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On the kernel theorem in hyperfunctions

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1 Introduction

Let V be an open subset in $\mathbb{R}^m$ and U an open subset in $\mathbb{R}^n$, m and n some fixed natural numbers. The Schwartz kernel theorem in distributions states that the linear continuous maps $T:\mathcal{D}(V)\to\mathcal{D}'(V)$ are precisely the maps for which we can find $\mathcal{K}\in\mathcal{D}'(U\times V)$ such that $T(u)(g)=\mathcal{K}(g\otimes u)$ for any $u\in C_0^\infty(V)$ and any $g\in C_0^\infty(U)$. Here $\mathcal{D}(V)$ denotes the space of $C_0^\infty(V)$ functions endowed with the Schwartz topology and $\mathcal{D}'(U)$ denotes the space of distributions on U. We shall also write $T(u)=\int_V \mathcal{K}(x,y)u(y)\,dy$ in this situation. and call $\mathcal{K}$ the kernel of T.

When we ask for a similar result in hyperfunctions, we encounter, right from the start, two difficulties which are not present in the distribution-case. The first is the lack of an appropriate test-function space in which functions have compact support. To circumvent this problem, we could then in principle either work with linear maps defined on $\mathcal{A}(V)$, or else with linear maps defined on $\mathcal{A}'(V)$. In the first case our starting space would be the closest thing there is in the analytic category to a test function space $\mathcal{D}(V)$, and in the second case we would conserve the "compact support" situation from the distributional kernel theorem. Of these two cases it is our opinion that the second one is the more natural one and it shall be this case which we discuss first later on. We also mention that while both, $\mathcal{A}(V)$ and $\mathcal{A}'(V)$, carry natural topologies, the two situations are not immediately related, since $\mathcal{A}(V)$ is not a subspace of $\mathcal{A}'(V)$. (Also see remark 1.3 below.)

The second difficulty in extending the kernel theorem to the analytic category comes from the fact that the canonical target space of our maps should be hyperfunctions, so that in fact we should study linear maps

$$T:\mathcal{A}'(V)\to\mathcal{B}(U),\ \text{respectively}\ T:\mathcal{A}(V)\to\mathcal{B}(U). \tag{1.1}$$

The problem inherent with hyperfunctions in this context is now that the space of (all) hyperfunctions on an open set admits no natural topology, so that for example it does not make sense to speak about linear continuous maps from $\mathcal{A}'(V)$ to $\mathcal{B}(V)$. Since it does not seem reasonable to look for a kernel representation for general linear operators (cf. in particular the examples in section

4) we have then to indicate a condition which is to replace the continuity requirement in the Schwartz kernel theorem. Fortunately, there is a variant of our problem in which there is no question about what the appropriate formulation for the kernel theorem should be. Indeed, this appears when we look at linear maps

$$T : \mathcal{A}'(V) \to \mathcal{A}'(Q), \text{ respectivley } T : \mathcal{A}(V) \to \mathcal{A}'(Q), \tag{1.2}$$

where Q is a compact orientable real-analytic variety. In this case, both $\mathcal{A}'(V)$ and $\mathcal{A}'(Q)$ and also $\mathcal{A}(V)$ have natural topologies and therefore there is a natural candidate for a "kernel theorem". The result which we can prove is in fact the following:

Theorem 1.1. *Let T be a linear continuous operator $T : \mathcal{A}'(V) \to \mathcal{A}'(Q)$. Then there is a kernel $\mathcal{K} \in \mathcal{B}(Q \times V)$ such that*

$$\{(x, y, 0, \eta); x \in Q, y \in V, \eta \neq 0\} \cap \mathrm{WF}_A \mathcal{K} = \emptyset \tag{1.3}$$

and

$$T(u) = \int_V \mathcal{K}(x, y) u(y)\, dy, \text{ for } u \in \mathcal{A}'(V). \tag{1.4}$$

The meaning of (1.4) is the one given in microlocal analysis to such expressions whenever (1.3) holds. Note that we identify analytic functionals and compactly supported hyperfunctions on open subsets of euclidean spaces by taking the standard volume elements, as usual. Moreover we also identify $\mathcal{A}'(Q)$ with $\mathcal{B}(Q)$ on a compact orientable manifold Q by fixing a volume element, even though there may be no standard choice for it.

Likewise, in the case $T : \mathcal{A}(V) \to \mathcal{A}'(Q)$ we have

Theorem 1.2. *Let $T : \mathcal{A}(V) \to \mathcal{A}'(Q)$ be a linear continuous operator. Then there is a kernel $\mathcal{K} \in \mathcal{A}'(Q \times V)$ such that*

$$T(u)(f) = \mathcal{K}(f \otimes u),\ \forall u \in \mathcal{A}(V),\ \forall f \in \mathcal{A}(Q). \tag{1.5}$$

