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REGULARITY OF $\mathcal{D}$-MODULES ASSOCIATED TO A SYMMETRIC PAIR: A CONJECTURE BY SEKIGUCHI

YVES LAURENT

1. Introduction

Let $G_{\mathbb{R}}$ be a real semi-simple Lie group, $\mathfrak{g}_{\mathbb{R}}$ its Lie algebra. A differential operator on $G_{\mathbb{R}}$ is bi-invariant if it is invariant under the left and the right actions of $G_{\mathbb{R}}$ on itself. A distribution T on $G_{\mathbb{R}}$ is an invariant eigendistribution if T is invariant under the adjoint action of $G_{\mathbb{R}}$ and T is an eigenvalue of each bi-invariant operator on $G_{\mathbb{R}}$. The characters of irreducible representations of $G_{\mathbb{R}}$ satisfy these properties. A classical theorem of Harish-Chandra asserts:

Theorem 1.1. *Any invariant eigendistribution is L^1_{loc}.*

After transfer to the Lie algebra by the exponential map and conjugation by a suitable function, T is solution of a holonomic $\mathcal{D}$-module $\mathcal{M}_\lambda$ defined on the complexification $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{R}}$. In this paper, we will consider only complex Lie groups and Lie algebras.

The module $\mathcal{M}_\lambda$ has been studied by R. Hotta and M. Kashiwara [2]. In particular, using a variant of Harish-Chandra theorem, they proved that this module is **regular holonomic**. Let us recall that a $\mathcal{D}$-module is holonomic if its characteristic variety is lagrangian while regularity is a generalization of Fuchsian differential equation. If $\mathcal{M}$ is holonomic regular, its formal solutions are convergent, its solutions holomorphic outside a hypersurface are meromorphic and in the real domain, hyperfunction solutions are distributions.

A natural extension of the action of a semi-simple Lie group on its Lie algebra is *a symmetric pair.* If $\mathfrak{g}$ is a reductive Lie algebra with an involution, it splits into its even and odd part, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The Lie group K associated to $\mathfrak{k}$ acts on $\mathfrak{p}$ and we say that $(\mathfrak{g}, \mathfrak{p})$ is a symmetric pair. The module $\mathcal{M}_\lambda$ may still be defined in this case.

J. Sekiguchi proved that Harish-Chandra's theorem is not true in general and defined a condition on symmetric pairs under which the theorem is true (for hyperfunctions and for distributions). He conjectured that the module is regular [9].

We want to show here that

Theorem 1.2. *The module $\mathcal{M}_\lambda$ is holonomic regular for any symmetric space.*

We will not give all the details of the proof which may be found in [7].

2. Microcharacteristic varieties and regularity

Let X be a complex analytic manifold and Y be submanifold of X. We denote by $p : T_Y X \to X$ its normal bundle and by $\pi : T^*_Y X \to X$ its conormal bundle. Remark that the duality between the fiber bundles $T_Y X$ and $T^*_Y X$ defines an isomorphism $T^* T_Y X \simeq T^* T^*_Y X$. We will also denote $\Lambda = T^*_Y X$.

Let $\mathcal{I}_Y$ be the ideal of $\mathcal{O}_X$ defining Y. Kashiwara's V-filtration [3] is defined on differential operators by:

$$V_k\mathcal{D}_X = \{ P \in \mathcal{D}_X|_Y \mid \forall l \in \mathbb{Z}, P\mathcal{I}_Y^l \subset \mathcal{I}_Y^{l+k} \} \tag{2.1}$$

(with $\mathcal{I}_Y^l = \mathcal{O}_X$ if $l \leq 0$).

In local coordinates such that $Y = \{ (x,t) \in X \mid t = 0 \}$, the operators x_i and $\frac{\partial}{\partial x_i}$ are of order 0 for the V- filtration while operators t_j are of order -1 and $\frac{\partial}{\partial t_j}$ of order $+1$.

