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Tamely ramified factors of zeta integrals for the Standard L-function of $U(2,1)$ *

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In the past three decades, integral expressions of many automorphic L-functions have been discovered and utilized for study of analytic properties of L-functions. But unfortunately, not so much investigation on ramified factors of these integrals has been accumulated, though it is indispensable to arithmetic study of L-functions. Recently there seems to be a movement of renovation of zeta integral method toward deeper arithmetic investigation beginning with low rank groups, say $GSp(4)$, $U(3)$.

It is reasonable to begin with the Standard L-function of $U(3)$. So far we have four different zeta integral expression for this L-function. That is a Rankin-Selberg integral [Ge-PS], a Shimura type integral [Shi], [Ge-PS], Murase-Sugano's integral by using their Shintani functions [Mu-Su] and the doubling integral [PS-Ra], [Tak].

The archimedean factor of the first integral was calculated by Koseki and Oda [K-O], where it is shown that the GCD of the integrals for all K-finite vectors turns out to be a product of three $\Gamma_{\mathbf{C}}$'s. Note that this type of zeta integral works only for generic cusp forms. As for the third integral, Tsuzuki calculated the archimedean component in a broader setting [Tsu].

In this note, we report some results on ramified factors of the first and second zeta integrals, which are recalled in §1. In §2, we calculate the archimedean component of Shimura type zeta integral. After normalization of Eisenstein series, we show that it is, up to elementary factors, a product of three $\Gamma_{\mathbf{C}}$'s for any discrete series π_∞ and satisfies a symmetric functional equation. In §3, we proceed into study of tamely ramified finite local factors. We begin with the case of Steinberg representation. By using Li's explicit formula [Li] of Whittaker function for Iwahori spherical vector, we compute the local component of Rankin-Selberg integral of Gelbart- Piatetski-Shapiro.

Contents

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1 Zeta integrals for the standard L-function

Note that we can obtain the same result without any loss of generality, even if we formulate the problem over an arbitrary totally real algebraic number field. So we take $\mathbb{Q}$ for our ground field.

Let E be an imaginary quadratic extension of $\mathbb{Q}$ and denote the non-trivial element of its Galois group by $\bar{\cdot}$. Put

$$G := \{g \in GL(3,E) \mid {}^t\bar{g}\begin{pmatrix} & & 1/\kappa \\ & 1 & \\ -1/\kappa & & \end{pmatrix} g = \begin{pmatrix} & & 1/\kappa \\ & 1 & \\ -1/\kappa & & \end{pmatrix}\},$$

where κ is an element of E such that $\mathrm{Tr}_{E/\mathbb{Q}}\kappa = 0$. This defines a quasi-split unitary group of three variables over $\mathbb{Q}$. We need a subgroup

$$H := \mathrm{Img}\Big(\iota : U(1,1) \ni \begin{pmatrix} \star & \star \\ \star & \star \end{pmatrix} \mapsto \begin{pmatrix} \star & & \star \\ & 1 & \\ \star & & \star \end{pmatrix} \in G\Big)$$

as the Euler subgroup for a Rankin-Selberg integral.

<Zeta integrals>

For a cusp form φ belonging to a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}) = U(3)_{\mathbb{A}}$, Gelbart and Piatetski-Shapiro introduced the following zeta integral

$$\mathcal{Z}(s;\varphi,\xi) := \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi|_H(h) E^{\xi,H}(s;h)\mathrm{d}h.$$

Here $E^{\xi,H}$ is an Eisenstein series on $H(\mathbb{A})$ constructed by a Hecke character ξ.

We denote a Shimura type zeta integral, first investigated by Shintani [Shi], by

$$\mathcal{Z}(s;\varphi,\theta,\xi) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(g)\theta(g)E^{\xi}(s;g)\mathrm{d}g.$$

Here E^{ξ} and θ are an Eisenstein series and a theta series on $G(\mathbb{A})$ respectively. And ξ is a Hecke character of E.

