ON THE SPACE OF STRINGS WITH PARTIALLY SUMMABLE LABELS

Shingo Okuyama
Takuma National College of Technology

Kazuhisa Shimakawa
Department of Mathematics, Okayama University

1. INTRODUCTION

In [3] we attached to any pair of a euclidean space $V$ and a partial abelian monoid $M$ a space $C(V, M)$ whose points are pairs $(c, a)$, where $c$ is a finite subset of $V$ and $a$ is a map $c \to M$, but $(c, a)$ is identified with $(c', a')$ if $c \subset c'$, $a'|c = a$, and $a'(v) = 0$ for $v \not\in c$. If $V$ is an orthogonal $G$-module and $M$ admits a $G$-action compatible with partial sum operations, where $G$ is a finite group, then $C(V, X)$ is a $G$-space with respect to the $G$-action $g(c, a) = (gc, gag^{-1})$, $g \in G$.

Let $I(\mathbb{R})$ be the partial abelian monoid consisting of bounded 1-dimensional submanifolds in the real line (see [1] and §3), and let

$$I(V, M) = C(V, I(\mathbb{R}) \wedge M)$$

for any partial abelian monoid with $G$-action $M$. The points of $I(V, M)$ can be represented by the pairs $(P, a)$, where $P$ is a finite union of parallel intervals,

$$\bigsqcup_{v \in c} \{v\} \times P(v) \subset V \times \mathbb{R} \quad (c \subset V, P(v) \in I(\mathbb{R}))$$

and $a$ is a map $c \to M$.

The purpose of this note is to show the following:

If $V$ is sufficiently large then there is a $G$-equivariant group completion map $C(V, M) \to I(V, M)$, and the correspondence $X \mapsto \{\pi_n I(V, X \wedge M); n \geq 0\}$ defines a $G$-equivariant generalized homology theory.

2. PARTIAL ABELIAN MONOIDS

Definition 1. A pointed $G$-space $M$ is called a partial abelian monoid with $G$-action, or $G$-partial monoid for short, if there are $G$-invariant subspaces $M_n$ of $M^n$ ($n \geq 0$) and $G$-equivariant maps, called partial sum operations,

$$M_n \to M, \quad (a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$$

satisfying the conditions below.

1. $M_0 = \{0\}$.
2. $M_1 \to M$ is the identity of $M$. 
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(3) Let $J_1, \cdots, J_r$ be pairwise disjoint subsets of $\{1, \ldots, n\}$ such that $\{1, \ldots, n\} = J_1 \cup \cdots \cup J_r$ holds. Let $(a_1, \ldots, a_n) \in M^n$ and suppose

$$\sum_{j \in J_k} a_j = a_{\lambda_k(1)} + \cdots + a_{\lambda_k(j_k)}$$

exists for each $k$, where $J_k = \{\lambda_k(1), \ldots, \lambda_k(j_k)\}$, $\lambda_k(1) < \cdots < \lambda_k(j_k)$. Then $(a_1, \ldots, a_n) \in M_n$ if and only if $(\sum_{j \in J_1} a_j, \ldots, \sum_{j \in J_r} a_j) \in M_r$ and we have

$$\sum_{1 \leq j \leq n} a_j = \sum_{j \in J_1} a_j + \cdots + \sum_{j \in J_r} a_j$$

whenever either the right or the left hand side sum exists.

Among the examples we have

- Let $A$ be a topological abelian monoid with $G$-action. Then any $G$-invariant subset $M$ of $A$ with $0 \in M$ can be regarded as an $G$-partial monoid by taking $M_n = \{(a_1, \ldots, a_n) \in M^n \mid a_1 + \cdots + a_n \in M\}$.
- Any pointed space $X$ is a $G$-partial monoid with respect to the trivial partial sum operations, i.e. folding maps $X_n = X \vee \cdots \vee X \to X$. In fact this is a special case of the previous example, as $X$ is a subset of the infinite symmetric product $\SP^\infty X$.
- Let $V$ be an infinite dimensional real inner product space on which $G$ acts through linear isometries. Then the Grassmannian $\Gr(V)$ of finite-dimensional subspaces of $V$ is a $G$-partial monoid with respect to the inner direct sum operations

$$\Gr(V)_n = \{(W_1, \ldots, W_n) \mid W_i \perp W_j, \ i \neq j \} \oplus \Gr(V)$$

