium of this economy is given by the point E in the diagram.

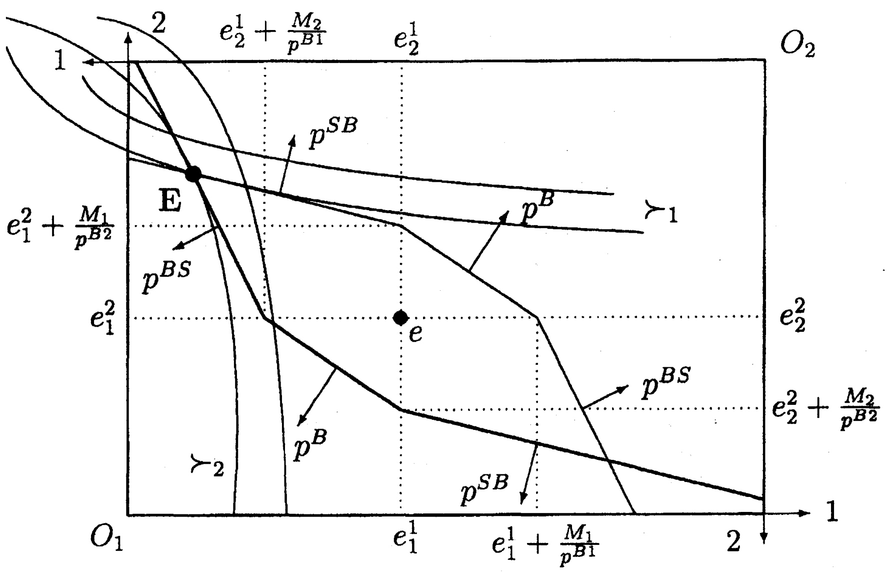

Figure 4: Monetary Equilibrium in an Edgeworth-Pareto Box Diagram

3 The Nature of Equilibrium and Implications

Basic Scenario of the Theorem

The formulation of the model in the previous section may be supported by a basic scenario described below. A market authority is organized and performs the operation of exchanging ℓ commodities for money at each point in time for a finite duration. The model delineates the terminal period only. The market authority plays the role of a "central banker" and collects money at the time of transactions. Fiat money need not be paper currency but could take the form of "electro-money" in the sense that it is composed of accounts held by traders at the central bank with debits and credits done by electronic devices. Overdrafts are not permitted and we require non-negative balances of money by traders.

Aside from the fact that money being the unit of account, a basic role of money in the model is transactional one at the terminal period. Although it is implicit, money also plays the role of a store of value carried over to the terminal period from the implicit previous time periods in the form of outside money balance.

The theorem asserts that if traders take buying and selling prices of commodities in markets as given, then there is a competitive market equilibrium with a positive monetary value even if no transaction costs are required provided that traders are better off with trades than without trades. This result confirms the conjecture of Duffie that his theorem establishing the existence of monetary equilibrium may obtain without transactions costs (see Duffie (1990, Theorem 1, p.87, and the second paragraph in p.92)).

Indeterminacy of Price Spreads

Let us briefly indicate the nature of buying and selling prices of commodities and the value of money at equilibrium that are given in the proof of the theorem. In our search for equilibrium buying and selling prices and the value of money we normalize buying prices p^B to the unit ball $\mathbb{B}^\ell$ in $\mathbb{R}^\ell$ and express selling prices p^S and value of money p^M relative to p^B. The concept of volume of trades is introduced as in Duffie (1990) to determine selling prices and the value of money. The volume of trades v associated with a transaction allocation $(x_1, \ldots, x_m) = \left((x_1^B, x_1^S) \ldots, (x_m^B, x_m^S)\right)$ is defined as the vector composed of the maximum number of total purchases or sales of each commodity in markets, that is,

$$v := \max\left\{\textstyle\sum_{i=1}^m x_i^B, \sum_{i=1}^m x_i^S\right\} \in \mathbb{R}_+^\ell$$

with the maximum taken coordinatewise. At equilibrium we have $\sum_{i=1}^m x_i^B = \sum_{i=1}^m x_i^S = v$.

Given the volume of trades $v \in \mathbb{R}_+^\ell$, selling prices p^S associated with buying prices $p^B \in \mathbb{B}^\ell$ are defined via bid-ask spread factor $\delta^j(p^B, v)$ given by

$$\delta^j(p^B, v) := \frac{sgn(p^{Bj})v^j}{1 + v^j}$$

for $j = 1, \ldots, \ell$. The bid-ask spread of commodity j is then $\delta^j(p^B, v)p^{Bj}$. In other words, the selling price p^{Sj} of commodity j is given by $(1 - \delta^j(p^B, v))p^{Bj}$. The absolute value of bid-ask spread factor of a commodity is strictly less than one, and it is monotonic in volumes of trades of that commodity. One might feel that the latter property is somewhat counter to what one might expect as a property of bid-ask spreads. Note, however, that the spirit of bid-ask spreads in the present model is that they originate in transactions costs which certainly accumulate as volumes of trades increase. After all a larger volume of trades conveys the importance or the higher value of money. But the higher the value of money is, the less one obtains from sales of given amounts of commodities, resulting in a bigger bid-ask spreads.