Remark 1.3. *The kernels in theorem 1.1, respectively theorem 1.2, are thus characterized by the condition (1.3), respectively by the fact that they have compact support. It should be mentioned that if $\mathcal{K} \in \mathcal{A}'(Q \times V)$, then (1.3) cannot hold. Assume in fact that there is $\mathcal{K} \in \mathcal{A}'(Q \times V)$ ($\mathcal{K} \neq 0$) satisfying* (1.3). *Then for an arbitrary vector $a \in \mathbb{R}^m$, there exists $b \in \mathbb{R}$ and $(x_0, y_0) \in \operatorname{supp} \mathcal{K}$ such that $\operatorname{supp} \mathcal{K} \subset \{(x, y); \langle y, a\rangle \leq b\}$ and that $\langle y_0, a\rangle = b$. But these conditions and the microlocal Holmgren theorem show that $(x_0, y_0, 0, \pm a) \in \mathrm{WF}_A \mathcal{K}$, which contradicts* (1.3). *This shows that the situations from theorem 1.1 and theorem 1.2 are mutually exclusive.*

In the following part of this report we now first concentrate on the case $T : \mathcal{A}'(V) \to \mathcal{B}(U)$, when U is open in $\mathbb{R}^n$ and V is open in $\mathbb{R}^m$. In view of the above results, it seems justified to look for kernel theorems when the kernel $\mathcal{K}$ satisfies the wave front set estimate

$$\{(x,y,0,\eta); x \in U, y \in V, \eta \neq 0\} \cap \mathrm{WF}_A \mathcal{K} = \emptyset, \tag{1.6}$$

and when T is defined by

$$T(u) = \int_V \mathcal{K}(x,y)u(y)\,dy, \text{ for } u \in \mathcal{A}'(V). \tag{1.7}$$

The meaning of (1.7) is of course again the one given by microlocal analysis.

We next want to show that operators T which are defined by a kernel $\mathcal{K}$ which satisfies (1.6) have some property which is in some sense related to continuity. To state this property we need a definition. For this purpose we consider a compact set $K \subset \mathbb{R}^n$ and an open convex cone $\Gamma \subset \mathbb{R}^n$. We denote by $\mathcal{B}_\Gamma(U)$ the space of hyperfunctions on U which are boundary values of analytic functions defined on $U \times i\Gamma_d$, where $\Gamma_d = \{t \in \Gamma; |t| < d\}$ and by $\mathcal{B}_\Gamma(K)$ the space of hyperfunctions defined in a neighborhood of K which are boundary values of analytic functions defined in a neighborhood of $K \times i\Gamma_d$ for some d.

Both spaces have natural topologies: we mention the one in the case of $\mathcal{B}_\Gamma(K)$. We may regard $\mathcal{B}_\Gamma(K)$ as the inductive limit of the spaces $\mathcal{A}(K_\varepsilon \times i\Gamma_d)$ for $\varepsilon > 0$ and $d > 0$ and the seminorms on $\mathcal{A}(K_\varepsilon \times i\Gamma_d)$ are given then by

$$|f|_A = \sup_{x \in A} |f(x)|, \tag{1.8}$$

where A runs through the family of compact sets in $K_\varepsilon \times i\Gamma_d$.

The main remark to justify our approach is now that we have the following result:

Proposition 1.4. *Let $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ be an operator of form (1.7) for some kernel $\mathcal{K}$ which satisfies* (1.6). *Also let Γ_j, $j = 1, \dots, n+1$, be open convex cones such that their duals form a covering of $\mathbb{R}^n$.*

If $K \subset\subset V$ and $K' \subset\subset U$ are fixed convex sets, we can then find $n+1$ linear continuous operators $T_j : \mathcal{A}'(K) \to \mathcal{B}_{\Gamma_j}(K')$ such that for all $u \in \mathcal{A}'(K)$ $T(u) = \sum_{j=1}^{n+1} T_j(u)$, on the interior of K'.

We now consider, conversely, the following

Definition 1.5. *Let $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ be a linear operator. We say that it is "semi-continuous" if it has the following property:*

for every $K \subset\subset V$ and every $K' \subset\subset U$ there exist open convex cones Γ_j and linear operators T_j, $j = 1, \dots, s$, such that $T_j : \mathcal{A}'(K) \to \mathcal{B}_{\Gamma_j}(K')$ are continuous and such that $T = \sum_{j=1}^{s} T_j$ on $K \times K'$.

An important point in this definition is of course that the ranges of the operators T_j into which T is split are not the same.

The following result is now the kernel theorem:

Theorem 1.6. *Let $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ be a linear operator. Then the following conditions are equivalent:*

i) T is semicontinuous.

ii) There is $\mathcal{K} \in \mathcal{B}(U \times V)$ which satisfies (1.6) such that

$$T(u) = \int_V \mathcal{K}(x, y)u(y)\, dy. \tag{1.9}$$

We also mention that one can reduce theorem 1.2 to theorem 1.6 with the aid of the following result:

Proposition 1.7. *Assume that Q is a compact oriented real-analytic manifold and let Let $\bigcup_j W_j$, $j = 1, \ldots, s$, be an open covering of Q. Then we have a finite system of linear continuous operators S_j, $j = 1, \ldots, s$, $S_j : \mathcal{B}(Q) \to \mathcal{B}(Q)$ such that $\sum_j S_j = \mathrm{id}_{\mathcal{B}(Q)}$ and that $\operatorname{supp} S_j u \subset W_j$ for any $u \in \mathcal{B}(Q)$.*

We now turn to the case of operators $T : \mathcal{A}(V) \to \mathcal{B}(U)$. We have already seen a result of this type for maps $T : \mathcal{A}(V) \to \mathcal{B}(Q)$ with Q a compact orientable manifold and now want to address the case of operators $T : \mathcal{A}(V) \to \mathcal{B}(U)$ with U open. The first questions to be discussed are how we want to associate linear maps with kernels and what substitute for continuity we shall allow.