By definition, the associated graded ring $gr_V\mathcal{D}_X = \bigoplus V_k\mathcal{D}_X/V_{k-1}\mathcal{D}_X$ operates on $\bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k-1}$. Let $\mathcal{O}_{[T_YX]}$ be the sheaf of holomorphic functions on T_YX which are polynomial in the fibers of $p : T_YX \to X$. There is a canonical isomorphism $\bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k-1} \simeq p_*\mathcal{O}_{[T_YX]}$ which defines an identification of $gr_V\mathcal{D}_X$ with $p_*\mathcal{D}_{[T_YX]}$, the sheaf of differential operators on T_YX with coefficients in $\mathcal{O}_{[T_YX]}$.

Let $P \in \mathcal{D}_X$ be a differential operator on X and denote by $\sigma_Y(P)$ its image in $gr_Y\mathcal{D}_X$. As $\sigma_Y(P)$ is an operator on T_YX, its principal symbol $\sigma(\sigma_Y(P))$ is a function on the cotangent bundle T^*T_YX hence as a function on $T^*\Lambda$ through the isomorphism $T^*T_YX \simeq T^*\Lambda$(see [6] for the details).

Then we say that the differential operator P is regular along $\Lambda = T_Y^*X$ if the order of $\sigma_Y(P)$ is equal to the order of P, (here the order is the usual order of differential operators) and define:

$$\sigma_{\Lambda(\infty,1)}(P) = \begin{cases} \sigma(\sigma_Y(P)), & \text{if P is regular;} \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Let $\mathcal{I}$ be a coherent ideal of $\mathcal{D}_X$ and $\mathcal{M}$ be the $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{I}$. As is well known, the characteristic variety of $\mathcal{M}$ is given by

$$Ch(\mathcal{M}) = \{ (x,\xi) \in T^*X \mid \forall P \in \mathcal{I}, \sigma(P)(x,\xi) = 0 \}$$

In the same way we define the microcharacteristic variety of $\mathcal{M}$ as the analytic subset of $T^*\Lambda$ given by

$$Ch_{\Lambda(\infty,1)}(\mathcal{M}) = \{ \lambda \in T^*\Lambda \mid \forall P \in \mathcal{I}, \sigma_{\Lambda(\infty,1)}(P)(\lambda) = 0 \}$$

This definition may be extended to any coherent $\mathcal{D}_X$-module $\mathcal{M}$ by means of a bifiltration (see [8][5][6]) but we will not use this here.

Assume now that $\mathcal{M}$ is holonomic. Then its characteristic variety is lagrangian and its irreducible components are generically the conormal to a submanifold of X. So if Λ is an irreducible component of the characteristic variety of $\mathcal{M}$, we say that $\mathcal{M}$ has regular singularities along Λ if $Ch_{\Lambda(\infty,1)}(\mathcal{M}) = \Lambda$ on a dense open subset of Λ. It has been proved in [5, Theorem 3.1.7.] that this definition is equivalent to the original one given by Kashiwara-Kawaï in [4, Definition 1.1.11.].

Definition 2.1. (Kashiwara-Kawaï [4]) A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is **regular** if it has regular singularities along each irreducible component of its characteristic variety.

3. Quasi-homogeneous microcharacteristic variety

The definitions of section 2 are associated to the canonical action of $\mathbb{C}$ on the fibers of T_Y^*X. Here we will give similar definitions in a quasi-homogeneous case. Intrinsic definitions have been given in [1] but for simplicity, we will give them here in local coordinates.

So let $Y = \{ (x_1, \ldots, x_p, t_1, \ldots t_d) \in X \mid t = 0 \}$, $(m_1, \ldots, m_d)$ **strictly positive** relatively prime integers and consider the vector field $\theta_m = \sum_{i=1}^{d} m_i t_i \frac{\partial}{\partial t_i}$.

We say that P is quasi-homogeneous of degree k if $[P, \theta_m] = kP$ and that P is of order k for the V^{θ_m}-filtration if P may be written as a (convergent) series $\sum_{l \leq k} P_l$ with P_l homogeneous of degree l. Remark that if all m_i are equal to 1, this filtration is the V-filtration (2.1).

