Asymptotic Analysis of Confluent Hypergeometric Partial Differential Equations in Many Variables

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1 Introduction

The confluent differential equation in one variable, known as the Kummer differential equation

$$x \frac{d^2 w}{dx^2} + (\gamma - x) \frac{dw}{dx} - \beta w = 0,$$

is studied by several authors in various ways. Among those, the so-called Borel-Laplace-Ecalle method is a powerful one, which is explained for example in [1]. This method is applicable to an analysis of the Humbert confluent hypergeometric differential equations $\Phi_2$ in 2 variables

$$x \frac{\partial^2 w}{\partial x^2} + y \frac{\partial w}{\partial y} + (\gamma - x) \frac{\partial w}{\partial x} - \beta w = 0,$$

and we can obtain formal solutions, asymptotic solutions and so-called Stokes multipliers (see [2], [3]).

It is also applicable to an asymptotic analysis of the Humbert confluent hypergeometric partial differential equations in $m(> 2)$ variables. Here, the author gives an overview of it.

2 Humbert confluent hypergeometric partial differential equations $\Phi_D$

The system of Humbert confluent hypergeometric partial differential equations $\Phi_D$ is as follows:

$$x_k \frac{\partial^2 u}{\partial x_k^2} + \sum_{i \neq k} x_i \frac{\partial^2 u}{\partial x_k \partial x_i} + (\gamma - x_k) \frac{\partial u}{\partial x_k} - \beta_k u = 0,$$

where $\beta_k (k = 1, \ldots, m)$ and $\gamma$ are not non-negative integers.

We consider this system in $M = (P^1(C))^m$. The system has irregular singularities on $H = \bigcup_{k=1}^{m} H_k$, where $H_k = P^1(C) \times \cdots \times \{\infty\} \times \cdots \times P^1(C)$.

For simplicity, let $p$ be a point in $H \setminus \bigcup_{k \neq l}(H_k \cap H_l)$, we consider the formal solutions and asymptotic solutions to $\Phi_D$ near the point.

**Proposition 1.** We have $(m + 1)$ linearly independent formal solutions. Among them, $(m - 1)$ formal solutions are convergent and 2 formal solutions are divergent.

Near a point $(\infty, x_2, \ldots, x_m)$ with bounded $x_2, \ldots, x_m$, we have divergent solutions of the following forms

$$e^{z_1 \beta_1 - \gamma} \hat{V}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, z_1^{-1}),$$

and

$$z_1^{-\beta_1} \hat{U}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, z_1^{-1}).$$

Here, we put

$$\hat{V}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m) = \sum_{n=0}^{\infty} P_n(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m) z_1^{-n},$$

and

$$\hat{U}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m) = \sum_{n=0}^{\infty} Q_n(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m) z_1^{-n}.$$
with the polynomials

\[
P_{n}(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, \gamma, x_{2}, \ldots, x_{m}) = \sum_{l=0}^{n} \frac{(\gamma - \beta_{1} + l)_{n-t}(1 - \beta_{1})_{n-t}}{(n-l)!\ell!} \left( \sum_{j_{2} + \ldots + j_{m} = \ell} \frac{(\beta_{2})_{j_{2}} \ldots (\beta_{m})_{j_{m}} l!}{j_{2}! \ldots j_{m}!} x_{2}^{j_{2}} \ldots x_{m}^{j_{m}} \right)
\]

and

\[
\hat{U}(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, \gamma, x_{2}, \ldots, x_{m}, x_{1}^{-1}) = \sum_{n=0}^{\infty} \frac{(\beta_{1})_{n}(\beta_{1} - \gamma + 1)_{n}}{n!} \Phi_{D}^{m-1}(\beta_{2}, \ldots, \gamma - \beta_{1}; \gamma - \beta_{1} - n; x_{2}, \ldots, x_{m})(-x_{1})^{-n},
\]

where \( \Phi_{D}^{m-1}(\beta_{2}, \ldots, \beta_{m}; \gamma - \beta_{1} - n; x_{2}, \ldots, x_{m}) \) is the Humbert confluent hypergeometric function in \((m-1)\) variables with the parameter \((\beta_{2}, \ldots, \beta_{m}; \gamma - \beta_{1} - n)\),

\[
\Phi_{D}^{m-1}(\beta_{2}, \ldots, \beta_{m}; \gamma - \beta_{1} - n; x_{2}, \ldots, x_{m}) = \sum_{j_{2}=0}^{\infty} \cdots \sum_{j_{m}=0}^{\infty} \frac{(b_{j_{2}}) \cdots (b_{j_{m}}) x_{2}^{j_{2}} \cdots x_{m}^{j_{m}}}{(\beta_{1} - \gamma - n)_{j_{2}+\cdots+j_{m}} j_{2}! \cdots j_{m}!}.
\]

In the above, we use the Pochhammer symbol \((b)_{s} = (b+1) \cdots (b+s-1)\).

