Generalization of operator type Shannon inequality and its reverse one

Takayuki Furuta

Abstract. We shall state the following generalization of operator type Shannon inequality and its reverse one as a simple corollary of parametric extensions of Shannon inequality in Hilbert space operators.

Let \( \{A_1, A_2, \ldots, A_n\} \) and \( \{B_1, B_2, \ldots, B_n\} \) be two sequences of strictly positive operators on a Hilbert space \( H \). If \( \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I \), then

\[
\sum_{j=1}^{n} S_2(A_j|B_j) \geq \left[ \sum_{j=1}^{n} B_j A_j^{-1} B_j \right] \log \left[ \sum_{j=1}^{n} B_j A_j^{-1} B_j \right] \geq \log \left[ \sum_{j=1}^{n} B_j A_j^{-1} B_j \right] \geq \sum_{j=1}^{n} S_1(A_j|B_j) \geq 0 \geq \sum_{j=1}^{n} S(A_j|B_j)
\]

where \( S_q(A|B) = A^\frac{1}{2} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^\frac{1}{2} \) for \( A > 0 \), \( B > 0 \) and any real number \( q \) and \( S(A|B) = S_0(A|B) = A^\frac{1}{2} (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^\frac{1}{2} \) which is the relative operator entropy of \( A > 0 \) and \( B > 0 \).

Our results can be considered as parametric extensions of the following celebrated Shannon inequality ([3],[5] and [233 p ,1]) which is very useful and so famous in information theory. Let \( \{a_1, a_2, \ldots, a_n\} \) and \( \{b_1, b_2, \ldots, b_n\} \) be two probability vectors. Then

\[
0 \geq \sum_{j=1}^{n} a_j \log b_j - \sum_{j=1}^{n} a_j \log a_j \text{ (see inequalities (2.4) of Corollary 2.4)}.
\]

§1 Introduction

First the Shannon inequality asserts: Let \( \{a_1, a_2, \ldots, a_n\} \) and \( \{b_1, b_2, \ldots, b_n\} \) be two probability vectors. Then
We remark that \( 0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} \) in (1.1) is equivalent to \( D = \sum_{j=1}^{n} a_j \log \frac{a_j}{b_j} \geq 0 \) which is the original number type Shannon inequality and this \( D \) is called “divergence” in [3] and [5].

In this paper we shall state parametric extensions of Shannon inequality and its reverse one in Hilbert space operators.

A bounded linear operator \( T \) on a Hilbert space \( H \) is said to be positive (denoted by \( T \geq 0 \)) if \( (Tx,x) \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is invertible and positive.

**Definition 1.1.** \( S_q(A|B) \) for \( A > 0, B > 0 \) and any real number \( q \) is defined by

\[
S_q(A|B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.
\]

We recall that \( S_0(A|B) = A^{\frac{1}{2}} (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} = S(A|B) \) is the relative operator entropy in [2] and \( S(A|I) = -A \log A \) is the usual operator entropy in [4].

**Definition 1.2.** \( A \mathfrak{h}_q B \) for \( A > 0 \) and \( B > 0 \) and any real number \( q \) is defined by

\[
A \mathfrak{h}_q B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q A^{\frac{1}{2}}
\]

and \( A \mathfrak{h}_p B \) for \( p \in [0,1] \) just coincides with \( A^p B \) which is well known as \( p \)-power mean.

We remark that \( S_1(A|B) = -S(B|A) \) and moreover \( S_q(A|B) = -S_{1-q}(B|A) \) for any \( q \).

Following after Definition 1.1, The original Shannon inequality can be expressed as follows:

\[
0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} = \sum_{j=1}^{n} a_j^{\frac{1}{2}} (\log a_j^{-\frac{1}{2}} b_j a_j^{-\frac{1}{2}}) a_j^{\frac{1}{2}} = \sum_{j=1}^{n} S(a_j|b_j).
\]

Consequently \( 0 \geq \sum_{j=1}^{n} S(a_j|b_j) \) in the original Shannon inequality can be extented to

\[
0 \geq \sum_{j=1}^{n} S(A_j|B_j) \text{ in operator version case (2.4) of Corollary 2.4, so that the form of (1.1) is convenient for operator type extension. We can summarize the following contrast:}
\]
The original Shannon inequality and its reverse one

\[ 0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} \geq -\log \sum_{j=1}^{n} \frac{a_j^2}{b_j}. \]

for \( a_j, b_j > 0 \) with \( 1 = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j. \)

The operator version Shannon inequality and its reverse one

\[ 0 \geq \sum_{j=1}^{n} S(A_j|B_j) \geq -\log \sum_{j=1}^{n} A_j B_j^{-1} A_j. \]

for \( A_j, B_j > 0 \) with \( I = \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j. \)

§2 Parametric extensions of operator reverse type Shannon inequality derived from two operator concave functions \( f_1(t) = \log t \) and \( f_2(t) = -t \log t \)

Firstly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function \( f(t) = \log t. \)

**Theorem 2.1.** Let \( p \in [0, 1] \) and also let \( \{A_1, A_2, \ldots, A_n\} \) and \( \{B_1, B_2, \ldots, B_n\} \) be two sequences of strictly positive operators on a Hilbert space \( H \) such that \( \sum_{j=1}^{n} A_j \#_p B_j \leq I, \) where \( I \) means the identity operator on \( H. \) Then

\[ (2.1) \quad \log \left[ \sum_{j=1}^{n} (A_j \#_{p+1} B_j) + t_0 (I - \sum_{j=1}^{n} A_j \#_p B_j) \right] - \log t_0 (I - \sum_{j=1}^{n} A_j \#_p B_j) \]

\[ \geq \sum_{j=1}^{n} S_p(A_j|B_j) \]

\[ \geq -\log \left[ \sum_{j=1}^{n} (A_j \#_{p-1} B_j) + t_0 (I - \sum_{j=1}^{n} A_j \#_p B_j) \right] + \log t_0 (I - \sum_{j=1}^{n} A_j \#_p B_j) \]

for fixed real number \( t_0 > 0, \) where \( S_p(A|B) \) is defined in Definition 1.1 and \( A \#_p B \) is defined in Definition 1.2.