If $P = \sum_{l \leq k} P_l$ with P_l homogeneous of degree l and $P_k \not\equiv 0$, we denote $\sigma_Y(P) = P_k$ and say that P is quasi-regular along Y if the order of $\sigma_Y(P)$ is equal to the order of P. We define $\sigma_\Lambda^{\theta_m}(\infty,1)(P)$ by the formula (2.2) and the microcharacteristic variety $Ch_\Lambda^{\theta_m}(\infty,1)(\mathcal{M})$.

Let $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ be a holonomic $\mathcal{D}_X$-module. If Λ is an irreducible component of the characteristic variety of $\mathcal{M}$, we say that $\mathcal{M}$ is quasi-regular along Λ if $Ch_\Lambda^{\theta_m}(\infty,1)(\mathcal{M}) = \Lambda$ on a dense open subset of Λ.

Theorem 3.1. [7, Corollary 1.4.4] *A holonomic module is regular if it is regular or quasi-regular along each irreducible component of its characteristic variety.*

This theorem has been proved in [7] using a ramification map $(t_1, \ldots, t_d) \mapsto (t_1^{m_1}, \ldots, t_d^{m_d})$ and the fact that regularity is preserved under inverse and direct image (Kashiwara-Kawaï [4]).

We will now show that, under suitable conditions, the microcharacteristic varieties are preserved under inverse images. Let $\varphi : Z \to X$ be an analytic map. A vector field u on Z is said to be *tangent to the fibers of* φ if $u(f \circ \varphi) = 0$ for all f in $\mathcal{O}_X$. A differential operator P on Z is said to be *invariant under* φ if there exists a differential operator A on X such that $P(f \circ \varphi) = A(f) \circ \varphi$ for all f in $\mathcal{O}_X$. If φ has a dense range in X, A is uniquely determined by P and we will denote by $A = \varphi_*(P)$ the image of P in $\mathcal{D}_X$ under this ring homomorphism.

Let $Z = \mathbb{C}^{p+d}$ and $\varphi : Z \to X = \mathbb{C}^d$ defined by $(\varphi_1, \ldots, \varphi_d)$ where φ_i is homogeneous of degree m_i. Let E be the Euler vector field and $\theta = \sum m_i t_i \frac{\partial}{\partial t_i}$.

Let $\mathcal{I}$ be an ideal of $\mathcal{D}_Z$ which is generated by all the vector fields tangent to the fibers of φ and by a finite set $(P_1, \ldots, P_l)$ of differential operators invariant under φ. Let $\mathcal{J}$ be the ideal of $\mathcal{D}_X$ generated by $(\varphi_*(P_1), \ldots, \varphi_*(P_l))$. Let $\mathcal{M} = \mathcal{D}_Z/\mathcal{I}$ and $\mathcal{N} = \mathcal{D}_X/\mathcal{J}$ and put on $\mathcal{M}$ and $\mathcal{N}$ the filtrations induced by $V\mathcal{D}_Z$ and $V^\theta\mathcal{D}_X$. The modules $\mathcal{M}$ and $\mathcal{N}$ are also provided by the filtrations induced by the usual filtrations (by the order) of $\mathcal{D}_Z$ and $\mathcal{D}_X$, we say that they are bi-filtrated.

Theorem 3.2. [7, Proposition 2.3.2.] *There exists a canonical morphism of $\mathcal{D}_Z$-modules $\mathcal{M} \to \varphi^*\mathcal{N}$ which is a morphism of bi-filtrated modules and an isomorphism at the points where φ is a submersion.*

Let $Y = \varphi^{-1}(\{0\})$ and x be a point of Y where φ is a submersion. In a neighborhood of x, Z is isomorphic to $X \times Y$ and if we fix such an isomorphism, θ which is a vector field on X may be considered as a vector field on Z. Remark that θ differ from E by a vector field tangent to φ. Then:

Corollary 3.3. *The microcharacteristic variety $Ch_Y(\infty,1)(\mathcal{M})$ is equal to $Ch^\theta(\infty,1)(\mathcal{M})$ in a neighborhood of x.*

4. Symmetric pairs

Let G be a complex reductive Lie group and $\mathfrak{g}$ its Lie algebra. Then $\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{c}$ is the center and $[\mathfrak{g}, \mathfrak{g}]$ is a semi-simple Lie algebra. For example, $\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C} \oplus \mathfrak{sl}_n(\mathbb{C})$.