**Proposition 2.** The divergent formal series

\[
\hat{V}(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, \gamma, x_{2}, \ldots, x_{m}, x_{1}^{-1})
\]

and

\[
\hat{U}(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, \gamma, x_{2}, \ldots, x_{m}, x_{1}^{-1})
\]

are of Gevrey order 1 as \(x_{1} \to \infty\) uniformly on a bounded domain \(D\) in the \((x_{2}, \ldots, x_{m})\)-space.

**Definition.** For a formal expression \(e^{\rho x_{1}} \sum_{n=0}^{\infty} c_{n}(x_{2}, \ldots, x_{m}) x_{1}^{-n}\) with a complex number \(\rho\), a non-negative integer \(A\) and a formal series \(\tilde{p}(x) = \sum_{n=0}^{\infty} c_{n}(x_{2}, \ldots, x_{m}) x_{1}^{-n}\), we define the Borel transform as follows,

\[
\hat{B}_{1}(e^{\rho x_{1}} \sum_{n=0}^{\infty} c_{n} x_{1}^{-\lambda-n})(\xi_{1}) = \sum_{n=0}^{\infty} c_{n} \frac{(\rho + \xi_{1})^{n+\lambda-1}}{\Gamma(n+\lambda)}.
\]

**Proposition 3.** The Borel transforms of divergent solutions are holomorphic functions in a domain in \(\mathbb{C}^{m}\), which are analytically prolongable.

In fact,

\[
\hat{B}_{1}(e^{\rho x_{1}} \sum_{n=0}^{\infty} c_{n} x_{1}^{-\lambda-n})(\xi_{1}) = \frac{1}{\Gamma(\gamma - \beta_{1})} (-\xi_{1})^{\beta_{1}-1}(1 + \xi_{1})^{\gamma - \beta_{1} - 1} \Phi_{D}^{m-1}(\beta_{2}, \ldots, \beta_{m}; \gamma - \beta_{1}; (1 + \xi_{1}) x_{2}, \ldots, (1 + \xi_{1}) x_{m}),
\]

\[
B_{1}(\hat{v})(\xi_{1}) = \hat{B}_{1}(e^{\rho x_{1}} \sum_{n=0}^{\infty} c_{n} x_{1}^{-\lambda-n})(\xi_{1})(\xi_{1})
\]

and

\[
B_{1}(\hat{u})(\xi_{1}) = \hat{B}_{1}(x_{1}^{-\beta_{1}} \hat{U}(\beta_{1}, \beta_{2}, \ldots, \beta_{m}, \gamma, x_{2}, \ldots, x_{m}, x_{1}^{-1}))(\xi_{1}),
\]
we have a relation
\[ \Gamma(\beta_1)B_1(\xi_1)(\xi_1) = \Gamma(\gamma - \beta_1)(-1)^{-\beta_1 + 1}B_1(\xi_1)(\xi_1), \]

**Definition.** Consider a function \( f(\xi_1, x_2, \ldots, x_m) \) which is holomorphic and exponentially small in a tubular neighborhood in the first variable and a bounded domain in the other variables. We define the generalized Laplace transforms of \( f(\xi_1, x_2, \ldots, x_m) \), as follows
\[
\int_{C(q, \theta)} \exp(-x_1\xi_1)f(\xi_1, x_2, \ldots, x_m)d\xi_1,
\]
where \( C(q, \theta) \) is a following path of integral. For a point \( q \) in the tubular neighborhood, \( C(q, \theta) \) is a path on which \( \arg(\xi_1 - q) \) is taken to be initially \( \theta \) and finally \( \theta + 2\pi \).