We start with a brief discussion of the relevant class of kernels and how an operator is associated with them. The first case to be considered is when we are given a kernel $\mathcal{K} \in \mathcal{A}'(U \times V)$ and associate with it a linear operator $T : \mathcal{A}(V) \to \mathcal{A}'(U)$ by the prescription $T(u)(f) = \mathcal{K}(f \otimes u)$. This is roughly speaking the case considered in theorem 1.2, but this class of operators is perhaps somewhat too special. More interesting is in fact the case when we consider a kernel $\mathcal{K} \in \mathcal{B}(U \times V)$ with the following property: for every $U' \subset\subset U$ we have that the set

$$[U' \times V] \cap \operatorname{supp} \mathcal{K} \text{ is relatively compact in } U \times V. \tag{1.10}$$

In analogy with the terminology used in the C^∞-category, such kernels are called "properly supported". (The condition above is equivalent to that the canonical projection $U \times V \to U$ restricted to $\operatorname{supp} \mathcal{K}$ becomes a proper map, i.e., $\mathcal{K}$ is a global section of the proper direct image of $\mathcal{B}$ by that projection.) With such kernels we can associate a linear map $T : \mathcal{A}(V) \to \mathcal{B}(U)$ by the formula

$$T(u) = \int_V \mathcal{K}(x, y)\, u(y)\, dy. \tag{1.11}$$

In the case $\mathcal{K} \in \mathcal{A}'(U \times V)$, we can see that (1.11) defines the same operator T as that by the duality $T(u)(f) = \mathcal{K}(f \otimes u)$ from Fubini theorem of integrations of hyperfunctions. Since the integral in (1.11) is defined locally in variables x, we can understand the operator T with a properly supported kernel $\mathcal{K}$ also via duality using cutting-off, as follows: we choose $U' \subset\subset U$ and denote $K = \overline{U'}$. Also choose $\mathcal{K}' \in \mathcal{A}'(U \times V)$ such that $\mathcal{K} = \mathcal{K}'$ on $U' \times V$, $\operatorname{supp} \mathcal{K}' \subset K \times V$. Such an analytic functional exists in view of (1.10). Next, we consider the map

$T' : \mathcal{A}(V) \to \mathcal{A}'(U)$ associated with $\mathcal{K}'$ by $(T'u)(f) = \mathcal{K}'(f \otimes u)$. Then for $u \in \mathcal{A}(V)$, $T(u) \in \mathcal{B}(U)$ defined by (1.11) can be characterized by the property that $T(u)$ is equal to $T'(u)$ on U'.

The next point is to discuss "semicontinuity" for operators defined by kernels. The definition which we shall use here is a variant of the one considered above for the case of operators defined on $\mathcal{A}'(V)$, namely

Definition 1.8. *Let $T : \mathcal{A}(V) \to \mathcal{B}(U)$ be a linear operator. We shall say that T is semicontinuous if for every $K \subset\subset U$ we can find open convex cones $\Gamma_j, j = 1, \ldots, s$, and linear continuous operators $T_j : \mathcal{A}(V) \to \mathcal{B}_{\Gamma_j}(K)$, such that for all $u \in \mathcal{A}(V)$ $T(u) = \sum_{j=1}^{s} T_j(u)$ on the interior of K.*

Proposition 1.9. *Let $\mathcal{K}$ be a kernel which satisfies the condition (1.10) for every $U' \subset\subset U$. The map $T : \mathcal{A}(V) \to \mathcal{B}(U)$ defined above is then semicontinuous.*

We now state the kernel theorem for maps $T : \mathcal{A}(V) \to \mathcal{B}(U)$.

Theorem 1.10. *Let $T : \mathcal{A}(V) \to \mathcal{B}(U)$ be linear and semicontinuous. Then there is a properly supported hyperfunction kernel $\mathcal{K} \in \mathcal{B}(U \times V)$ such that $T(u) = \int_U K(x, y) u(y)\, dy$.*

The proofs of the results in this paper shall be published elsewhere.

