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DECAY AND REGULARITY FOR DISPERSIVE EQUATIONS WITH NON-POLYNOMIAL SYMBOLS

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1. INTRODUCTION

The aim of this paper is to provide a new method to prove and extend the result by Hoshiro [1] on global smoothing estimates for dispersive equations such as Schrödinger equations. Our method will produce time local estimates for operators with not only polynomial symbols as well as give corresponding time global estimates. For the purpose, the Egorov-type theorem via canonical transformation in the form of a class of Fourier integral operators, together with their weighted L^2-boundedness is used.

We consider (Fourier integral) operators, which can be globally written in the form

$$Tu(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i\phi(x,y,\xi)}p(x,y,\xi)u(y)\,dyd\xi\quad(x\in\mathbb{R}^n),\tag{1.1}$$

where $p(x,y,\xi)$ is an amplitude function and $\phi(x,y,\xi)$ is a real-valued phase function. If we take

$$\phi(x,y,\xi)=x\cdot\xi-y\cdot\psi(\xi)$$

as a special case, we have

$$Tu(x)=F^{-1}[(Fu)(\psi(\xi))](x),$$

where F (F^{-1} *resp.*) denotes the (inverse *resp.*) Fourier transformation. Hence we have the relation

$$T\cdot\sigma(D)=a(D)\cdot T,\qquad a(D)=(\sigma\circ\psi)(D),\tag{1.2}$$

for constant coefficient operators $\sigma(D)$ and $a(D)$. This fact is known as a special case of Egorov's theorem, which we modify to allow for the exact calculus. By choosing a phase function appropriately, properties of the operator $a(D)$ can be extracted from those of the operator $\sigma(D)$.

We mention a boundedness theorem which will appear in [4]. For $k\in\mathbb{R}$, let $L^2_k(\mathbb{R}^n)$ be the set of measurable functions f such that the norm

$$\|f\|_{L^2_k(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}\left|\langle x\rangle^k f(x)\right|^2dx\right)^{1/2};\quad \langle x\rangle^k=\left(1+|x|^2\right)^{k/2}$$

is finite. Then we have the following:

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Theorem 1.1. *Let the operator T be defined by (1.1) with $\phi(x,y,\xi) = x\cdot\xi + \varphi(y,\xi)$. Assume that*

$$|\det \partial_y \partial_\xi \varphi(y,\xi)| \geq C > 0,$$

and all the derivatives of entries of $\partial_y\partial_\xi\varphi$ are bounded. Also assume that

$$\left|\partial_\xi^\alpha \varphi(y,\xi)\right| \leq C_\alpha \langle y \rangle \quad (\forall |\alpha| \geq 1),$$
$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma p(x,y,\xi)\right| \leq C_{\alpha\beta\gamma} \langle x \rangle^{-|\alpha|} \quad (\forall \alpha,\beta,\gamma)$$

or

$$\left|\partial_y^\alpha \partial_\xi^\beta \varphi(y,\xi)\right| \leq C_\alpha \langle y \rangle^{1-|\alpha|} \quad (\forall \alpha, |\beta| \geq 1),$$
$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma p(x,y,\xi)\right| \leq C_{\alpha\beta\gamma} \langle y \rangle^{-|\beta|} \quad (\forall \alpha,\beta,\gamma).$$

Then T is bounded on $L^2_k(\mathbb{R}^n)$ for any $k \in \mathbb{R}$.

2. Smoothing effects of dispersive equations

We consider the following dispersive equation:

$$(2.1) \qquad \begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0, \\ \qquad\qquad u(0,x) = \varphi(x) \in L^2(\mathbb{R}^n), \end{cases}$$

where $a = a(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ is a real-valued function. The solution $u(t,x)$ to equation (2.1) can be expressed as

$$u(t,x) = e^{ita(D)}\varphi.$$

We assume that $a(\xi) = a_m(\xi) + r(\xi)$ for large ξ, where $a_m(\lambda\xi) = \lambda^m a_m(\xi)$ for $\lambda > 0$, $\xi \neq 0$, and $r(\xi)$ is a smooth symbol of order $m-1$ satisfying $|\partial^\alpha r(\xi)| \leq C\langle\xi\rangle^{m-1-|\alpha|}$ for all multiindices α. Equation (2.1) with $a(\xi) = |\xi|^2$ is the Schrödinger equation. First we have the following time local estimate:

Theorem 2.1. *Suppose $m \geq 1$, $s > 1/2$, and $T > 0$. Assume that $\nabla a(\xi) \neq 0$ if $a(\xi) = 0$ and $|\xi|$ is large. Then we have the estimate*

$$\int_0^T \left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D)} \varphi(x) \right\|^2_{L^2(\mathbb{R}^n_x)} dt \leq C \|\varphi\|^2_{L^2(\mathbb{R}^n)}.$$

As a corollary, we have the following result obtained by Hoshiro [1] for polynomials $a(\xi)$. He used Mourre's method which is known in spectral and scattering theories.