**Definition 2.** For given $G$-partial monoids $M$ and $N$, the smash product $M \wedge N$ is a $G$-partial monoid whose partial sums are generated by the distributivity relations:

$$c_1 \wedge d + \cdots + c_k \wedge d = (c_1 + \cdots + c_k) \wedge d, \quad (c_1, \ldots, c_k) \in M_k$$

$$c \wedge d_1 + \cdots + c \wedge d_l = c \wedge (d_1 + \cdots + d_l), \quad (d_1, \ldots, d_l) \in N_l$$

**Example 3.** If $X$ is a pointed space then for any $G$-partial monoid $M$ we have

$$(X \wedge M)_n = X \wedge M_n, \quad n \geq 0$$

3. THE SPACE OF PARALLEL STRINGS WITH LABELS

If $J$ is a bounded interval in the real line $\mathbb{R}$ then its endpoint, say $a$, is called a closed endpoint if $a \in J$, and an open endpoint otherwise. Thus $J$ is a closed (resp. open) interval if its two endpoints are closed (resp. open), and is a half open interval if $J$ has a closed endpoint and an open endpoint.

Following [1], we denote by $I(\mathbb{R})$ the space of all bounded $1$-dimensional submanifolds of the real line $\mathbb{R}$, including the empty set. An element of $I(\mathbb{R})$ can be written as a union, say $P = J_1 \cup \cdots \cup J_r$, of finite number of pairwise disjoint bounded intervals. Here we may suppose $J_i < J_{i+1}$ holds for $1 \leq i < r$, that is to say, $x \in J_i$ and $y \in J_{i+1}$ yields $x < y$. But the union $J_i \cup J_{i+1}$ in this expression can be replaced by a single interval $J$ if $J_i \cup J_{i+1} = J$ is a connected interval, and
$J_i$ can be removed if $J_i$ is a half open interval of length 0. The latter means that half open intervals are collapsible to the empty set.

Let $I(\mathbb{R})_+$ be the subset of $I(\mathbb{R})$ consisting of those elements $J_1 \cup \cdots \cup J_r$ such that every $J_i$ is an closed interval. Then $I(\mathbb{R})$ is a partial abelian monoid with respect to the superimposition,

$$I(\mathbb{R})_n \to I(\mathbb{R}) \quad (P_1, \ldots, P_n) \mapsto P_1 \cup \cdots \cup P_n,$$

where $I(\mathbb{R})_n$ consists of those $(P_1, \ldots, P_n) \in I(\mathbb{R})^n$ such that $P_i \cap P_j = \emptyset$, $i \neq j$, and $I(\mathbb{R})_+$ is a partial submonoid of $I(\mathbb{R})$.

**Definition 4.** For an orthogonal $G$-module $V$ and a $G$-partial monoid $M$, we put

$$I(V, M) = C(V, I(\mathbb{R}) \wedge M), \quad I_+(V, M) = C(V, I(\mathbb{R})_+ \wedge M).$$

We call $I(V, M)$ the space of parallel strings in $V$ with labels in $M$.

To relate $I(V, M)$ with $C(V, M)$, we introduce the map $b: I(\mathbb{R})_+ \to C(\mathbb{R})$ which takes $J_1 \cup \cdots \cup J_r$ to the finite set $\{bJ_1, \ldots, bJ_r\}$ consisting of the barycenters of $J_i$. One easily observes that the natural map

$$I_+(V, M) \to C(V, C(\mathbb{R}) \wedge M)$$

induced by $b: I(\mathbb{R})_+ \to C(\mathbb{R})$, is a homotopy equivalence.

We also have

**Lemma 5.** If $V$ is sufficiently large then the inclusion

$$C(V, C(\mathbb{R}) \wedge M) \to C(V \times \mathbb{R}, M)$$

is a $G$-homotopy equivalence.

