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ON THE SPACE OF STRINGS WITH PARTIALLY SUMMABLE LABELS

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1. Introduction

In [3] we attached to any pair of a euclidean space V and a partial abelian monoid M a space $C(V, M)$ whose points are pairs (c, a), where c is a finite subset of V and a is a map $c\to M$, but (c, a) is identified with (c', a') if $c\subset c'$, $a'|c=a$, and $a'(v)=0$ for $v\notin c$. If V is an orthogonal G-module and M admits a G-action compatible with partial sum operations, where G is a finite group, then $C(V, X)$ is a G-space with respect to the G-action $g(c, a)=(gc, gag^{-1})$, $g\in G$.

Let $I(\mathbb{R})$ be the partial abelian monoid consisting of bounded 1-dimensional submanifolds in the real line (see [1] and §3), and let

$$I(V, M)=C(V, I(\mathbb{R})\wedge M)$$

for any partial abelian monoid with G-action M. The points of $I(V, M)$ can be represented by the pairs (P, a), where P is a finite union of parallel intervals,

$$\textstyle\coprod_{v\in c}\{v\}\times P(v)\subset V\times \mathbb{R}\quad (c\subset V,\ P(v)\in I(\mathbb{R}))$$

and a is a map $c\to M$.

The purpose of this note is to show the following:

If V is sufficiently large then there is a G-equivariant group completion map $C(V, M)\to I(V, M)$, and the correspondence $X\mapsto\{\pi_{n}I(V, X\wedge M);n\geq 0\}$ defines a G-equivariant generalized homology theory.

2. Partial Abelian Monoids

Definition 1. A pointed G-space M is called a partial abelian monoid with G-action, or *G-partial monoid* for short, if there are G-invariant subspaces M_{n} of M^{n} $(n\geq 0)$ and G-equivariant maps, called *partial sum operations*,

$$M_{n}\to M,\quad (a_{1}, \ldots, a_{n})\mapsto a_{1}+\cdots+a_{n}$$

satisfying the conditions below.

(1) $M_{0}=\{0\}$.
(2) $M_{1}\to M$ is the identity of M.

(3) Let $J_1, \cdots, J_r$ be pairwise disjoint subsets of $\{1,\dots,n\}$ such that $\{1,\dots,n\} = J_1 \cup \cdots \cup J_r$ holds. Let $(a_1,\dots,a_n) \in M^n$ and suppose
$$\textstyle\sum_{j\in J_k} a_j = a_{\lambda_k(1)} + \cdots + a_{\lambda_k(j_k)}$$
exists for each k, where $J_k = \{\lambda_k(1),\dots,\lambda_k(j_k)\}$, $\lambda_k(1) < \cdots < \lambda_k(j_k)$. Then $(a_1,\dots,a_n) \in M_n$ if and only if $\left(\sum_{j\in J_1} a_j,\dots,\sum_{j\in J_r} a_j\right) \in M_r$ and we have
$$\textstyle\sum_{1\le j\le n} a_j = \sum_{j\in J_1} a_j + \cdots + \sum_{j\in J_r} a_j$$
whenever either the right or the left hand side sum exists.

Among the examples we have

- Let A be a topological abelian monoid with G-action. Then any G-invariant subset M of A with $0 \in M$ can be regarded as an G-partial monoid by taking $M_n = \{(a_1,\dots,a_n) \in M^n \mid a_1 + \cdots + a_n \in M\}$.
- Any pointed space X is a G-partial monoid with respect to the trivial partial sum operations, i.e. folding maps $X_n = X \vee \cdots \vee X \to X$. In fact this is a special case of the previous example, as X is a subset of the infinite symmetric product $\mathrm{SP}^\infty X$.
- Let V be an infinite dimensional real inner product space on which G acts through linear isometries. Then the Grassmannian $\mathrm{Gr}(V)$ of finite-dimensional subspaces of V is a G-partial monoid with respect to the inner direct sum operations
$$\mathrm{Gr}(V)_n = \{(W_1,\dots,W_n) \mid W_i \perp W_j,\ i \ne j\} \xrightarrow{\oplus} \mathrm{Gr}(V)$$

Definition 2. For given G-partial monoids M and N, the smash product $M \wedge N$ is a G-partial monoid whose partial sums are generated by the distributivity relations:
$$c_1 \wedge d + \cdots + c_k \wedge d = (c_1 + \cdots + c_k) \wedge d, \qquad (c_1,\dots,c_k) \in M_k$$
$$c \wedge d_1 + \cdots + c \wedge d_l = c \wedge (d_1 + \cdots + d_l), \qquad (d_1,\dots,d_l) \in N_l$$

Example 3. If X is a pointed space then for any G-partial monoid M we have
$$(X \wedge M)_n = X \wedge M_n, \quad n \ge 0$$

3. The Space of Parallel Strings with Labels

If J is a bounded interval in the real line $\mathbb{R}$ then its endpoint, say a, is called a closed endpoint if $a \in J$, and an open endpoint otherwise. Thus J is a closed (resp. open) interval if its two endpoints are closed (resp. open), and is a half open interval if J has a closed endpoint and an open endpoint.