Corollary 2.2. *Suppose $m \geq 1$, $s > 1/2$, and $T > 0$. Assume that $\nabla a_m(\xi) \neq 0$ for $|\xi| = 1$. Then we have the estimate*

$$\int_0^T \left\| \langle x \rangle^{-s} \langle D_x \rangle^{(m-1)/2} e^{ita(D)} \varphi(x) \right\|^2_{L^2(\mathbb{R}^n_x)} dt \leq C \|\varphi\|^2_{L^2(\mathbb{R}^n)}.$$

The proof of Theorem 2.1 will be outlined in Section 4. Let us now explain how it implies Corollary 2.2. First, the assumption $\nabla a_m(\xi) \neq 0$ for $|\xi| = 1$ is equivalent to the assumption of Theorem 2.1, so we obtain the desired estimate for large frequencies. For small frequencies it follows by a general functional analytic argument under no additional assumptions on $a(\xi)$ due to the boundedness of the time interval (see Section 4 or [1]).

We have the time global estimate if we assume $\nabla a(\xi) \neq 0$ for small ξ also.

Theorem 2.3. *Suppose $m \geq 1$ and $s > 1/2$. Assume that $a \in C^\infty(\mathbb{R}^n)$ and that $\nabla a(\xi) \neq 0$ if $a(\xi) = 0$. Then we have the estimate*

$$\int_0^\infty \left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D)} \varphi(x) \right\|^2_{L^2(\mathbb{R}^n_x)} dt \leq C \|\varphi\|^2_{L^2(\mathbb{R}^n)}.$$

There are obstructions for the time global version of the estimate of Corollary 2.2 due to the small frequencies, see, e.g. [1] or [6].

3. Main tool

Based on the argument in the introduction, we will now describe the main tool for the proofs of theorems of the previous section.

Let $\Gamma, \tilde{\Gamma} \subset \mathbb{R}^n$ be open sets and $\psi : \Gamma \to \tilde{\Gamma}$ be a C^∞-diffeomorphism. We assume

$$C^{-1} \leq |\det \partial\psi(\xi)| \leq C \quad (\xi \in \Gamma), \tag{3.1}$$

for some $C > 0$, and all the derivatives of entries of the $n \times n$ matrix $\partial\psi$ are bounded. We set formally

$$Iu(x) = F^{-1}\left[Fu(\psi(\xi))\right](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi(\xi))} u(y)\, dy d\xi,$$

$$I^{-1}u(x) = F^{-1}\left[Fu(\psi^{-1}(\xi))\right](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))} u(y)\, dy d\xi.$$

The operators I and I^{-1} can be justified by using cut-off functions $\gamma \in C^\infty(\Gamma)$ and $\tilde{\gamma} = \gamma \circ \psi^{-1} \in C^\infty(\tilde{\Gamma})$ which satisfy $\operatorname{supp}\gamma \subset \Gamma$, $\operatorname{supp}\tilde{\gamma} \subset \tilde{\Gamma}$. We set

$$\begin{aligned} I_\gamma u(x) &= F^{-1}\left[\gamma(\xi) Fu(\psi(\xi))\right](x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\Gamma} e^{i(x\cdot\xi - y\cdot\psi(\xi))} \gamma(\xi) u(y)\, dy d\xi, \\ I_{\tilde{\gamma}}^{-1} u(x) &= F^{-1}\left[\tilde{\gamma}(\xi) Fu(\psi^{-1}(\xi))\right](x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\tilde{\Gamma}} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))} \tilde{\gamma}(\xi) u(y)\, dy d\xi, \end{aligned} \tag{3.2}$$

and we have the expressions

$$I_\gamma = \gamma(D) \cdot I = I \cdot \tilde{\gamma}(D), \quad I_{\tilde{\gamma}}^{-1} = \tilde{\gamma}(D) \cdot I^{-1} = I^{-1} \cdot \gamma(D),$$

and the identities

$$I_\gamma \cdot I_{\tilde{\gamma}}^{-1} = \gamma(D)^2, \quad I_{\tilde{\gamma}}^{-1} \cdot I_\gamma = \tilde{\gamma}(D)^2.$$

On account of (1.2), we have the formula

$$I_\gamma \cdot \sigma(D) = a(D) \cdot I_\gamma, \quad a(\xi) = (\sigma \circ \psi)(\xi). \tag{3.3}$$

We have the boundedness on weighted spaces by Theorem 1.1.

Proposition 3.1. *The operators I_γ and $I_{\tilde{\gamma}}^{-1}$ defined by* (3.2) *are $L_k^2(\mathbb{R}^n)$-bounded for any $k \in \mathbb{R}$.*

4. Proof of Theorem 2.1

We will now outline the proof of Theorem 2.1. The details will appear in [5] which also includes the proof of Theorem 2.3.