**Proof.** Let $i: \mathbb{R} \to V^G$ be an linear embedding, and define a homotopy $h: I \times \mathbb{R} \to V$ by $h(t, x) = (1 - t)i(x)$. If we write $h(t) = h(t, x)$ then $h_0 = i$ and $h_1$ is the constant map with value 0. Let $l$ be a $G$-linear isometry $V \times V \to V$. Then there is a homotopy

$$H: I_+ \wedge C(V \times \mathbb{R}, M) \to C(V \times \mathbb{R}, M)$$

such that $H_t = H(t, -)$ is induced by the composite

$$V \times \mathbb{R} \xrightarrow{1 \times \text{diag}} V \times \mathbb{R} \times \mathbb{R} \xrightarrow{1 \times h_1 \times 1} V \times V \times \mathbb{R} \xrightarrow{l \times 1} V \times \mathbb{R},$$

One easily observes that

1. $H$ restricts to a map $I_+ \wedge C(V, C(\mathbb{R}) \wedge M) \to C(V, C(\mathbb{R}) \wedge M)$,
2. $\text{Im} H_0 \subset C(V, C(\mathbb{R}) \wedge M)$, and
3. $H_1$ is induced by the linear isometry $l' \times 1: V \times \mathbb{R} \to V \times \mathbb{R}$, where $l'$ is the composite $V = V \times \{0\} \subset V \times V \xrightarrow{l'} V$.

Since the space of linear isometries of $V$ is contractible, $l'$ is equivariantly isotopic to the identity through $G$-linear isometries. Thus we have $H_0 \simeq H_1 \simeq 1$, hence $H_0$ induces a homotopy inverse to the inclusion $C(V, C(\mathbb{R}) \wedge M) \to C(V \times \mathbb{R}, M)$.

**Corollary 6.** If $V$ is sufficiently large then there is a map of Hopf $G$-spaces $\lambda: I_+(V, M) \to C(V, M)$, which is natural in $M$ and is a $G$-homotopy equivalence.
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Proof. Choose a $G$-linear isometry $l : V \times \mathbb{R} \cong V$, and define $\lambda$ as the composite
\[
I_+(V, M) \overset{\cong}{\rightarrow} C(V, C(\mathbb{R}) \wedge M) \overset{\subset}{\rightarrow} C(V \times \mathbb{R}, M) \overset{\iota}{\rightarrow} C(V, M)
\]

\[
\square
\]

4. MAIN RESULTS

To state the main results we introduce the $G$-category $\text{Top}(G)$ consisting of all pointed $G$-spaces and pointed maps, with $G$ acting on morphisms by conjugation. As we showed in [2], any $G$-equivariant continuous functor $T$ of $\text{Top}(G)$ into itself is accompanied with pairings
\[
X \wedge T Y \rightarrow T(X \wedge Y), \quad TX \wedge Y \rightarrow T(X \wedge Y)
\]
natural in both $X$ and $Y$. One easily observes from this that $T$ preserves $G$-homotopies, and there is a suspension natural transformation $T(X) \rightarrow \Omega^W T(\Sigma^W X)$ defined for any finite dimensional orthogonal $G$-module $W$.

Suppose $V$ is linearly and equivariantly isometric to the direct product of countably many copies of the regular representation of $G$ over the real number fields. Such a $G$-module $V$ is said to be sufficiently large. Then we have

Theorem 7. In the diagram of Hopf $G$-spaces
\[
C(V, M) \overset{\lambda}{\leftarrow} I_+(V, M) \overset{\rho}{\rightarrow} I(V, M),
\]
where $\rho$ is induced by the inclusion $I(\mathbb{R})_+ \subset I(\mathbb{R})$, we have
\begin{enumerate}
    \item $\lambda$ is a $G$-homotopy equivalence.
    \item $\rho$ is a $G$-equivariant group completion, that is to say, $\rho^H : I_+(V, M)^H \rightarrow I(V, M)^H$ is a group completion for every subgroup $H$ of $G$.
\end{enumerate}