Following [1], we denote by $I(\mathbb{R})$ the space of all bounded 1-dimensional submanifolds of the real line $\mathbb{R}$, including the empty set. An element of $I(\mathbb{R})$ can be written as a union, say $P = J_1 \cup \cdots \cup J_r$, of finite number of pairwise disjoint bounded intervals. Here we may suppose $J_i < J_{i+1}$ holds for $1 \le i < r$, that is to say, $x \in J_i$ and $y \in J_{i+1}$ yields $x < y$. But the union $J_i \cup J_{i+1}$ in this expression can be replaced by a single interval J if $J_i \cup J_{i+1} = J$ is a connected interval, and

J_i can be removed if J_i is a half open interval of length 0. The latter means that half open intervals are collapsible to the empty set.

Let $I(\mathbb{R})_+$ be the subset of $I(\mathbb{R})$ consisting of those elements $J_1 \cup \cdots \cup J_r$ such that every J_i is an closed interval. Then $I(\mathbb{R})$ is a partial abelian monoid with respect to the *superimposition*,

$$I(\mathbb{R})_n \to I(\mathbb{R}) \quad (P_1, \dots, P_n) \mapsto P_1 \cup \cdots \cup P_n,$$

where $I(\mathbb{R})_n$ consists of those $(P_1, \dots, P_n) \in I(\mathbb{R})^n$ such that $P_i \cap P_j = \emptyset$, $i \neq j$, and $I(\mathbb{R})_+$ is a partial submonoid of $I(\mathbb{R})$.

Definition 4. For an orthogonal G-module V and a G-partial monoid M, we put

$$I(V, M) = C(V, I(\mathbb{R}) \wedge M), \quad I_+(V, M) = C(V, I(\mathbb{R})_+ \wedge M)$$

We call $I(V, M)$ *the space of parallel strings in V with labels in M*.

To relate $I(V, M)$ with $C(V, M)$, we introduce the map $b\colon I(\mathbb{R})_+ \to C(\mathbb{R})$ which takes $J_1 \cup \cdots \cup J_r$ to the finite set $\{bJ_1, \dots, bJ_r\}$ consisting of the barycenters of J_i. One easily observes that the natural map

$$I_+(V, M) \to C(V, C(\mathbb{R}) \wedge M)$$

induced by $b\colon I(\mathbb{R})_+ \to C(\mathbb{R})$, is a homotopy equivalence.

We also have

Lemma 5. *If V is sufficiently large then the inclusion*

$$C(V, C(\mathbb{R}) \wedge M) \to C(V \times \mathbb{R}, M)$$

is a G-homotopy equivalence.

Proof. Let $i\colon \mathbb{R} \to V^G$ be an linear embedding, and define a homotopy $h\colon I \times \mathbb{R} \to V$ by $h(t, x) = (1 - t)i(x)$. If we write $h_t(x) = h(t, x)$ then $h_0 = i$ and h_1 is the constant map with value 0. Let l be a G-linear isometry $V \times V \to V$. Then there is a homotopy

$$H\colon I_+ \wedge C(V \times \mathbb{R}, M) \to C(V \times \mathbb{R}, M)$$

such that $H_t = H(t, -)$ is induced by the composite

$$V \times \mathbb{R} \xrightarrow{1\times\text{diag.}} V \times \mathbb{R} \times \mathbb{R} \xrightarrow{1\times h_t \times 1} V \times V \times \mathbb{R} \xrightarrow{l\times 1} V \times \mathbb{R},$$

One easily observes that

(1) H restricts to a map $I_+ \wedge C(V, C(\mathbb{R}) \wedge M) \to C(V, C(\mathbb{R}) \wedge M)$,
(2) $\operatorname{Im} H_0 \subset C(V, C(\mathbb{R}) \wedge M)$, and
(3) H_1 is induced by the linear isometry $l' \times 1\colon V \times \mathbb{R} \to V \times \mathbb{R}$, where l' is the composite $V = V \times \{0\} \subset V \times V \xrightarrow{l} V$.