<Unfolding and local integrals>

By using the multiplicity one result on Whittaker models and an unfolding procedure, the Rankin-Selberg integral decomposes into a product of local integrals:

$$\mathcal{Z}(s;\varphi,\xi) = \prod_v \mathcal{Z}_v(s; W_\psi^\pi, \Phi_\xi^{(s)}),$$

with

$$\mathcal{Z}_v(s; W_\psi^\pi, \Phi_\xi^{(s)}) := \int_{Z_{N,v}\backslash H_v} W_\psi^{\pi_v}|_{H_v}(h_v)\Phi_\xi^{(s)}(h_v)\mathrm{d}h_v.$$

Here $Z_{N,v}$ is the center of the maximal nilpotent subgroup N_v of G_v, $W_\psi^{\pi_v}$ is a Whittaker vector for π_v and $\Phi_\xi^{(s)}$ is a special section of the principal series $\mathrm{Ind}_P^H(\mu|\cdot|^s)$ of H induced up from its Borel subgroup $\iota(\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right))$. Note that this integral vanishes unless φ is a generic cusp form.

Similarly, by using the multiplicity one result on Fourier-Jacobi models (cf. [B-PS-R]) and an unfolding procedure, the Shimura type integral decomposes into a product of local integrals:

$$\mathcal{Z}(s;\varphi,\theta,\xi) = \prod_v \mathcal{Z}_v(s;W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)})$$

with

$$\mathcal{Z}_v(s;W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)}) := \int_{R_v\backslash G_v} \left\langle W_{(\xi\eta)^*}^{\Theta_v}(g_v) \middle| W_\eta^{\pi_v}(g_v)\right\rangle_\eta \Phi_{\sigma\xi}^{(s)}(g_v)dg_v.$$

Here R_v modulo the center is the stabilizer subgroup of $Z_{N,v}$ in a Borel subgroup B of G for the adjoint action. And W^X means a generalized Whittaker vector in Fourier-Jacobi model for X (see §2) and $\Phi_{\sigma\xi}^{(s)}$ is a section of the principal series $\mathrm{Ind}_B^G(\sigma\xi|\cdot|^s)$ of G.

<Unramified components>

Over the places where everything is unramified, the local components $\mathcal{Z}_v(s;W_\psi^\pi,\Phi_\xi^{(s)})$ and $\mathcal{Z}_v(s;W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)})$ of these zeta integrals were computed by Gelbart-Piatetski-Shapiro, Shintani and Gelbart-Rogawski respectively.

Proposition 1 **([Ge-PS] §4)**

$$\mathcal{Z}_v(s;W_\psi^\pi,\Phi_\xi^{(s)}) = L_v(s;\pi_v\otimes\xi_v)$$

Proof. Use Casselman-Shalika formula. □

Proposition 2 **([Ge-Ro] §8, [Shi])**

$$\mathcal{Z}_v(s;W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)}) = \frac{L_v(s+\frac{1}{2};\pi_v\otimes\xi_v)}{L_{E,v}(s+1;\xi_v)L_v(2s+1;(\xi|_{\mathbf{Q}}\gamma_{E/\mathbf{Q}})_v)}$$

Proof. Use recursion relations, coming from the Hecke action, for unramified generalized Whittaker vector in Fourier-Jacobi model for π_p. □

Note the local factor $L_v(s;\pi_v\otimes\xi_v)$ is given by

$$L_v(s;\pi_v\otimes\xi_v) = L_{E,v}(s;\xi_v)L_v(2s;\xi_v\nu)L_v(2s;\xi_v/\nu).$$

Here ν is the unramified character to define the unramified principal series π_v (see §3).

2 The archimedean factors

In this section, we calculate the archimedean component $\mathcal{Z}_\infty(s;W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)})$ of a Shimura type integral to have a local functional equation and a nice expression, after recalling a result of Koseki and Oda [K-O].

<Rankin-Selberg integral >

Koseki and Oda used their explicit formula for Whittaker functions on $SU(2,1)$. Their result looks quite complicated. But after rewriting the result by our coordinate, it turns out that their GCD of archimedean integral $\mathcal{Z}_\infty^\xi(s;W,\Phi)$'s can be written as follows.