Theorem 8. The correspondence $X \rightarrow I(V, X \wedge M)$ is a $G$-equivariant continuous functor of $\text{Top}(G)$ into itself and we have
\begin{enumerate}
    \item For any orthogonal $G$-module $W$ the suspension map $I(V, X \wedge M) \rightarrow \Omega^W I(V, \Sigma^W X \wedge M)$ is a weak $G$-homotopy equivalence.
    \item There is an $RO(G)$-graded $G$-homology theory $h^G_\ast(-)$ such that
    \[
h^G_n(X) = \pi_n I(V, X \wedge M)^G
    \]
    holds for any $X$ and $n \geq 0$.
\end{enumerate}

5. BRIEF OUTLINE OF THE PROOFS

Outline of the Proof of Theorem 7. We need to show that
\[
(5.1) \quad \rho^H : I_+(V, M)^H \rightarrow I(V, M)^H \text{ is a group completion}
\]
holds for every subgroup $H$ of $G$.

Observe that $V$ is an $H$-universe for any subgroup $H$ of $G$. Hence (5.1) for general $H$ follows from the special case $H = G$. that $\rho^H : I_+(V, M)^H \rightarrow I(V, M)^H$ is a group completion follows from the case $H = G$. But the usual argument using the notion of orbit type family enables us to reduce the proof of this problem to the case where $G$ is trivial. Thus we may assume $G = 1$ and $V = \mathbb{R}^\infty$. Recall from
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[3] that there is a weak equivalence of Hopf spaces \( \Phi: D(M) \to C(\mathbb{R}^\infty, M) \), where \( D(M) \) is the realization of the diagonal

\[ [k] \mapsto S_k N_k \mathcal{Q}(M) = N_k \mathcal{Q}(S_k M) \]

associated with the bisimplicial set of the total singular complex of the nerve of a permutative category \( \mathcal{Q}(M) \), whose space of objects is \( \coprod_{p \geq 0} M^p \), and whose morphisms from \( (a_i) \in M^p \) to \( (b_j) \in M^q \) are maps of finite sets \( \theta: \{1, \ldots, p\} \to \{1, \ldots, q\} \) such that \( b_j = \sum_{i \in \theta^{-1}(j)} a_i \) holds for \( 1 \leq j \leq q \).

Since \( \Phi \) is natural in \( M \), Theorem 7 follows from

**Proposition 9.** The natural map \( D(I_+^\times \mathbb{R}) \wedge M \to D(I(\mathbb{R}) \wedge M) \), induced by the inclusion \( I_+^\times \mathbb{R} \subset I(\mathbb{R}) \), is a group completion.

The rest of this section is devoted to the proof of this proposition.

Given a map of topological monoids \( f: D \to D' \) let \( B(D, D') \) denote the realization of the category \( B(D, D') \) whose space of objects is \( D' \) and whose space of morphisms is the product \( D \times D' \), where \( (d, d') \in D \times D' \) is regarded as a morphism from \( d' \) to \( f(d) \cdot d' \). Then there is a sequence of maps

\[ D' = B(0, D') \to B(D, D') \to B(D, 0) = BD \]

induced by the maps \( 0 \to D \) and \( D' \to 0 \) respectively. Observe that \( BD \) is the standard classifying space of the monoid \( D \) and \( B(D, D) \) is contractible when \( f \) is the identity.

Let \( D = D(I_+^\times \mathbb{R}) \wedge M \), \( D' = D(I(\mathbb{R}) \wedge M) \), and let \( i: D \to D' \) be the map induced by the inclusion \( I_+^\times \mathbb{R} \to I(\mathbb{R}) \). Then we have a commutative diagram

\[
\begin{array}{ccc}
D & \longrightarrow & B(D,D) \\
\downarrow i & & \downarrow B(1,i) \\
D' & \longrightarrow & B(D,D')
\end{array}
\]

in which the upper and the lower sequences are associated with the identity and the inclusion \( i: D \to D' \), respectively.