Since the space of linear isometries of V is contractible, l' is equivariantly isotopic to the identity through G-linear isometries. Thus we have $H_0 \simeq H_1 \simeq 1$, hence H_0 induces a homotopy inverse to the inclusion $C(V, C(\mathbb{R}) \wedge M) \to C(V \times \mathbb{R}, M)$. $\square$

Corollary 6. *If V is sufficiently large then there is a map of Hopf G-spaces $\lambda\colon I_+(V, M) \to C(V, M)$, which is natural in M and is a G-homotopy equivalence.*

Proof. Choose a G-linear isometry $l\colon V \times \mathbb{R} \cong V$, and define λ as the composite

$$I_+(V, M) \xrightarrow{b_*} C(V, C(\mathbb{R}) \wedge M) \xrightarrow{\subset} C(V \times \mathbb{R}, M) \xrightarrow{l_*} C(V, M)$$

□

4. Main Results

To state the main results we introduce the G-category $\mathrm{Top}(G)$ consisting of all pointed G-spaces and pointed maps, with G acting on morphisms by conjugation. As we showed in [2], any G-equivariant continuous functor T of $\mathrm{Top}(G)$ into itself is accompanied with pairings

$$X \wedge TY \to T(X \wedge Y), \quad TX \wedge Y \to T(X \wedge Y)$$

natural in both X and Y. One easily observes from this that T preserves G-homotopies, and there is a suspension natural transformation $T(X) \to \Omega^W T(\Sigma^W X)$ defined for any finite dimensional orthogonal G-module W.

Suppose V is linearly and equivariantly isometric to the direct product of countably many copies of the regular representation of G over the real number fields. Such a G-module V is said to be *sufficiently large*. Then we have

Theorem 7. *In the diagram of Hopf G-spaces*

$$C(V, M) \xleftarrow{\lambda} I_+(V, M) \xrightarrow{\rho} I(V, M),$$

where ρ is induced by the inclusion $I(\mathbb{R})_+ \subset I(\mathbb{R})$, we have

(1) *λ is a G-homotopy equivalence.*
(2) *ρ is a G-equivariant group completion, that is to say, $\rho^H\colon I_+(V, M)^H \to I(V, M)^H$ is a group completion for every subgroup H of G.*

Theorem 8. *The correspondence $X \to I(V, X \wedge M)$ is a G-equivariant continuous functor of* $\mathrm{Top}(G)$ *into itself and we have*

(1) *For any orthogonal G-module W the suspension map $I(V, X \wedge M) \to \Omega^W I(V, \Sigma^W X \wedge M)$ is a weak G-homotopy equivalence.*
(2) *There is an $RO(G)$-graded G-homology theory $h^G_\bullet(-)$ such that*

$$h^G_n(X) = \pi_n I(V, X \wedge M)^G$$

holds for any X and $n \geq 0$.

5. Brief Outline of the Proofs

Outline of the Proof of Theorem 7. We need to show that

(5.1) $\qquad \rho^H\colon I_+(V, M)^H \to I(V, M)^H$ is a group completion

holds for every subgroup H of G.

Observe that V is an H-universe for any subgroup H of G. Hence (5.1) for general H follows from the special case $H = G$. that $\rho^H\colon I_+(V, M)^H \to I(V, M)^H$ is a group completion follows from the case $H = G$. *But the usual argument using the notion of orbit type family enables us to reduce the proof of this problem to the case where G is trivial.* Thus we may assume $G = 1$ and $V = \mathbb{R}^\infty$. Recall from

[3] that there is a weak equivalence of Hopf spaces $\Phi\colon D(M) \to C(\mathbb{R}^\infty, M)$, where $D(M)$ is the realization of the diagonal

$$[k] \mapsto S_k N_k \mathcal{Q}(M) = N_k \mathcal{Q}(S_k M)$$

associated with the bisimplicial set of the total singular complex of the nerve of a permutative category $\mathcal{Q}(M)$, whose space of objects is $\coprod_{p\geq 0} M^p$, and whose morphisms from $(a_i) \in M^p$ to $(b_j) \in M^q$ are maps of finite sets $\theta\colon \{1,\dots,p\} \to \{1,\dots,q\}$ such that $b_j = \sum_{i\in\theta^{-1}(j)} a_i$ holds for $1 \leq j \leq q$.

Since Φ is natural in M, Theorem 7 follows from

Proposition 9. *The natural map $D(I_+(\mathbb{R}) \wedge M) \to D(I(\mathbb{R}) \wedge M)$, induced by the inclusion $I_+(\mathbb{R}) \subset I(\mathbb{R})$, is a group completion.*

The rest of this section is devoted to the proof of this proposition.