Proposition 3 ([K-O] Theorem 6.8) *Let $L^{\xi}(s; D_{\Lambda}^{(1,1)})$ be the GCD of the family*

$$\{\mathcal{Z}_{\infty}^{\xi}(s; W, \Phi_{\xi,\phi}^{(s)}) \mid W : K_{\infty}\text{-finite Whittaker vector}, \phi \in \mathcal{S}(\mathbb{C}^2)\}.$$

See §3 for $\Phi_{\xi,\phi}^{(s)}$. Then the GCD $L^{\xi}(s; D_{\Lambda}^{(1,1)})$ is given as follows.
(1) *When $\Lambda_1 + \Lambda_3 \geq m \geq \Lambda_2 + \Lambda_3$,*

$$2^{-s}\Gamma(s+t+\Lambda_1-\frac{m}{2})\Gamma(s+t-\Lambda_2+\frac{m}{2})\begin{cases} \Gamma(s+t-\Lambda_3+\frac{m}{2}) & \text{when } m \geq 0, 0 \geq \Lambda_3 \text{ or when } m < 0, m \geq \Lambda_3 \\ \Gamma(s+t+\frac{m}{2}) & \text{when } m \geq \Lambda_3 > 0 \\ \Gamma(s+t-\frac{m}{2}) & \text{when } 0 \geq \Lambda_3 > m \\ \Gamma(s+t+\Lambda_3-\frac{m}{2}) & \text{when } m \geq 0, \Lambda_3 > 0 \text{ or when } m < 0, \Lambda_3 > m \end{cases}$$

(2) *When $\Lambda_2 + \Lambda_3 \geq m$,*

$$2^{-s}\Gamma(s+t+\Lambda_1-\frac{m}{2})\Gamma(s+t-\Lambda_3-\frac{m}{2})\begin{cases} \Gamma(s+t-\Lambda_2+\frac{m}{2}) & \text{when } m \geq 0, 0 \geq \Lambda_2 \text{ or when } m < 0, m \geq \Lambda_2 \\ \Gamma(s+t+\frac{m}{2}) & \text{when } m \geq \Lambda_2 > 0 \\ \Gamma(s+t-\frac{m}{2}) & \text{when } 0 \geq \Lambda_2 > m \\ \Gamma(s+t+\Lambda_2-\frac{m}{2}) & \text{when } m \geq 0, \Lambda_2 > 0 \text{ or when } m < 0, \Lambda_2 > m \end{cases}$$

(3) *When $m \geq \Lambda_1 + \Lambda_3$,*

$$2^{-s}\Gamma(s+t-\Lambda_2+\frac{m}{2})\Gamma(s+t-\Lambda_3+\frac{m}{2})\begin{cases} \Gamma(s+t-\Lambda_1+\frac{m}{2}) & \text{when } m \geq 0, 0 \geq \Lambda_1 \text{ or when } m < 0, m \geq \Lambda_1 \\ \Gamma(s+t+\frac{m}{2}) & \text{when } m \geq \Lambda_1 > 0 \\ \Gamma(s+t-\frac{m}{2}) & \text{when } 0 \geq \Lambda_1 > m \\ \Gamma(s+t+\Lambda_1-\frac{m}{2}) & \text{when } m \geq 0, \Lambda_1 > 0 \text{ or when } m < 0, \Lambda_1 > m \end{cases}$$

□

Note that in some cases the GCD in the above list may vanish by virtue of K_{∞}-type compatibility. A natural question arises here. Is it possible to regain the third missing Harish-Chandra parameter Λ_i and to obtain a local functional equation by normalizing the Eisenstein series $E^{\xi,H}$ on H? We will study this problem in the near future.