**Lemma 10.** The natural map \( D \to \Omega BD \) is a group completion.

This follows from the fact that \( D \) is a homotopy commutative, hence admissible, monoid.

**Lemma 11.** The lower sequence in the diagram (5.2) is a homotopy fibration sequence with contractible total space \( B(D,D') \).

Proposition 9 can be deduced from these lemmas, because \( D \to D' \) is equivalent to the group completion map \( D \to \Omega BD \) under the equivalence \( D' \simeq \Omega BD \).

**Outline of the Proof of Theorem 8.** We need the following

**Proposition 12.** Let \( T \) be a \( G \)-equivariant continuous functor of the category of pointed \( G \)-spaces and pointed maps to itself. Suppose \( T \) satisfies the following conditions.
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C1: \( T^* = * \)
C2: \( \|T(X_\bullet)\| \cong_G T(\|X_\bullet\|) \) for any simplicial objects \( X_\bullet \)
C3: \( T(X \cup Y) \cong_G TX \times TY \)
C4: \( T(G/H_+ \wedge X) \cong_G \text{Map}(G/H, TX) \) for any subgroup \( H \).

Suppose further that \( TX^H \) is grouplike for any \( X \) and any subgroup \( H \) of \( G \).
Then we have

(a) \( TX \cong_G \Omega^W T(\Sigma^W X) \) for any real \( G \)-module \( W \),
(b) \( X \mapsto \{ \pi_n TX^G \} \) defines an \( RO(G) \)-gradable equivariant homotopy theory on the category of pointed \( G \)-spaces.

Proof. As \( TX^H \) is grouplike for any \( H \) the natural map \( TX \to \Omega T(\Sigma X) \) is a weak \( G \)-equivalence. Hence by the argument similar to the proof of [3, Theorem 2.12]
\[
TA \to TX \to T(X \cup CA)
\]
is a \( G \)-fibration sequence up to weak \( G \)-equivalence for any pair of pointed \( G \)-spaces \((X, A)\). It also follows by the property of \( G \)-equivariant continuous functor that \( T \) preserves \( G \)-homotopies. Thus (a) implies (b).

To prove (a) we need only show that the correspondence \( S \mapsto T(S \wedge X) \), where \( S \) is any pointed finite \( G \)-set, defines a special \( \Gamma_G \)-space in the sense of [2], that is, the natural map
\[
T(S \wedge X) \to \text{Map}_0(S, TX)
\]
is a \( G \)-equivalence. But this follows from the conditions C3 and C4.

Let \( TX = I(V, X \wedge M) \). We shall show that \( T \) satisfies the conditions C1, C2, C3 and C4. This of course implies Theorem 8.

Clearly the condition C1 holds, and C2 is a routine exercise. That C3 holds is proved as follows.

To verify C4 we shall show that \( T(G/H_+ \wedge X) \to \text{Map}(G/H, TX) \) has a \( G \)-homotopy inverse \( \rho \) defined as follows:

1. Choose a \( G \)-embedding \( G/H \to V \) and a linear \( G \)-isometry \( V \times V \to V \).
2. For given \( f : G/H \to TX \) write \( f(gH) = (c(gH), a(gH)) \), where \( c(gH) \subset V \), \( a(gH) : c(gH) \to X \wedge M \wedge I(\mathbb{R}) \).
3. Define \( \tilde{\epsilon} \) to be the image of \( \bigcup \{gH\} \times c(gH) \) under the embedding
\[
\iota : G/H \times V \subset V \times V \to V
\]
4. Define \( \tilde{\alpha} : \tilde{\epsilon} \to G/H_+ \wedge X \wedge I(\mathbb{R}) \wedge M \) by
\[
\tilde{\alpha}(\iota(gH, \xi)) = gH \wedge a(gH)(\xi), \quad \xi \in c(gH)
\]
5. \( \rho : \text{Map}(G/H, TX) \to T(G/H_+ \wedge X) \) is given by \( \rho(f) = (\tilde{\epsilon}, \tilde{\alpha}) \).

REFERENCES