2 Uniqueness theorems

The arguments in the proofs are in most cases "local". In the case of the kernel-theorems this means that we shall construct some local kernels which must then be glued together. This is done with the aid of some "uniqueness theorems" which we shall mention in this section. The argument in one of the uniqueness theorems can be based on a regularity result, which is a corollary of a theorem of F.Bastin-P.Laubin and which we mention separately since it seems of independent interest. We recall at first the result of F.Bastin-P.Laubin (cf. [1]):

Theorem 2.1. *Let $\mathcal{K} \in \mathcal{A}'(\mathbb{R}^n \times \mathbb{R}^m)$ and $y^0 \in \mathbb{R}^m$, $\eta^0 \in \mathbb{R}^m$, such that*

$$(x, y^0, 0, -\eta^0) \notin \mathrm{WF}_A\, \mathcal{K} \tag{2.1}$$

for every $x \in \mathbb{R}^n$. If $x^0 \in \mathbb{R}^n$, $\xi^0 \in \mathbb{R}^n$ and $(x^0, \xi^0) \notin \mathrm{WF}_A(\int \mathcal{K}(x, y) g(y) dy)$ for every $g \in \mathcal{B}(\mathbb{R}^m)$ satisfying

$$\mathrm{WF}_A\, g \subset \{(y^0, t\eta^0); t > 0\} \tag{2.2}$$

then $(x^0, y^0, \xi^0, -\eta^0) \notin \mathrm{WF}_A\, \mathcal{K}$.

Note that this theorem also holds in the case $\eta^0 = 0$, when the assumption (2.1) no more makes sense and provided that we replace condition (2.2) by the condition that the test function g runs through the elements in $\mathcal{A}(\mathbb{R}^m)$. See also [1].

The following result is a corollary of theorem 2.1:

Theorem 2.2. *Let $\mathcal{K} \in \mathcal{B}(U \times V)$ be a kernel which satisfies the wave front set estimate (1.6). Assume that the operator $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ defined in (1.7) actually maps $\mathcal{A}'(V)$ into $\mathcal{A}(U)$. Then $\mathcal{K}$ is real-analytic on $U \times V$.*

Theorem 2.3. *Let $\mathcal{K} \in \mathcal{B}(U \times V)$ be a kernel which satisfies the wave front set estimate (1.6). If $U' \subset\subset U$ and $V' \subset\subset V$ are given such that $T(u) = 0$ on U' for every $u \in \mathcal{A}'(V')$, then $\mathcal{K}$ must vanish on $U' \times V'$.*

(In view of theorem 2.2, $\mathcal{K}$ is real-analytic on $U' \times V'$. Then the result is obvious.)

It is also possible to give an alternative proof of theorem 2.3 which is not based on theorem 2.2. Since the statement is local, it is no loss of generality to assume that $U' = U, V' = V$. Thus we may prove that $T(\mathcal{A}'(U)) = \{0\}$ implies $\mathcal{K} = 0$ on $U \times V$. The argument is based on a variant of the Malgrange-Zerner theorem for hyperfunctions with real-analytic parameters, which was proved by T.Oshima and independently by K.Kataoka. (For a proof of this result, which was not published by T.Oshima, respectively K.Kataoka, see Theorem 4.4.7' in A.Kaneko [8].)

Theorem 2.4 (Oshima-Kataoka). *Let $\mathcal{K}(x,y) \in \mathcal{B}(U \times V)$ be a hyperfunction with real-analytic parameter y satisfying*

$$\mathcal{K}(x,y)|_{y=y^0} = 0, \textit{ for any } y^0 \in V.$$

Then $\mathcal{K} = 0$ on $U \times V$.

To apply this result, we first recall that by definition (cf. [8]) it is said that $\mathcal{K}$ is a "hyperfunction with real-analytic parameter y" precisely when the condition (1.6) holds. Moreover, we recall that if (1.6) holds, then the restrictions $\mathcal{K}(x,y)|_{y=y^0}$ are well-defined in microlocal analysis. We also note that $\mathcal{K}(x,y)|_{y=y^0} = T(\delta_{y^0})(x)$, where δ_{y^0}, the Dirac distribution at y^0. The assumption in theorem 2.3 now implies that $T(\delta_{y^0})(x) = 0$, so $\mathcal{K}(x, y^0) = 0$ for every y^0. The theorem of Oshima-Kataoka therefore gives the desired result. Actually, the second argument also gives the following strengthened form of theorem 2.3:

Theorem 2.5. *Let $\mathcal{K} \in \mathcal{B}(U \times V)$ be a kernel which satisfies the wave front set estimate (1.6). If $U' \subset\subset U$ and $V' \subset\subset V$ are given such that $T(u) = 0$ on U' for every $\delta_y, y \in V'$, then $\mathcal{K}$ must vanish on $U' \times V'$. In particular it follows then that $T(u) = 0$ on U' for every $u \in \mathcal{A}'(V')$.*

Proposition 2.6. *Let $\mathcal{K} \in \mathcal{A}'(U \times V)$ and consider the operator $T : \mathcal{A}(V) \to \mathcal{A}'(U)$ given by $Tu = \int_V \mathcal{K}(x,y)u(y)\,dy$. Also consider an open set $U' \subset\subset U$ such that*

$$Tu(x) = 0 \textit{ for } x \in U', \textit{ for any } u \in \mathcal{A}(V). \tag{2.3}$$

Then $\mathcal{K}$ is real-analytic on $U' \times V$ and therefore must vanish there.