If $X \in \mathfrak{g}$, AdX is the endomorphism of $\mathfrak{g}$ given by $Y \mapsto [X, Y]$. The Killing form on $[\mathfrak{g}, \mathfrak{g}]$ is defined by $\kappa_0(X, Y) = Trace(AdX AdY)$. Let us choose a non-degenerate G-invariant symmetric bilinear form κ on $\mathfrak{g}$ such that its restriction to $[\mathfrak{g}, \mathfrak{g}]$ is κ_0. Let ϑ be an involutive automorphism on $\mathfrak{g}$ preserving κ, we define $\mathfrak{k} = Ker(\vartheta - I)$ and $\mathfrak{p} = Ker(\vartheta + I)$.

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the pair $(\mathfrak{g}, \mathfrak{p})$ or $(\mathfrak{g}, \vartheta)$ is called a symmetric pair. Here $\mathfrak{k}$ is a reductive Lie algebra and K, the subgroup of G associated to $\mathfrak{k}$, acts on $\mathfrak{p}$ by the adjoint action.

Example 4.1. (diagonal) If G_0 is a reductive Lie group, $\mathfrak{g}_0$ its Lie algebra, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$ and $\vartheta(x, y) = (y, x)$, K is equal to G_0 acting on its Lie algebra $\mathfrak{g}_0$.

Example 4.2. If $G = Sl_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ with $\vartheta(A) = -tr(A)$, we find that $\mathfrak{k}$ is the Lie algebra of antisymmetric matrices, and $K = SO_n(\mathbb{C})$ acts on $\mathfrak{p}$ the set of symmetric matrices.

We will now defines the holonomic $\mathcal{D}$-modules associated to a symmetric pair. As $\mathfrak{p}$ is a vector space, $T\mathfrak{p} \simeq \mathfrak{p} \times \mathfrak{p}$. Any $A \in \mathfrak{k}$ defines a vector field on $\mathfrak{p}$ tangent to the orbits by the map $\tau(A)(X) = (X, [X, A])$ from $\mathfrak{p}$ to $T\mathfrak{p} \simeq \mathfrak{p} \times \mathfrak{p}$. The action of $\tau(A)$ on a function f on $\mathfrak{p}$ is given by:

$$\tau(A)(f)(X) = \frac{d}{dt} f(\exp(-tA).X)|_{t=0}$$

The set $\tau(\mathfrak{k})$ of all vector fields $\tau(A)$ for $A \in \mathfrak{k}$ generates the vector fields tangent to the orbits.

Let $\mathfrak{p}^*$ the dual of $\mathfrak{p}$ (as a $\mathbb{C}$-vector space), then an element P of $\mathbb{C}[\mathfrak{p}^*]$ (polynomial functions on $\mathfrak{p}^*$) defines a differential operator with constant coefficients on $\mathfrak{p}$ by $P(\xi) \mapsto P(\frac{\partial}{\partial x})$. The set $\mathbb{C}[\mathfrak{p}^*]^K$ of polynomials on $\mathfrak{p}^*$ invariant under the action of K is very simple by Chevalley theorem: there are algebraically independent invariant polynomials $q_1, \ldots, q_d$ such that $\mathbb{C}[\mathfrak{p}^*]^K$ is equal to $\mathbb{C}[q_1, \ldots, q_d]$, hence it is isomorphic to the algebra of polynomials $C[t_1, \ldots, t_d]$.

Example 4.3. In the case of Gl_n the invariant functions are the polynomials in the coefficients of the characteristic polynomial.

Let F be a finite codimensional ideal of $\mathbb{C}[\mathfrak{p}^*]^K$. For example, if $\lambda \in \mathfrak{p}^*$, then $F_\lambda = \{ P - P(\lambda) \mid P \in \mathbb{C}[\mathfrak{p}^*]^K \}$ is finite codimensional.

Definition 4.4. The module $\mathcal{M}_F$ is the quotient of $\mathcal{D}_X$ by the ideal generated by $\tau(\mathfrak{k})$ and F.