<generalized Whittaker vector>

Different from the $GL_2 \cong U(1,1)$ case, the maximal nilpotent subgroup N_v of our $G_v \cong U(3)$ is not abelian, is isomorphic to the Heisenberg group. The unitary dual $N_v^{\wedge}$ consists not only of unitary characters ψ but also of infinite dimensional irreducible unitary representations ρ. So considering $\mathrm{Hom}_{N_v}(\pi_v|_{N_v}, \rho)$ seems to be natural. But this intertwining space is infinite dimensional. This is the reason why the bigger group $R_v \cong U(1) \ltimes N_v$ is introduced. Then we have the correct intertwining space $\mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{R_v}^{G_v}\eta)$, i.e. *the Fourier-Jacobi model* of π_v. Multiplicity one result $\dim_{\mathbb{C}} \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{R_v}^{G_v}\eta) \leq 1$ has last been published in [B-PS-R]. Here $\eta := \widetilde{\chi} \otimes \omega\rho$ is an irreducible unitary representation of R_v induced from ρ. When $v = \infty$, η is parameterized by $(\widetilde{\mu}, \ell) \in (\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}) \times (\mathbb{Z}\backslash\{0\})$. We give an explicit formula for the generalized Whittaker vectors coming from Fourier-Jacobi model of π_∞, when π_∞ is a cohomological unitarizable representation of $U(2,1)$.

Proposition 4 *The moderate growth generalized Whittaker vector belonging to the minimal K_∞-type of π_∞ is given as follows by expanding by the Fock and theGel'fand-Zetlin basis.*

$$W_\eta^{\pi,\tau_\lambda^*}|_A(a) = \sum_{k=-\lambda_2}^{-\lambda_1} c_k^\pi(a)\Big(\Big[\begin{array}{c}\widetilde{\mu},\ell\\ j_k^{\widetilde{\mu}}\end{array}\Big] \otimes \Big|\begin{array}{c}-\lambda_2, \ -\lambda_1\\ k\end{array}\Big\rangle^{-\lambda_0}\Big).$$

Here $j_k^{\widetilde{\mu}}$ is $k + \lambda_1 + \lambda_2 - \widetilde{\mu} - (\mathrm{sgn}\ell)\frac{1}{2}$.

i) *When π is a discrete series representation $D_\Lambda^{p,q}$ with Blattner parameter $\lambda = [\lambda_1, \lambda_2; \lambda_0]$.*

i-1) *The case of large discrete series $D_\Lambda^{1,1}$ i.e. contributes to $H^{(1,1)}$.*
The generalized Whittaker model exists exactly when $\ell > 0$ and $\lambda_1 \geq \widetilde{\mu} + \frac{1}{2}$, or when $\ell < 0$ and $\lambda_2 \leq \widetilde{\mu} - \frac{1}{2}$.

$$c_k^\pi(a_y) = \gamma_k^{\mathrm{La,sgn}\ell}(\lambda) \cdot y^{-\lambda_2+\lambda_1-1} W_{\kappa,\mu}(2\pi|\ell|y^2)$$

with

$$\kappa = (\mathrm{sgn}\ \ell)\frac{-k + 2\widetilde{\mu} - c^\pi}{2}, \quad \mu = \frac{-k + 2\lambda_0 - c^\pi}{2}.$$

Here $c^\pi = \lambda_1 + \lambda_2 + \lambda_0$.

i-2) *The case of holomorphic discrete series $D_\Lambda^{2,0}$ i.e. contributes to $H^{(2,0)}$.*
The generalized Whittaker model exists exactly when $\ell > 0$ and $\lambda_1 \geq \widetilde{\mu} + \frac{1}{2}$.

$$c_k^\pi(a_y) = \gamma_k^{\mathrm{Hol}}(\lambda) \cdot y^{-2-\lambda_0-k} e^{-\pi\ell y^2}.$$

i-3) *The case of anti-holomorphic discrete series $D_\Lambda^{0,2}$ which contributes to $H^{(0,2)}$.*
The generalized Whittaker model exists exactly when $\ell < 0$ and $\lambda_2 \leq \widetilde{\mu} - \frac{1}{2}$.

$$c_k^\pi(a_y) = \gamma_k^{\mathrm{AnH}}(\lambda) \cdot y^{-2+\lambda_0+k} e^{\pi\ell y^2}.$$

ii) *When π is a cohomological unitarizable representation $A_\mathfrak{q}(\lambda)$ which contributes to H^1. For these representations the indices $j_k^{\widetilde{\mu}}$ of Fock basis are always zero.*

ii-1) *The case of lowest weight module, i.e. contributes to $H^{(1,0)}$.*
The generalized Whittaker model exists exactly when $\ell > 0$.

$$c_k^\pi(a_y) = \gamma_k^{\mathrm{low}}(\lambda) \cdot y^{-2-\alpha(\gamma)-k} e^{-\pi\ell y^2}.$$

ii-2) *The case of highest weight module, i.e. contributes to $H^{(0,1)}$.*
The generalized Whittaker model exists exactly when $\ell < 0$.

$$c_k^\pi(a_y) = \gamma_k^{\mathrm{hst}}(\lambda) \cdot y^{-2+\alpha(\gamma)+k} e^{\pi\ell y^2}.$$

For normalizing constants $\gamma_k^*(\lambda)$, see [I].