Given a map of topological monoids $f\colon D \to D'$ let $B(D, D')$ denote the realization of the category $\mathcal{B}(D, D')$ whose space of objects is D' and whose space of morphisms is the product $D \times D'$, where $(d, d') \in D \times D'$ is regarded as a morphism from d' to $f(d)\cdot d'$. Then there is a sequence of maps

$$D' = B(0, D') \to B(D, D') \to B(D, 0) = BD$$

induced by the maps $0 \to D$ and $D' \to 0$ respectively. Observe that BD is the standard classifying space of the monoid D and $B(D, D)$ is contractible when f is the identity.

Let $D = D(I_+(\mathbb{R}) \wedge M)$, $D' = D(I(\mathbb{R}) \wedge M)$, and let $i\colon D \to D'$ be the map induced by the inclusion $I_+(\mathbb{R}) \to I(\mathbb{R})$. Then we have a commutative diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d, "i"'] & B(D,D) \arrow[r] \arrow[d, "B(1,i)"] & BD \arrow[d, equal] \\ D' \arrow[r] & B(D,D') \arrow[r] & BD \end{tikzcd} \tag{5.2}$$

in which the upper and the lower sequences are associated with the identity and the inclusion $i\colon D \to D'$, respectively.

Lemma 10. *The natural map $D \to \Omega BD$ is a group completion.*

This follows from the fact that D is a homotopy commutative, hence admissible, monoid.

Lemma 11. *The lower sequence in the diagram (5.2) is a homotopy fibration sequence with contractible total space $B(D, D')$.*

Proposition 9 can be deduced from these lemmas, because $D \to D'$ is equivalent to the group completion map $D \to \Omega BD$ under the equivalence $D' \simeq \Omega BD$.

Outline of the Proof of Theorem 8. We need the following

Proposition 12. *Let T be a G-equivariant continuous functor of the category of pointed G-spaces and pointed maps to itself. Suppose T satisfies the following conditions.*

C1: $T* = *$
C2: $\|T(X_\bullet)\| \simeq_G T(\|X_\bullet\|)$ *for any simplicial objects* $X_\bullet$
C3: $T(X \vee Y) \simeq_G TX \times TY$
C4: $T(G/H_+ \wedge X) \simeq_G \mathrm{Map}(G/H, TX)$ *for any subgroup* H.

Suppose further that TX^H *is grouplike for any* X *and any subgroup* H *of* G. *Then we have*

(a) $TX \simeq_G \Omega^W T(\Sigma^W X)$ *for any real* G*-module* W,
(b) $X \mapsto \{\pi_n TX^G\}$ *defines an* $RO(G)$*-gradable equivariant homology theory on the category of pointed* G*-spaces.*

Proof. As TX^H is grouplike for any H the natural map $TX \to \Omega T(\Sigma X)$ is a weak G-equivalence. Hence by the argument similar to the proof of [3, Theorem 2.12]

$$TA \to TX \to T(X \cup CA)$$

is a G-fibration sequence up to weak G-equivalence for any pair of pointed G-spaces (X, A). It also follows by the property of G-equivariant continuous functor that T preserves G-homotopies. Thus (a) implies (b).

To prove (a) we need only show that the correspondence $S \mapsto T(S \wedge X)$, where S is any pointed finite G-set, defines a special Γ_G-space in the sense of [2], that is, the natural map

$$T(S \wedge X) \to \mathrm{Map}_0(S, TX)$$

is a G-equivalence. But this follows from the conditions C3 and C4. □

Let $TX = I(V, X \wedge M)$. We shall show that T satisfies the conditions C1, C2, C3 and C4. This of course implies Theorem 8.

Clearly the condition C1 holds, and C2 is a routine exercise. That C3 holds is proved as follows.

To verify C4 we shall show that $T(G/H_+ \wedge X) \to \mathrm{Map}(G/H, TX)$ has a G-homotopy inverse ρ defined as follows:

(1) Choose a G-embedding $G/H \to V$ and a linear G-isometry $V \times V \to V$.
(2) For given $f\colon G/H \to TX$ write $f(gH) = (c(gH), a(gH))$, where $c(gH) \subset V$, $a(gH)\colon c(gH) \to X \wedge M \wedge I(\mathbb{R})$.
(3) Define $\tilde{c}$ to be the image of $\bigcup\{gH\} \times c(gH)$ under the embedding

$$\iota\colon G/H \times V \subset V \times V \to V$$

(4) Define $\tilde{a}\colon \tilde{c} \to G/H_+ \wedge X \wedge I(\mathbb{R}) \wedge M$ by

$$\tilde{a}(\iota(gH, \xi)) = gH \wedge a(gH)(\xi), \quad \xi \in c(gH)$$

(5) $\rho\colon \mathrm{Map}(G/H, TX) \to T(G/H_+ \wedge X)$ is given by $\rho(f) = (\tilde{c}, \tilde{a})$.

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