3 Instances of semicontinuity

Example 3.1. *The simplest case of semicontinuous operators is when we know that $T : \mathcal{A}'(V) \to \mathcal{B}_\Gamma(U)$ (for some given convex cone $\Gamma \subset \dot{\mathbb{R}}^n$), assuming that T is continuous. It is in this form in which kernels in hyperfunction theory traditionally appear.*

Remark 3.2. *The identity map $I : \mathcal{A}'(U) \to \mathcal{A}'(U)$ is semicontinuous. (This is trivial but it also follows from the fact that the identity is given by the kernel $\mathcal{K}(x, y) = \delta(x - y)$.) More generally, we have:*

Proposition 3.3. *Let $T : \mathcal{A}'(V) \to \mathcal{A}'(U)$ be a linear operator.*
I. Then there are equivalent:

a) T is semicontinuous as an operator $\mathcal{A}'(V) \to \mathcal{B}(U)$.

b) T is continuous as an operator $\mathcal{A}'(V) \to \mathcal{A}'(U)$.

II. In both cases, the following happens:
B) for every $K \subset\subset V$ there is $K' \subset\subset U$ such that $\operatorname{supp} T(u) \subset K'$ *if $u \in \mathcal{A}'(K)$.*

Also the following variants of the preceding result show that semicontinuity is a useful notion:

Proposition 3.4. *Let $X \subset \mathcal{B}(U)$ be a locally convex topological vector space and consider a linear operator $T : \mathcal{A}'(V) \to X$. Then the following two statements are equivalent in the cases $X = \mathcal{L}^2(U)$, $X = \mathcal{A}(U)$ and $X = \mathcal{D}'_L(U)$, where $\mathcal{D}'_L(U)$ are Denjoy-Carleman ultradistributions associated with some fixed sequence $L = \{L_j\}$, assuming that the corresponding Denjoy-Carleman class is non-quasianalytic:*

a) T is semicontinuous as an operator from $\mathcal{A}'(V)$ to $\mathcal{B}(U)$.

b) T is continuous as an operator from $\mathcal{A}'(V)$ to X.

We now turn to a situation which is heuristically speaking a combination of $T : \mathcal{A}'(V) \to \mathcal{A}'(U)$ and $T : \mathcal{A}'(V) \to \mathcal{A}(U)$. At first we shall exhibit a rather general space of hyperfunctions which has a natural topology. For simplicity we shall not strive at maximum generality.

Let $U \subset \mathbb{R}^n$ be open and fix $K' \subset\subset U$ compact in U. We consider the space $\mathcal{B}_{K'}(U) = \{f \in \mathcal{B}(U); \operatorname{singsupp} f \subset K'\}$. A topology can be introduced on $\mathcal{B}_{K'}(U)$ as follows: we fix $\varphi \in C_0^\infty(U)$ to be identically one in a neighborhood of K', and consider the map $f \to (f_{|\complement K'}, \varphi f)$ as a map

$$\mathcal{B}_{K'}(U) \to \begin{pmatrix} \mathcal{A}(U \cap \complement K') \\ \times \\ \mathcal{A}'(K'') \end{pmatrix},$$

where K'' is the support of φ. The topology in $\mathcal{B}_{K'}(U)$ is then the weakest one which makes this continuous.

Remark 3.5. *The topology defined above does not depend on the choice of φ.*

In fact, it is clearly a locally convex topology and we can choose as a fundamental family of seminorms generating the topology on $\mathcal{B}_K(U)$ the seminorms on $\mathcal{A}(U \cap \complement K')$, to which we add all seminorms of form $f \to |\varphi f|_q$, where q runs through the family of seminorms on $\mathcal{A}'(U)$. Only the latter will be affected if we change φ and if φ and φ' are two $C_0^\infty(U)$ functions which are identically one in a neighborhood of K', then $|\varphi f|_q \leq |\varphi' f|_q + |(\varphi - \varphi')f|_q \leq |\varphi' f|_q + \sup_{x \in \mathrm{supp}(\varphi - \varphi')} |f(x)|$. (Or similar.)

Comment on the topology of $\mathcal{A}'(\tilde{K})$ when $\tilde{K}$ is a compact. Let q be a seminorm given in the following way:

$$|u|_q = \sup_{g \in \mathcal{M}} |u(g)|,$$

where $\mathcal{M}$ is some previously fixed bounded set in $\mathcal{A}(\tilde{K})$. In particular, cf. the characterization of bounded sets in $\mathcal{A}(K)$ above, there is $\varepsilon > 0$ so that $g \in \mathcal{M}$ implies $g \in \mathcal{A}(K_\varepsilon)$ and for every $\varepsilon' < \varepsilon$ a constant $c > 0$ such that

$$|g(x)| \leq c, \forall x \in K_{\varepsilon'}, \forall g \in \mathcal{M}.$$

We conclude that if f is a function supported in $\{x; \varphi'(x) - \varphi(x) \neq 0\}$ that $|f|_q \leq |\int g(x)f(x)\,dx| \leq c \sup |f(x)|$.