**Proposition 4.** The Laplace transforms of Borel transforms of divergent solutions are holomorphic functions in a suitable angular domain with the summit \( p \) in \( P^1(C) \times C^{m-1} \), where they are actual solutions to the system \( \Phi_D \) with asymptotic expansions of Gevrey order 1. Here, the asymptotic expansions coincide with the divergent solutions, respectively.

In fact, for
\[-2\pi < \theta < 0,
\]
the Laplace integral
\[
\frac{1}{\Gamma(\beta_1)} \int_{C(-1, \theta)} \exp(-x_1\xi_1)(-\xi_1)^{\beta_1 - 1}(1 + \xi_1)^{\gamma - \beta_1 - 1} \times \Phi_D^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \cdots, (1 + \xi_1)x_m)d\xi_1
\]
is defined and represents a holomorphic function in the first variable \( x_1 \) in the angular domain (mod. \( 2\pi \))
\[
\frac{\pi}{2} < \arg(-\xi_1x_1) < \frac{3\pi}{2},
\]
namely,
\[-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,
\]
because \( \exp(-x_1\xi_1) \) tends to 0 as \( \xi_1 \) tends to the infinity. By considering the analytic prolongation, we obtain an actual solution \( v \) in the angular domain
\[-\frac{5\pi}{2} < \arg x_1 < \frac{\pi}{2}.
\]

For
\[-\pi < \theta < \pi,
\]
the Laplace integral
\[
\frac{1}{\Gamma(\beta_1)} \int_{C(0, \theta)} \exp(-x_1\xi_1)(\xi_1)^{\beta_1 - 1}(1 + \xi_1)^{\gamma - \beta_1 - 1} \times \Phi_D^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \cdots, (1 + \xi_1)x_m)d\xi_1
\]
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\[
\frac{\pi}{2} < \arg(-\xi_1x_1) < \frac{3\pi}{2},
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namely,
\[-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,
\]
because \( \exp(-x_1\xi_1) \) tends to 0 as \( \xi_1 \) tends to the infinity. By considering the analytic prolongation, we obtain an actual solution \( u \) in the angular domain
\[-\frac{3\pi}{2} < \arg x_1 < \frac{3\pi}{2}.
\]
Proposition 5 We have fundamental systems of solutions

\[(e_1 u, e_2 v, w_2, \ldots, w_m)\]

in the angular domain

\[-\frac{3\pi}{2} < \arg x_1 < \frac{\pi}{2}.\]

and

\[(e_1 u, e_2 v(x_1 e^{-2i\pi})e^{2i\pi(-\gamma + \beta_1)}, w_2, \ldots, w_m)\]

in the angular domain

\[-\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}.\]

where

\[w_2 = x_1^{-\beta_1} x_2^{\beta_1-\gamma+1} h_2, \ldots, w_m = x_1^{-\beta_1} x_m^{\beta_1-\gamma+1} h_m\]

with holomorphic functions \(h_2, \ldots, h_m\) at the point \(p\).

Then, we have the relations

\[(e_1 u, e_2 v(x_1 e^{-2i\pi})e^{2i\pi(-\gamma + \beta_1)})\]

\[= (e_1 u, e_2 v)(\begin{array}{c} 1 \\ 0 \\ 1 \end{array})\]

in the angular domain

\[-\frac{\pi}{2} < \arg x_1 < \frac{\pi}{2}.\]

and

\[(e_1 u(x_1 e^{-2i\pi})e^{2i\pi(-\beta_1)}, e_2 v(x_1 e^{-2i\pi})e^{2i\pi(-\gamma + \beta_1)})\]

\[= (e_1 u, e_2 v(x_1 e^{-2i\pi})e^{2i\pi(-\gamma + \beta_1)})(\begin{array}{c} 1 \\ 0 \\ c_{21} \end{array})\]

in the angular domain

\[\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}.\]

In the above, we use the following constants

\[\begin{align*}
e_1 &= (e^{2i\pi\beta_1} - 1)^{-1}, \\
e_2 &= (e^{2i\pi(\gamma - \beta_1)} - 1)^{-1}, \\
c_{12} &= \frac{-2i\pi}{\Gamma(1 - \beta_1)\Gamma(\gamma - \beta_1)}, \\
c_{21} &= \frac{-2i\pi e^{i\pi(\gamma - 2\beta_1)}}{\Gamma(\beta_1)\Gamma(1 - \gamma + \beta_1)}.\end{align*}\]

References