Proposition 4.5. *$\mathcal{M}_F$ is a holonomic $\mathcal{D}_{\mathfrak{p}}$-module*

The characteristic variety of $\mathcal{M}$ is a subset of $T^*\mathfrak{p}$ which isomorphic to $\mathfrak{p} \times \mathfrak{p}^*$. If $\mathfrak{p}^*$ is identified to $\mathfrak{p}$ by the non degenerate bilinear form κ, the characteristic variety may be defined as a subset of $\mathfrak{p} \times \mathfrak{p}$. Let $\mathfrak{n}$ be the set of nilpotent elements of $\mathfrak{p}$, then

$$Ch(\mathcal{M}) \subset \{ (X, Y) \in \mathfrak{p} \times \mathfrak{p} \mid [X, Y] = 0, Y \in \mathfrak{n} \}$$

Proof. Let $B : \mathfrak{p} \to \mathfrak{p}^*$ the isomorphism given by $\kappa(X,Y) = \langle X, B(Y)\rangle$.

Let $(X,Y) \in Ch(\mathcal{M})$, then for all $A \in \mathfrak{k}$, $\langle [A,X], B(Y)\rangle = 0$. So we have $\kappa(A,[X,Y]) = \kappa([A,X],Y) = \langle [A,X], B(Y)\rangle = 0$ and as κ is non degenerate on $\mathfrak{k}$, this implies $[X,Y]=0$.

On the other hand, the graduate of F is F_0 the set of invariant polynomials vanishing at 0, and the common roots of these polynomial are exactly the nilpotent elements. □

Now our main theorem is

Theorem 4.6. *The $\mathcal{D}_\mathfrak{p}$-module $\mathcal{M}_F$ is holonomic regular.*

This shows in particular that all hyperfunctions solutions of this system of partial differential equations are distributions.

A very simple example: $\mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ is the set of matrices $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$.

regular orbits are given by $\{x^2+yz=a\}$, $a \neq 0$ and there are 2 nilpotent orbits $\{x^2+yz=0, (x,y,z) \neq (0,0,0)\}$ and $\{(0,0,0)\}$.

$\mathbb{C}[\mathfrak{g}]^G$ is the set of functions $f(x^2+yz)$

$\mathbb{C}[\mathfrak{g}^*]^G$ is the set of functions $f(\xi^2+4\eta\zeta)$

$\tau(\mathfrak{g})$ is generated by the 3 vector fields

$u = 2x\frac{\partial}{\partial y} - z\frac{\partial}{\partial x}$

$v = 2x\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}$

$w = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$

Then the module $\mathcal{M}_\lambda$ is given by the equations

$$u, v, w, (\frac{\partial}{\partial x})^2 + 4\frac{\partial}{\partial y}\frac{\partial}{\partial z} - \lambda$$

Its characteristic variety is the union of the zero section of $T^*\mathfrak{g}$ and of the closure of the conormal to the non-zero nilpotent orbit.

5. Sketch of proof for the main theorem

We shall give here an idea of the proof of the main theorem, details may be found in [7].

The first step is a reduction to the nilpotent points. Any $X \in \mathfrak{p}$ has a Jordan decomposition, that is may be written in a unique way as $X = S + N$ where S is semi-simple, N is nilpotent and $[S,N]=0$. Then $\mathfrak{g}^S = \{ Z \in \mathfrak{g} \mid [Z,S] = 0 \}$ is a reductive Lie algebra and $(\mathfrak{g}^S, \mathfrak{p}^S)$ is a symmetric pair of dimension strictly lower to the dimension of the initial pair if $S \neq 0$.

In a neighborhood of S, $(\mathfrak{g},\mathfrak{p})$ is isomorphic to the product of $(\mathfrak{g}^S,\mathfrak{p}^S)$ by the orbit of S and $\mathcal{M}_F$ is isomorphic to the product of a module $\mathcal{N}_{F'}$ of the same kind on $\mathfrak{p}^S$ by holomorphic functions on the orbit.