Remark 3.6. *Let $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ be a map such that for all $K \subset\subset V$ there is $K' \subset\subset V$ for which $T(\mathcal{A}'(K)) \subset \mathcal{B}_{K'}(U)$. The "induced" maps $T_K : \mathcal{A}'(K) \to \mathcal{B}_{K'}(U)$ are then continuous precisely if for every $\varepsilon > 0$ and every $\varphi \in C_0^\infty(U)$ which is identically one in a neighborhood of K' and has support in K'_ε the two maps $T'_K : \mathcal{A}'(K) \to \mathcal{A}(\complement K')$ and $T''_K : \mathcal{A}'(K) \to \mathcal{A}'(K'_\varepsilon)$, $T''_K(u) = \varphi T(u)$ are continuous.*

We conclude the section introducing a new variant of semicontinuity and stating the most relevant results on it.

We recall the following definition of Bros-Iagolnitzer [3] (in the presentation, we follow [10]):

Definition 3.7. *a) A set $\Omega \subset \mathbb{C}^n$ is called a "profile" if it is open and if $x \in \Omega$ implies that* $\mathrm{Re}\, x + it \,\mathrm{Im}\, x \in \Omega$ *for every $t \in \mathbb{R}_+$. The set* $\{\mathrm{Re}\, x; x \in \Omega\}$ *is called the base of Ω. In this paper we shall always assume that $\Omega \subset U + i\mathbb{R}^n$ for some open set $U \subset \mathbb{R}^n$ and that the base of Ω is U.*

b) A tuboid with profile Ω is an open set $W \subset \mathbb{C}^n$ such that for every compact set $K \subset \Omega$ there is a constant $d > 0$ such that the set $\{\mathrm{Re}\, x + it \,\mathrm{Im}\, x; x \in K, 0 < t < d\}$ *lies in W.*

Remark 3.8. *Let Ω be a profile with base U and fix $x^0 \in U$. Then there is $\varepsilon > 0$ and an open convex cone $\Gamma \subset \mathbb{R}^n$ such that $\{x; |\operatorname{Re} x - x^0| < \varepsilon, \operatorname{Im} x \in \Gamma\} \subset \Omega$.*

Definition 3.9. *For a given profile Ω we denote by $\mathcal{A}(\tilde{\Omega})$ the union of the sets $\mathcal{A}(W)$ where W runs through the set of tuboids with profile Ω. $\mathcal{A}(\tilde{\Omega})$ has a natural topology: the inductive limit topology of the $\mathcal{A}(W)$.*

Lemma 3.10. *Let $\mathcal{K}(x, y) \in \mathcal{B}(U \times V)$ be a hyperfunction satisfying (1.6), and $\Gamma_j \subset \dot{\mathbb{R}}^n$, $j = 1, \ldots, J$, proper open convex cones such that their duals form a covering of $\dot{\mathbb{R}}^n$. Then there exist open sets $\Omega_j \subset \mathbb{C}^{n+m}$ and holomorphic functions $F_j(x, y) \in \mathcal{O}(\Omega_j)$, $(j = 1, \ldots, J)$ satisfying the following properties.*

(1) For any open set $W \subset\subset U \times V$, we can take open convex cones $\Lambda_j \subset \mathbb{R}^{n+m}$ $(j = 1, \ldots, J)$ with $\Lambda_j \supset \Gamma_j \times \{0\}$ and a positive constant $d > 0$ for which

$$\Omega_j \supset \{(x, y) \in W + i\Lambda_j; |\operatorname{Im} x| < d, |\operatorname{Im} y| < d\}$$

holds for any j.

(2) $\mathcal{K}$ is written as a sum of boundary values of F_j's, i.e.,

$$\sum_j b(F_j) = \mathcal{K} \text{ on } U \times V.$$

Here each $b(F_j)$ is defined, locally in the $\operatorname{Re}(x, y)$ variables, as the boundary value from the region $\operatorname{Im}(x, y) \in \Lambda_j$ when $\operatorname{Re}(x, y) \in W$, where W and Λ_j are as in (1).

Definition 3.11. *Let $T : \mathcal{A}(V) \to \mathcal{B}(U)$. We say that T is semicontinuous with respect to profiles, if we can find profiles $\Omega_1, \ldots, \Omega_s$ and linear continuous maps $T_j : \mathcal{A}(V) \to \mathcal{A}(\tilde{\Omega}_j)$ such that $T(u) = \sum_{j=1}^{s} b(T_j(u))$ for every $u \in \mathcal{A}(V)$.*

Theorem 3.12. *The conditions i) and ii) in theorem 1.6 are also equivalent to the condition*
iii) T is semicontinuous with respect to profiles.