<Shimura type integral >

Theta functions on $U(3)$ is, by definition, automorphic forms obtained by restriction of theta functions on $\widetilde{Sp_6}$. That is Kazhdan lift [Kaz] attached to the dual reductive pair $(U(1), U(3))$ in $\widetilde{Sp_6}$. It is known that Kazhdan lift is everywhere non-tempered. So the archimedean component Θ_∞ of automorphic representation generated by Kazhdan lift is isomorphic to $A_{\mathfrak{q}}(\lambda)$ with $\mathfrak{q} \neq \mathfrak{b}$. There are two possibility for choice of archimedean splitting on $U(2,1)$ in $\widetilde{Sp_6}(\mathbb{R})$, i.e. $\alpha(\gamma) = 0$ or 1. Here we record odd splitting case only. For $\alpha(\gamma) = 0$ case, see [Is]. By using our explicit formula (Proposition 5) for the generalized Whittaker vectors, we can calculate the archimedean component of Shimura type integral.

Proposition 5 *Assume the archimedean component π_∞ of cuspidal representation generated by φ is discrete series representation $D_\Lambda^{p,q}$ with Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$. The archimedean zeta integral vanishes unless*

$$\lambda_2 \le \widetilde{\mu} + (\mathrm{sgn}\ell)\frac{1}{2} \le \lambda_1.$$

When $\alpha(\gamma) = 1$, the archimedean zeta integral is given as follows, if it does not vanish.

1) *When π is a holomorphic discrete series $D_\Lambda^{2,0}$ and the parameter ℓ of $\eta = \widetilde{\chi}\otimes(\omega_\psi\rho_\psi)$ is positive,*

$$\mathcal{Z}_\infty(s; W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)}) = \frac{(-1)^{\widetilde{\mu}+\frac{1}{2}-\lambda_2}(\dim\tau_\lambda - 1)!}{2\ell^{s+\frac{1}{2}+\frac{t}{2}+\frac{\Lambda_1-\Lambda_3}{2}}}\Gamma_{\mathbf{C}}(s+\frac{1}{2}+\frac{t}{2}-\Lambda_3+\frac{m}{2}).$$

2) *When π is an anti-holomorphic discrete series $D_\Lambda^{0,2}$ and η has the negative parameter $\ell < 0$,*

$$\mathcal{Z}_\infty(s; W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)}) = \frac{(-1)^{\widetilde{\mu}-\frac{1}{2}-\lambda_1}(\dim\tau_\lambda - 1)!}{2(-\ell)^{s+\frac{1}{2}+\frac{t}{2}+\frac{\Lambda_2-\Lambda_3}{2}}}\Gamma_{\mathbf{C}}(s+\frac{1}{2}+\frac{t}{2}+\Lambda_3+1+\frac{m}{2}).$$

3) *When π is a large discrete series $D_\Lambda^{1,1}$, there are two subcases.*

3+) *If the parameter ℓ of η is positive,*

$$\mathcal{Z}_\infty(s; W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)}) = \frac{(-1)^{\widetilde{\mu}+\frac{1}{2}-\lambda_1}(\dim\tau_\lambda - 1)!}{2\ell^{s+\frac{1}{2}+\frac{t}{2}+\frac{\Lambda_1-\Lambda_2}{2}-1}}c_+P_+(s-\frac{1}{2}+\frac{t}{2})\Gamma_{\mathbf{C}}(s+\frac{1}{2}+\frac{t}{2}-\Lambda_2-1+\frac{m}{2}).$$