Secondly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function \( f(t) = -t \log t. \)

**Theorem 2.2.** Let \( p \in [0, 1] \) and also let \( \{A_1, A_2, \ldots, A_n\} \) and \( \{B_1, B_2, \ldots, B_n\} \) be two sequences of strictly positive operators on a Hilbert space \( H \) such that \( \sum_{j=1}^{n} A_j \#_p B_j \leq I, \) where \( I \) means the identity operator on \( H. \) Then
(2.2) \[ \sum_{j=1}^{n} S_{p+1}(A_j|B_j) \]

\[ \geq \left[ \sum_{j=1}^{n} (A_j h_{p+1} B_j) + t_0 (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \log \left[ \sum_{j=1}^{n} (A_j h_{p+1} B_j) + t_0 (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \]

\[-t_0 \log t_0 (I - \sum_{j=1}^{n} A_j h_p B_j) \quad \text{for fixed real number } t_0 > 0,\]

and

(2.2') \[ \sum_{j=1}^{n} S_{p-1}(A_j|B_j) \]

\[ \leq - \left[ \sum_{j=1}^{n} (A_j h_{p-1} B_j) + t_0 (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \log \left[ \sum_{j=1}^{n} (A_j h_{p-1} B_j) + t_0 (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \]

\[ + t_0 \log t_0 (I - \sum_{j=1}^{n} A_j h_p B_j) \quad \text{for fixed real number } t_0 > 0,\]

where \( S_q(A|B) \) is defined in Definition 1.1 and \( A h_q B \) is defined in Definition 1.2.

We shall state the following result which can be shown by combining Theorem 2.1 with Theorem 2.2.

**Corollary 2.3.** Let \( p \in [0, 1] \) and also let \( \{A_1, A_2, \ldots, A_n\} \) and \( \{B_1, B_2, \ldots, B_n\} \) be two sequences of strictly positive operators on a Hilbert space \( H \) such that \( \sum_{j=1}^{n} A_j h_p B_j \leq I \), where \( I \) means the identity operator on \( H \). Then

(2.3) \[ \sum_{j=1}^{n} S_{p+1}(A_j|B_j) \]

\[ \geq \left[ \sum_{j=1}^{n} (A_j h_{p+1} B_j) + (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \log \left[ \sum_{j=1}^{n} (A_j h_{p+1} B_j) + (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \]

\[ \geq \log \left[ \sum_{j=1}^{n} (A_j h_{p+1} B_j) + (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \]

\[ \geq \sum_{j=1}^{n} S_{p}(A_j|B_j) \]

\[ \geq - \log \left[ \sum_{j=1}^{n} (A_j h_{p-1} B_j) + (I - \sum_{j=1}^{n} A_j h_p B_j) \right] \]
\[
\geq - \left[ \sum_{j=1}^{n} (A_j \mathfrak{h}_{p-1} B_j) + (I - \sum_{j=1}^{n} A_j \mathfrak{h}_{p} B_j) \right] \log \left[ \sum_{j=1}^{n} (A_j \mathfrak{h}_{p-1} B_j) + (I - \sum_{j=1}^{n} A_j \mathfrak{h}_{p} B_j) \right] \\
\geq \sum_{j=1}^{n} S_{p-1}(A_j | B_j)
\]

where \( S_q(A|B) \) is defined in Definition 1.1 and \( A \mathfrak{h}_q B \) is defined in Definition 1.2.

Corollary 2.3 easily implies the following result which can be considered as operator version of Shannon inequality and its reverse one.

**Corollary 2.4.** Let \( \{A_1, A_2, \ldots, A_n\} \) and \( \{B_1, B_2, \ldots, B_n\} \) be two sequences of strictly positive operators on a Hilbert space \( H \). If \( \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I \), then

\[
\sum_{j=1}^{n} S_2(A_j | B_j) \geq \sum_{j=1}^{n} B_j A_j^{-1} B_j \geq \log \sum_{j=1}^{n} B_j A_j^{-1} B_j \geq \sum_{j=1}^{n} S_0(A_j | B_j) \geq 0 \geq \sum_{j=1}^{n} S_{-1}(A_j | B_j).
\]

**Remark 2.1.** We recall \( S_q(A|B) \) for \( A > 0, B > 0 \) and any real number \( q \) as follows:

\[
S_q(A|B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{q} (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.
\]

By an easy calculation we have

\[
\frac{d}{dq}\bigl[S_q(A|B)\bigr] = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{q} [\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}]^{2} A^{\frac{1}{2}} \geq 0,
\]

so that \( S_q(A|B) \) is an increasing function of \( q \), and it is interesting to point out that the decreasing order of the positions of \( \sum_{j=1}^{n} S_2(A_j | B_j), \sum_{j=1}^{n} S_1(A_j | B_j), \sum_{j=1}^{n} S(A_j | B_j) \), and \( \sum_{j=1}^{n} S_{-1}(A_j | B_j) \) in (2.4) of Corollary 2.4 is quite reasonable since \( \sum_{j=1}^{n} S(A_j | B_j) = \sum_{j=1}^{n} S_0(A_j | B_j) \).

This paper will appear elsewhere with complete proofs.
References


Takayuki Furuta
Department of Mathematical Information Science, Faculty of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
e-mail: furuta@rs.kagu.tus.ac.jp