By induction on the dimension, $\mathcal{N}_{F'}$ is regular hence $\mathcal{M}_F$ is regular in a neighborhood Ω of S. As $\mathcal{M}_F$ is invariant under the action of K, it is regular in a neighborhood of the orbits which meet Ω. The nilpotent orbits are conic hence S is in the closure of the orbit of X and thus $\mathcal{M}_F$ is regular in a neighborhood of X.

So we may now assume that $S = 0$, that is that X is nilpotent. We have to prove that $\mathcal{M}$ is regular or quasi-regular along the nilpotent orbits and the second step will be to consider the null orbit.

We consider the V-filtration associated to 0, that is given by the Euler vector field of $\mathfrak{p}$, $E = \sum x_i \frac{\partial}{\partial x_i}$. Here all is linear hence $\Lambda = T^*_{\{0\}}\mathfrak{p}$ is identified to $\mathfrak{p}$ and $T^*\Lambda$ is identified to $T^*\mathfrak{p} = \mathfrak{p} \times \mathfrak{p}^*$.

Let us calculate the symbol $\sigma_{\Lambda(\infty,1)}(\tau(P))$ for the operators of $\tau(\mathfrak{k})$ and F:

a) If $A \in \mathfrak{k}$, by definition $\tau(A)(f)(X) = \frac{d}{dt}f(\exp(-tA).X)|_{t=0}$ and $E(f)(X) = \frac{d}{dt}f(tX)|_{t=0}$ hence they commute. So $\tau(A)$ is homogeneous of order 0 for the V-filtration and by definition $\sigma_{\Lambda(\infty,1)}(\tau(A)) = \sigma(\tau(A))$.

b) If P is an operator with constant coefficients, the V-filtration at 0 is the usual filtration and again $\sigma_{\Lambda(\infty,1)}(P) = \sigma(P)$. So we have

$$Ch_{\Lambda(\infty,1)}(\mathcal{M}) \subset \{(X,Y) \in \mathfrak{p} \times \mathfrak{p} \mid [X,Y] = 0, Y \in \mathfrak{n}\} \qquad (5.1)$$

Remark that $T^*_{\{0\}}\mathfrak{p}$ is not a component of the characteristic of $\mathcal{M}$ (because there are always some Y which is not nilpotent) hence we do not have to verify that $Ch_{\Lambda(\infty,1)}(\mathcal{M}) \subset \Lambda$.

Now we use inclusion (5.1) to show that $\mathcal{M}^F_\lambda$ is regular along the other nilpotent orbits. A nilpotent orbit $\mathfrak{S}$ is conic hence defined by homogeneous functions $\varphi_1, \ldots, \varphi_p$, φ_i being homogeneous of degree m_i.

In fact, it is known [9], that we can choose coordinates (x,t) in a neighborhood of $X \in \mathfrak{S}$ such that $\varphi_i = t_i$ and $\theta = \sum m_i t_i \frac{\partial}{\partial t_i}$ is equal to E modulo vector fields tangent to the orbits.

Applying the inverse image theorem (corollary 3.3) we deduce that

$$Ch^\theta{}_{(\infty,1)}(\mathcal{M}) = Ch_{\Lambda(\infty,1)}(\mathcal{M}) \subset \{(X,Y) \in \mathfrak{p} \times \mathfrak{p} \mid [X,Y] = 0, Y \in \mathfrak{n}\}$$

Suppose that $T^*_{\mathfrak{S}}\mathfrak{p}$ is an irreducible component of the characteristic variety $Ch(\mathcal{M}_F)$ and let x^* be a generic point of $T^*_{\mathfrak{S}}\mathfrak{p}$, that is a point which does not belong to other irreducible components of $Ch(\mathcal{M}_F)$. We have $T^*_{\mathfrak{S}}\mathfrak{p} \subset Ch(\mathcal{M}_F) \subset (\mathfrak{p} \times \mathrm{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$ and as they have the same dimension, they are equal generically. So $Ch^\theta{}_{(\infty,1)}(\mathcal{M}_F) = T^*_{\mathfrak{S}}\mathfrak{p}$ generically on $T^*_{\mathfrak{S}}\mathfrak{p}$ and $\mathcal{M}$ is quasi-regular along the orbit.

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