3−) *If the parameter ℓ of η is negative,*

$$\mathcal{Z}_\infty(s; W_\eta^\pi, W_{(\xi\eta)^*}^\Theta, \Phi_{\sigma\xi}^{(s)}) = \frac{(-1)^{\widetilde{\mu}-\frac{1}{2}-\lambda_1}(\dim\tau_\lambda - 1)!}{2(-\ell)^{s+\frac{1}{2}+\frac{t}{2}+(\Lambda_1-\Lambda_2)}}c_-P_-(s+\frac{1}{2}+\frac{t}{2})\Gamma_{\mathbf{C}}(s+\frac{1}{2}+\frac{t}{2}+\Lambda_1-\frac{m}{2}).$$

Here $P_\pm$ are polynomials in s and $c_\pm$ are constants, see [Is]. □

<Normalization and Local functional equation >

In order to have a local functional equation in a symmetric form, we normalize the intertwining operator $A_\infty(s)$ and a section $\Phi_\infty^{(s)}$. The intertwining operator is defined by

$$(A_v(s).\Phi_v^{(s)})(g) := \int_{N_v} \Phi_v^{(s)}(w^{-1}ng)dn.$$

where w is the longest Weyl element. Normalize this $A_v(s)$ as

$$\mathcal{A}_v(s) := \varepsilon_v(s;\xi,\psi)\frac{L_v(1-s;\xi)}{L_v(s;\xi)}\cdot\varepsilon_v(2s;\xi|_{\mathbb{Q}}\gamma,\psi)\frac{L_v(1-2s;\xi|_{\mathbb{Q}}\gamma)}{L_v(2s;\xi|_{\mathbb{Q}}\gamma)}A_v(s),$$

then $\mathcal{A}_v(-s)\mathcal{A}_v(s) = \mathrm{Id}$, i.e. self-adjoint.

Lemma 6 *Assume $\mu = m < 0$ and $\alpha(\gamma) = 1$, then we have*

$$\mathcal{A}_\infty(s).\left[d(s+\frac{t}{2})\Phi_\infty^{(s)}\right] = \varepsilon_{\mu,\gamma}\times\left[d(-s-\frac{t}{2})\Phi_\infty^{(-s)}\right]$$

for the section $\Phi_\infty^{(s)}$ belonging to the corner K_∞-type. Here

$$\varepsilon_{\mu\gamma} = (-1)^{\mu+\frac{\alpha(\gamma)}{2}} = (-1)^m\sqrt{-1}$$

and

$$d(s+\frac{t}{2}) := 2^{s+\frac{t}{2}}\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\mu-\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\frac{|m|}{2}).$$

□

Moreover if we normalize as

$$\widehat{\mathcal{Z}}_\infty(s;W_\eta^\pi,W_{(\xi\eta)^*}^\Theta,\Phi_{\sigma\xi}^{(s)}) := \mathcal{Z}_\infty(s;W_\eta^\pi,W_{(\xi\eta)^*}^\Theta,d(s')\Phi_{\sigma\xi}^{(s)}),$$

where $s' = s+\frac{t}{2}$, then we have a clean functional equation and a nice expression of the archimedean zeta integral.

Theorem 7 *Assume $\mu = m < 0$ and $\alpha(\gamma) = 1$. The archimedean component of normalized zeta integral satisfies a local functional equation*

$$\widehat{\mathcal{Z}}_\infty(-s;W_\eta^\pi,W_{(\xi\eta)^*}^\Theta,\mathcal{A}_\infty(s).\Phi_{\sigma\xi}^{(s)}) = \varepsilon_{\mu,\gamma}\cdot\widehat{\mathcal{Z}}_\infty(s;W_\eta^\pi,W_{(\xi\eta)^*}^\Theta,\Phi_{\sigma\xi}^{(s)}),$$

and is, up to simple factors appearing in Proposition 5, of the following form.
1) *When π is a holomorphic discrete series $D_\Lambda^{2,0}$ and the parameter ℓ is positive,*

$$\Gamma_{\mathbb{C}}(s+\frac{1}{2}+\frac{t}{2}-\Lambda_3+\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\Lambda_1-\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\frac{|m|}{2}).$$