Bid-ask spreads are designed so that a positive buying price commands a positive spread and a negative price a negative spread. This means that for desired commodities buyers pay more than sellers receive, and for undesired commodities, in order to have them disposed, sellers pay more than buyers receive.

We note that the bid-ask spread factor $\delta^j(p^B, v)$ which is defined by $\delta^j(p^B, v) = sgn(p^{Bj})v^j/(1+v^j)$ is not the only functional form that would allow the existence of a monetary equilibrium. In this sense there is an indeterminacy problem of bid-ask spread factor which in turn implies the indeterminacy of selling prices and the value of money.

Now, agents will in general take advantage of bid-ask spreads unless their arbitrage transactions are unprofitable due to existing transactions costs. Therefore, the fact that the theorem is true seems to point to the need for strengthening the equilibrium concept adopted by us and others in the context of a general equilibrium model with transactions costs. It is implicitly assumed in the literature (see, e.g., Foley (1970) and Duffie (1990, p.90)) that transactions costs are severe enough to prevent arbitrage over bid-ask spreads. What need to be done thus seems to be to require explicitly bid-ask spreads of prices be arbitrage free in the sense that arbitrage transactions are not profitable vis-à-vis transactions costs incurred by such activity. It would lead to rethinking of the formulation of the model with individual transactions technologies.

Value of Money

In the existence proof of Yamazaki (2004), given buying prices p^B, selling prices p^S are determined according to bid-ask spreads given in the previous paragraph. The value p^M of money is set so that the value of existing stock of outside money is exactly equal to the total sum of the bid-ask spreads resulting from the volume v of trades in markets; that is,

$$p^M = \frac{\sum_{j=1}^{\ell} \delta^j(p^B, v)p^{Bj}v^j}{M}.$$

It means that agents use outside money to finance the excess of their payments over receipts in order to clear the results of their transactions with the market authority in commodity markets.

By the hypothesis of the theorem the initial endowments allocation is not Pareto optimal. Thus, agents do wish to engage in trades provided bid-ask spreads are sufficiently small. Bid-ask spreads introduced above have the property that they can be made arbitrarily small in neighborhoods of the initial endowments allocation, that is, in neighborhoods of zero volume of trades. It therefore follows that there will be a positive volume of trades at equilibria. By the way value of money is determined in our model, a positive volume of trades induces a positive terminal value of money.

Let $|p^B| := \left(|p^{B1}|, \ldots, |p^{B\ell}|\right)$. If each component of v is large, $p^M M$ is approximately equal to $|p^B| \cdot v$ or $p^M M \approx |p^B| \cdot v$. Thus, for and $t > 0$, we have

$$tp^M \approx \frac{|p^B| \cdot tv}{M}\ .$$

It shows that when each component of the volume of trades is large, the value of money is homogeneous of degree 1 in the volume of trades v.

Further Remarks

At this point some comments concerning bounds for feasible transaction allocations may be in order. In general equilibrium models with transaction technologies (mentioned earlier in the introduction), feasible transactions are bounded because transaction technologies do set bounds on feasible transactions by requiring transaction costs in terms of commodities. In our approach this way of setting bounds on transactions is not available. Thus, in principle, feasible consumptions may be bounded in our setting and yet transaction vectors may become arbitrarily large, creating a difficulty in an existence proof. The idea of our approach is that as long as buying prices differ from selling prices their spreads impose de facto transactions costs in buying and selling activities that force feasible transactions to be bounded. Whether spreads between buying and selling prices are non-zero or not depends upon two factors: one is the volume of trades and the other is whether buying and selling prices are non-zero. The non-zero volume of trades is insured by non-Pareto optimality of the initial endowment allocation as is well known. We introduced an assumption on preference distribution called

Possibility of Individual Utility Enhancement for Feasible Allocations. This assumption warrants strict positivity or strict negativity of prices.

Nevertheless, the theorem may at first seem counterintuitive. Since the model deals with the terminal period, it must be transaction demands that give rise to the positivity of monetary value. Market transactions of commodities must require costs if they are to motivate transactions demands for money. Thus, a natural question might arise: Why is it that one can establish the existence of monetary equilibrium without requiring transactions costs in an explicit way? The answer, however, seems to be straightforward. It is because we assumed that agents take buying and selling prices as given no matter how big bid-ask spreads are.

We would like to note that writers such as Lerner (1947), Hahn (1971), Starr (1974), Heller (1974), and others have either simply forced agents to demand fiat money at the terminal period or motivated such holdings requiring the payment of exogenously given lump sum taxes. In this context we could regard bid-ask spreads of prices in the present model as *endogenous indirect taxes* by the authority. Spread factor δ^j then represents a quantity tax levied on commodity j. One is thus led to feel that it is natural to have existence of a monetary equilibrium in a framework without explicit transactions costs. And a conceptual difficulty pointed out in the previous paragraph would not arise in this interpretation.

A final remark may be due. Our model does not suggest why fiat money may offer transactional advantages as a medium of exchange, but given a positive value for money the advantages are not difficult to imagine, considering for instance the work of Kiyotaki and Wright (1989), and Ostroy and Starr (1973). Instead the model presented here merely suggests why outside fiat money may have positive value in the first place even in a situation where explicit transactions costs do not exist.

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