The argument is based on

Proposition 3.13. *Let $\tilde{\Omega}$ be a tuboid with profile Ω. Let $T : \mathcal{A}'(V) \to \mathcal{A}(\tilde{\Omega})$ be a linear continuous map. Fix a compact set K in V and $x^0 \in U$. Then there is $\varepsilon > 0$, $d > 0$ and an open convex one $\Gamma \subset \mathbb{R}^n$ such that $T(\mathcal{A}'(K)) \subset \mathcal{A}(\{x, |\operatorname{Re} x - x^0| < \varepsilon, \operatorname{Im} x \in \Gamma_d\})$ and such that the map $T : \mathcal{A}'(K) \to \mathcal{A}(\{x, |\operatorname{Re} x - x^0| < \varepsilon, \operatorname{Im} x \in \Gamma_d\})$ is continuous.*

4 Comments and remarks; examples

The history of kernel theorems in Lebesgue spaces and in abstract functional analysis is long and need not be recalled here. We recall however that the notion of nuclear topological vector spaces is deeply related to such theorems. A number of results on kernel theorems are related in some way or another to the Schwartz kernel theorem mentioned in the introduction. Extensions to different classes of generalized functions have been given in [14], [9], [4], [5], [15]. In particular, [14], [4], [5], [15] study the kernel theorem in Fourier-hyperfunctions. The result of [16] is closer in spirit to our own results, although it refers apparently to kernel theorems for compactly supported cohomology classes rather than hyperfunctions. We should also mention that in most situations linear maps $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ in pde will be constructed in such a way that the kernel of the map can be read off (perhaps modulo smoothing kernels, or just microlocally) from the construction. To some extent our main results say that we also have the converse: reasonable maps $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ will exist only if we can find some kernel for them. It is also clear from this that it is easy to give interesting examples of kernels in pde which are as in theorem 1.6.

If, on the other hand, no additional condition on a linear map $T : \mathcal{A}'(V) \to \mathcal{B}(U)$ is assumed, we cannot expect the map to have a kernel. Before we give explicit examples of linear operators without a kernel, we start with some general considerations. Consider the subspace $\mathcal{A}'(\mathbb{R}\setminus\{0\}) \subset \mathcal{A}'(\mathbb{R})$, and let $T : \mathcal{A}'(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$ be a non-trivial linear map which vanishes on $\mathcal{A}'(\mathbb{R} \setminus \{0\})$ (or, what is the same, assume that $T(\delta_y) = 0$ when $y \neq 0$). Suppose that T is given by a kernel $K(x, y) \in \mathcal{B}(\mathbb{R} \times \mathbb{R})$ which satisfies the wave front set estimate $\{(x, y, 0, \eta); x \in \mathbb{R}, y \in \mathbb{R}, \eta \neq 0\} \cap \mathrm{WF}_A\, \mathcal{K} = \emptyset$, which is (1.6) in the present situation. Then the restriction $K(x, y)|_{y\neq 0}$ of the kernel corresponds to the restriction $T_{|\mathcal{A}'(\mathbb{R}\setminus\{0\})}$ of the map, which is identically zero. Thus from theorem 2.3 we have $K(x, y)_{|y\neq 0} = 0$, which means that the support of K is included in $\{(x, y); y = 0\}$.

Holmgren's theorem, together with this support estimate and the wave front set estimate above, shows that $K = 0$ holds also on $\{y = 0\}$, which contradicts the non-triviality of T. In order to find linear operators $T : \mathcal{A}'(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$ without a kernel, we may thus simply construct linear operators which vanish on $\mathcal{A}'(\mathbb{R} \setminus \{0\})$.

To apply this we consider a linear basis $\{g_\iota, \iota \in I\}$ in $\mathcal{A}'(\mathbb{R})$ which contains as a subbasis a basis in the space $\mathcal{A}'(\mathbb{R}\setminus\{0\})$ (or, alternatively, comprizes δ_y, $y \neq 0$) and also the δ-function at zero. Also consider a linear map $L : \mathcal{A}'(\mathbb{R}) \to \mathcal{A}'(\mathbb{R})$ defined by the condition $L(\delta) = \delta$, $L(g) = 0$ for every $g \in \{g_\iota, \iota \in I\} \setminus \{\delta\}$. By the very construction, T vanishes on $\mathcal{A}'(\mathbb{R} \setminus \{0\})$ so in view of the above discussion, $\mathcal{K}$ must vanish identically, contrary to the assumption that $L(\delta) = \delta$.

Also in our second example, which is more natural, we start with some preparations. We consider $f \neq 0$ in $C_0^\infty(U)$ and denote by $\tilde{T} : \mathcal{E}'(U) \to \mathcal{E}'(U)$ the operator of multiplication with f. We claim that there is no semicontinuous extension of this operator to an operator $T : \mathcal{A}'(U) \to \mathcal{B}(U)$. Assume in fact that there were such an extension T. T were then given by a kernel $\mathcal{K}$ which had to satisfy the wave front set estimate (1.6) and therefore $\mathcal{K}$ were a hyperfunction in $U \times U$ having y as a real-analytic parameter. The restriction of

$\mathcal{K}(x,y)$ to $y = y^0$ were equal to $\int_U \mathcal{K}(x,y)\delta_{y^0}(y)\,dy$, so we could conclude at first that $\mathcal{K}(x,y^0)$ vanished near x when $x \notin \operatorname{supp} f$, (see section 2) and then, applying the theorem of Kataoka-Oshima, that the support of $\mathcal{K}$ would have to lie in $K \times U$, where $K = \operatorname{supp} f$. In particular, T would define a continuous operator from $\mathcal{A}'(U)$ to $\mathcal{A}'(K)$ and it is also immediate that if $u \in \mathcal{A}'(U) = 0$ in a neighborhood of $\operatorname{supp} f$, then $T(u) = 0$. To reach a contradiction, we now observe that from all this and the uniqueness theorem we could conclude that $\operatorname{supp} \mathcal{K}$ were compact, which is incompatible with the wave front set estimate (1.6) for $\mathcal{K}$.