We may assume $\operatorname{supp}\hat{\varphi} \subset \{\xi; |\xi| \geq R\}$ for some large $R > 0$. In fact, if $\operatorname{supp}\hat{\varphi} \subset \{\xi; |\xi| \leq R\}$, we have

$$\begin{aligned}
&\int_0^T \left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D)} \varphi(x) \right\|_{L^2(\mathbb{R}_x^n)}^2 dt \\
\leq &\int_0^T \left\| |D_x|^{(m-1)/2} e^{ita(D)} \varphi(x) \right\|_{L^2(\mathbb{R}_x^n)}^2 dt \\
\leq &CT \left\| |\xi|^{(m-1)/2} \hat{\varphi}(\xi) \right\|_{L^2(\mathbb{R}^n)}^2 \\
\leq &CTR^{m-1} \|\varphi\|_{L^2(\mathbb{R}^n)}^2
\end{aligned}$$

by Plancherel's theorem.

Furthermore, by the microlocalization and the rotation, we may assume $\operatorname{supp}\hat{\varphi} \subset \Gamma$, where $\Gamma \subset \mathbb{R}^n \setminus 0$ is a sufficiently small conic neighborhood of $e_n = (0, \ldots 0, 1)$. By formula (3.3) and Proposition 3.1, it is sufficient to find $\psi : \Gamma \to \tilde{\Gamma}$ which satisfies (3.1) and to find $\sigma(\eta)$ such that $a(\xi) = (\sigma \circ \psi)(\xi)$ $(\xi \in \Gamma)$, and then show the result by replacing $a(D)$ by $\sigma(D)$, assuming $\operatorname{supp}\hat{\varphi} \subset \tilde{\Gamma}$.

Since $a(\xi)$, $\nabla a(\xi)$ behaves like $a_m(\xi)$, $\nabla a_m(\xi)$ for large ξ, respectively, we have $\nabla a_m(e_n) \neq 0$ by the assumption and the Euler's identity $a_m(\xi) = (\xi/m) \cdot \nabla a_m(\xi)$. We have the following two possibilities:

- **(i):** $\partial_n a_m(e_n) \neq 0$. Then, by Euler's identity, we have $a_m(e_n) \neq 0$. Hence, in this case, we may assume that $a(\xi)(> 0)$ and $\partial_n a(\xi)$ are bounded away from 0 for $\xi \in \Gamma$ and $|\xi| \geq R$.
- **(ii):** $\partial_n a_m(e_n) = 0$. Then there exits $j \neq n$ such that $\partial_j a_m(e_n) \neq 0$. Hence, in this case, we may assume $\partial_1 a(\xi)$ is bounded away from 0 for $\xi \in \Gamma$ and $|\xi| \geq R$.

Case (i). We take

$$\sigma(\eta) = \eta_n^m, \quad \psi(\xi) = (\xi_1, \ldots, \xi_{n-1}, a(\xi)^{1/m}).$$

Then we have $a(\xi) = (\sigma \circ \psi)(\xi)$ and

$$\det \partial\psi(\xi) = \begin{vmatrix} E_{n-1} & 0 \\ * & (1/m)a(\xi)^{1/m-1}\partial_n a(\xi) \end{vmatrix}$$

satisfies (3.1), where E_{n-1} is the identity matrix of order $n-1$. The estimate for $\sigma(D) = D_n^m$ is given by the following (see Kenig, Ponce and Vega [2, p.56] in the case $m = 2$):

Proposition 4.1. *In the case $n = 1$, we have*

$$\sup_{x\in\mathbb{R}} \left\| |D_x|^{(m-1)/2} e^{itD_x^m} f(x) \right\|_{L^2(\mathbb{R}_t)} \le C\|f\|_{L^2(\mathbb{R}_x)}.$$

Hence we have

$$\left\| \langle x\rangle^{-s} |D_n|^{(m-1)/2} e^{it\sigma(D)} \varphi(x) \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

for $s > 1/2$. Here we have used the trivial inequality $\langle x\rangle^{-s} \le \langle x_n\rangle^{-s}$, Schwarz's inequality, and Plancherel's theorem. Since ψ maps Γ into another small conic neighborhood of e_n, $|\xi_n|$ is equivalent to $|\xi|$ there. Hence we have the estimate

$$\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{it\sigma(D)} \varphi(x) \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

by Theorem 1.1 with $\varphi(y,\xi) = -y\cdot\xi$.

Case (ii). We take

$$\sigma(\eta) = \eta_1\eta_n^{m-1}, \quad \psi(\xi) = \left(a(\xi)\xi_n^{1-m}, \xi_2, \dots, \xi_n\right)$$

Then we have $a(\xi) = (\sigma\circ\psi)(\xi)$ and

$$\det\partial\psi(\xi) = \begin{vmatrix} \partial_1 a(\xi)\xi_n^{1-m} & * \\ 0 & E_{n-1} \end{vmatrix}$$

satisfies (3.1). The estimate for $\sigma(D) = D_1 D_n^{m-1}$ was given by the following (see Linares and Ponce [3, p.528] in the case $m = 2$):

Proposition 4.2. *In the case $n = 2$, we have*

$$\sup_{y\in\mathbb{R}} \left\| |D_x|^{(m-1)/2} e^{itD_x^{m-1}D_y} f(x,y) \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x)} \le C\|f\|_{L^2(\mathbb{R}^2_{x,y})}.$$

Similarly to the case (i), we have

$$\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{it\sigma(D)} \varphi(x) \right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

for $s > 1/2$.

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