2) *When π is an anti-holomorphic discrete series $D_\Lambda^{0,2}$ and the negative parameter $\ell < 0$,*

$$\Gamma_{\mathbb{C}}(s+\frac{1}{2}+\frac{t}{2}+\Lambda_3+1+\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\Lambda_2-1-\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\frac{|m|}{2}).$$

3) *When π is a large discrete series $D_\Lambda^{1,1}$, there are two subcases.*
   3+) *If the parameter ℓ of η is positive,*

$$\Gamma_{\mathbb{C}}(s+\frac{1}{2}+\frac{t}{2}-\Lambda_2-1+\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\Lambda_1-\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\frac{|m|}{2}).$$

   3−) *If the parameter ℓ of η is negative,*

$$\Gamma_{\mathbb{C}}(s+\frac{1}{2}+\frac{t}{2}+\Lambda_1-\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\Lambda_2-1-\frac{m}{2})\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\frac{|m|}{2}).$$

Here $(t,m)\in\mathbb{C}\times\mathbb{Z}$ is the parameter of ξ. □

The Langlands parameterization of irreducible admissible representations of real reductive group says the archimedean factor should be

$$L_\infty(s;\pi_\infty\otimes\xi_\infty) \;:=\; \prod_{i=1}^{3}\Gamma_{\mathbb{C}}(s+\frac{t}{2}+\Lambda_i+\frac{|m|}{2}),$$

with the Harish-Chandra parameter $(\Lambda_1,\Lambda_2,\Lambda_3)$ for $\pi_\infty \cong D_\Lambda^{p,q}$.

So the ratio $\widehat{\mathcal{Z}}_\infty(s;W_\eta^\pi,W_{(\xi\eta)^*}^\Theta,\Phi_{\sigma\xi}^{(s)})/L_\infty(s;\pi_\infty\otimes\xi_\infty)$ is rational function, sometimes polynomial, in s. Here is a question. Is it possible to find an appropriate K_∞-finite generalized Whittaker vector by which the zeta integral $\widehat{\mathcal{Z}}_\infty(s;W_\eta^\pi,W_{(\xi\eta)^*}^\Theta,\Phi_{\sigma\xi}^{(s)})$ expresses local factor $L_\infty(s;\pi_\infty\otimes\xi_\infty)$ itself?

3 The case of Steinberg representation

In this section, we calculate the local component of Rankin-Selberg integral for the Iwahori spherical Whittaker vector in the Steinberg representation. From now on we denote p for a fixed finite place v and $\mathfrak{p}$ for the place of E which lies over p. Assume $E_\mathfrak{p}/\mathbb{Q}_p$ is unramified extension. Let G_p denote the $\mathbb{Q}_p$-valued points $G(\mathbb{Q}_p)$ of G and J the Iwahori subgroup of the hyperspecial maximal compact subgroup $K_p := G(\mathbb{Z}_p)$, defined as $J \bmod p = B(\mathbb{Z}_p/p\mathbb{Z}_p)$, i.e.

$$J = \begin{pmatrix} \mathcal{O}_\mathfrak{p} & \mathcal{O}_\mathfrak{p} & \mathcal{O}_\mathfrak{p} \\ \mathfrak{p} & \mathcal{O}_\mathfrak{p} & \mathcal{O}_\mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O}_\mathfrak{p} \end{pmatrix}.$$

<Iwahori spherical vectors>

We recall Borel's characterization of Iwahori spherical representations.

Proposition 8 ([Bo]) *For an irreducible admissible representation π_p of G_p, the followings are equivalent.*

(i) π_p has a non-zero Iwahori spherical vector, i.e. $\pi_p^J \neq \{0\}$.

(ii) π_p is a subquotient of unramified principal series $I_\nu(s)$.

Here $I_\nu(s) := \mathrm{Ind}_{B_p}^{G_p}\nu|\cdot|^s\,(\nu'\circ det)$ with unramified characters ν, ν'. □

The reducing points of $I_\nu(s)$ is known for our group G_p.

Proposition 9 *The principal series representation $I_\nu(s)$ is reducible exactly when $s = \pm1, \pm\frac{1}{2}, 0$.* □

Especially when $s=+1$, we have

$$0 \to St(\nu') \to I_\nu(+1) \to \nu'\circ det \to 0.$$

The representation $St(\nu')$ is the Steinberg representation.

