On the $SL(3, \mathbb{R})$ action on 4-sphere

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Abstract
We construct the smooth $SL(3, \mathbb{R})$ actions on $S^4$. That is we solve the problem of F.Uchida ([6] and (P2) in [7]).

1 Introduction
In 1981 ([5]), F. Uchida gave the example of the orthogonal $SO(4)$-action on the 6-sphere $S^6$ which is not extendable to any continuous $SL(4, \mathbb{R})$-action. In 1985 ([6]), he studied the action $\psi$ of $SO(3)$ on the 4-sphere $S^4$ which coming from the adjoint action on the vector space $\text{sym}(3) \cong \mathbb{R}^5$, where $\text{sym}(3) = \{ A \in M_3(\mathbb{R}) : \text{trace}(A) = 0, A = A^t \}$ and $S^4 = \{ A \in \text{sym}(3) : \text{trace}(A^t A) = 1 \}$. Then he constructed a continuous $SL(3, \mathbb{R})$-action on $S^4$ which extends this $SO(3)$-action $\psi$. However this action is not smooth. It is still open whether this $SO(3)$-action can be extended to a smooth $SL(3, \mathbb{R})$-action or not. In this paper we construct the smooth $SL(3; \mathbb{R})$-action on $S^4$ which extends $\psi$.

2 Structure
First, remember the orbit structure of the $SO(3)$-action $\psi$ on $S^4$. The orbit space of this action is closed interval $[-\frac{1}{3\sqrt{6}}, \frac{1}{3\sqrt{6}}]$. Put the projection $\pi : S^4 \to [-\frac{1}{3\sqrt{6}}, \frac{1}{3\sqrt{6}}]$ which induced from a determinant of matrix. We can easily check $\pi^{-1}(\pm \frac{1}{3\sqrt{6}})$ are the singular orbits $\mathbb{R}P(2)$ and other orbits are the principal orbits $SO(3)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. 

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3 Construction

Let us construct the smooth action. Consider the natural smooth $SL(3, \mathbb{R})$-action on $CP(2) \cong \mathbb{C}^3 - \{0\}/\mathbb{C}^*$. Then the restricted $SO(3)$-action has just two singular orbits $S^2$ and $\mathbb{R}P(2)$ and the other orbits are principal orbit $SO(3)/\mathbb{Z}_2$. Moreover this action commute with complex conjugation. Remember the quotient space of $CP(2)$ by complex conjugation is $S^4$ ([2]). Hence this action induces the smooth $SL(3, \mathbb{R})$-action $\Psi$ on $S^4$ because of the commutativity $SL(3, \mathbb{R})$-action and the complex conjugation on $CP(2)$. Consider the restriction $\Psi$ to $SO(3)$. Then this restricted action has just two singular orbits $\mathbb{R}P(2)$ and the other orbits are principal orbit $SO(3)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. From the classification ([1]), the effective $SO(3)$-action on $S^4$ which has this orbit structure is unique up to equivariant diffeomorphic. So this restricted action is equivariant diffeomorphic to $\psi$. Therefor the $SL(3, \mathbb{R})$-action $\Psi$ is smooth and extended action of $\psi$.

4 Classification problem

Does $SO(3)$-action $\psi$ extend to a smooth $SL(3, \mathbb{R})$-action? This problem was solved in this paper. The answer was Yes. To solve this problem, we can consider the classification problem about a smooth $SL(3, \mathbb{R})$-action on $S^4$.

In 1974 ([3]), C. R. Schneider succeeded in the classification of the real analytic $SL(2, \mathbb{R})$-action on $S^2$. In 1979 ([4]), F. Uchida classified the real analytic $SL(n, \mathbb{R})$-action on $S^n$ for $n \geq 3$. Moreover, in 1981 ([5]), he succeeded in the classification of the real analytic $SL(n, \mathbb{R})$-action on $S^m$ for $5 \leq n \leq m \leq 2n - 2$. As is well known, $SL(n, \mathbb{R})$-action on $S^{n-1}$ is unique up to equivalence and $SL(n, \mathbb{R})$-action on $S^m$ for $m \leq n - 2$ is trivial. Therefore the case $(n, m) = (3, 4), (4, 5), (4, 6)$ and $m \geq 2n - 1$ (for $SL(n, \mathbb{R})$-action on $S^m$) are still open. The author would like to solve the case $(n, m) = (3, 4), (4, 5), (4, 6)$ near the future.

References