This also immediately leads to a new example of a linear operator without kernel: in fact, it suffices to extend the operator $\tilde{T}$ defined above to a linear operator on all of $\mathcal{A}'(U)$ in an arbitrary way. By the above such an operator will have no kernel.

We now turn our attention to semicontinuous operators $T : \mathcal{A}(V) \to \mathcal{B}(U)$. Our interest lies here in maps with a rather "large" image, for example we would like to have that $T(\mathcal{A}(V))$ is not a subset in $\mathcal{D}'(U)$, since otherwise we would remain in the realm of classical kernel theorems. A simple idea to construct such maps is the following: we consider $f \in C_0^\infty(V)$ and let J be some infinite order operator $\mathcal{B}(V) \to \mathcal{B}(U)$; then the operator $u \to J(fu)$ will in general map real-analytic functions into hyperfunctions. While this example is not related to any explicit application, it is based on a simple idea of producing examples of operators $T : \mathcal{A}(V) \to \mathcal{B}(U)$ which we can also apply in other occasions: we let T be a superposition of an operator from real-analytic functions to C^∞-functions, with an operator which transforms C^∞- functions to hyperfunctions. Since the main interests of the present authors lies in pde, we shall apply this principle to some question in pde, although we admit of course that also this example is somewhat artificial. We should also say perhaps beforehand that it seems difficult to think of operators which are useful in pde and which transform real-analytic functions into general hyperfunctions, without that the image of the operator were already contained in some class of Gevrey-ultradistributions and indeed our operator will transform real-analytic functions into Gevrey-2 ultradistributions.

To study such a situation we denote by $q(D_x, D_t)$ the operator $(\partial/\partial t)^2 - i(\partial/\partial x)$ and consider the weakly hyperbolic Cauchy problem

$$q(D_x, D_t)u(x,t) = f(x,t),\ u(x,0) = g_0(x),\ u_t(x,0) = g_1(x). \tag{4.1}$$

We denote by T the map which sends (f, g_0, g_1) to the solution u of this problem and by $T_1(f) = T(f,0,0)$, $T_2(g_0) = T(0,g_0,0)$, $T_3(g_1) = T(0,0,g_1)$.

Lemma 4.1. *We denote by X the space of Gevrey-2 ultradistributions. Then we have that*

a) $T_1(C^\infty(\mathbb{R}^2)) \subset X$, $T_2(C^\infty(\mathbb{R})) \subset X$, $T_3(C^\infty(\mathbb{R})) \subset X$,
but

b) there are functions $f \in C_0^\infty(\mathbb{R}^2)$, $g_0, g_1 \in C_0^\infty(\mathbb{R})$, such that there is no $\delta > 0$ for which $T_1(f) \in \mathcal{D}'(\{(x,t); x \in \mathbb{R}, |t| < \delta\})$, respectively $T_2(g_0) \in \mathcal{D}'(\{(x,t); x \in \mathbb{R}, |t| < \delta\})$, respectively $T_3(g_1) \in \mathcal{D}'(\{(x,t); x \in \mathbb{R}, |t| < \delta\})$.

It also follows that there is a $C_0^\infty(\mathbb{R})$ function g such that $T_1(tg_x)$ is not a distribution for $|t| < \delta$, whatever $\delta > 0$ we choose.

To apply lemma 4.1, we consider a $C_0^\infty(\mathbb{R})$-function g for which $T_1(tg_x)$ is not a distribution on $|t| < \delta$ for some $\delta > 0$. We fix a small neighborhood W of some point x^0 such that $w = 1 + tg_x(x)$, possibly after shrinking δ suitably, does not vanish on $\tilde{W} = W \times \{t; |t| < \delta\}$ and such that $T_1(tg_x)$ is not a distribution on $\tilde{W}$. Denote then by f the function $f(x,t) = [(\partial/\partial t)w(x,t)]/w(x,t)$. It is $C^\infty(\tilde{W})$ and satisfies by construction the Cauchy problem $w_t(x,t) - f(x,t)w(x,t) = 0$, $w(x,0) = 1$.

We can now finally consider the system

$$\begin{aligned} q(D_x, D_t)v - w &= 0, \\ w_t + f(x,t)w &= 0, \end{aligned} \tag{4.2}$$

with Cauchy conditions $v_{|t=0} = v_{t|t=0} = 0$, $w_{|t=0} = h$. The second equation together with the initial condition for w completely determines w, and the first equation, together with the remaining Cauchy conditions, becomes an inhomogeneous Cauchy problem for v with vanishing Cauchy data. The solution to this problem is not a distribution when $h \equiv 1$. The map which sends real-analytic functions h into the solution v of the aforementioned Cauchy problem (or, for that matter, into the restriction to $t = t^0$ of that solution) will then have values in Gevrey-2 ultradistributions, but the solutions will not in general be distributions.

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