Lemma 10 *The Steinberg representation $St(\nu')$ has unique Iwahori spherical vector up to constant multiple, i.e.* $\dim_{\mathbb{C}} St(\nu')^J = 1$. *Moreover the vector is given by* $\Phi^{St} := -p\Phi_e + \Phi_w$, *i.e.* $St(\nu')^J = \mathbb{C}\,\Phi^{St}$. *Here* Φ_s *denote the characteristic function of* JsJ *with* $s \in W = \{e := \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, w := \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}\}$.

Proof. Just trace the argument of Casselman in [Cas]. □

<Whittaker vectors>

Let

$$\Lambda_\nu \; : \; I_\nu(s) \to \mathbb{C}$$

be the Whittaker functional with respect to a non-degenerate character ψ_N of N_p, i.e.

$$\Lambda_\nu(R(n).\phi) = \psi_N(n)\Lambda_\nu(\phi)$$

for $\forall n \in N_p$ and for $\forall \phi \in I_\nu(s)$. Here $R(*)$ means the left regular representation.

Fortunately, it can be easily checked that the Whittaker vector coming from the Iwahori spherical vector Φ^{St} coincides with Jian-Shu Li's $W_\nu^{(2)}$ up to constant, i.e.

$$\Lambda_\nu(R(g).\Phi^{St}) \;=\; (\text{const.})W_\nu^{(2)}(g).$$

Li obtained an explicit formula for four J-spherical Whittaker vectors $W_\nu^{(i)}$ $(i = 1, \ldots, 4)$ on arbitrary quasi-split reductive group G_p. $W_\nu^{(1)}$ is K_p-spherical and his explicit formula is Casselman-Shalika formula. We need $W_\nu^{(2)}$ for our purpose, and write down Li's formula in the case of $G_p \cong U(3)$.

Proposition 11 ([Li]) *We denote* $\#(E_p/O_p)$ *by* q_E. *For* $k \in \mathbb{Z}_{\geq}$,

$$W_\nu^{(2)}(\begin{pmatrix} p^k & & \\ & 1 & \\ & & p^{-k} \end{pmatrix}) \;=\; \Big\{\Big(1 - q_E^{-1}\nu(p)\Big)\Big(1 + p^{-1}\nu(p)\Big)\nu(p)^{k+1} + \Big(q_E^{-1} - \nu(p)\Big)\Big(p^{-1} + \nu(p)\Big)\nu(p)^{-k-1}\Big\}\frac{|p^k|^2}{\nu(p) - \nu(p^{-1})}$$

□

<Zeta integral >

Now we compute the local factor $\mathcal{Z}_\infty(s; W_\psi^\pi, \Phi_\xi^{(s)})$ of Rankin-Selberg integral for the Steinberg representation $St(\nu')$ by a standard procedure.

Theorem 12 *When* $\pi_p \cong St(\nu')$ *and* W_p *is the Iwahori spherical Whittaker vector* $W_\nu^{(2)}$ *in* π_p,

$$\mathcal{Z}_p(s; W_\psi^\pi, \Phi_\xi^{(s)}) \quad \text{expresses} \quad L_{E,p}(s;\xi_v)L_p(2s;\xi_v\nu).$$

Proof. By the Iwasawa decomposition $H_p = Z_{N,p}A_pK_H$ with $K_H := K_p \cap H_p$, the local integral is

$$\mathcal{Z}_p(s; W_\psi^\pi, \Phi_\xi^{(s)}) = \int_{A_p} \Big(\int_{K_H} W_\nu^{(2)}(\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} k)\ \Phi_\xi^{(s)}(k)\mathrm{d}k \Big) \xi(a)|a|^{2s} \frac{\mathrm{d}a}{|a|^2}.$$

For the inner integral, use the decomposition

$$K_H = \iota\Big(\Gamma_0(\mathfrak{p}) \sqcup \coprod_{x \bmod \mathfrak{p}} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \Gamma_0(\mathfrak{p})\Big)$$

For the outer integral, insert Li's explicit formula (Proposition 11) and section of the form,

$$\Phi_{\xi,\phi}^{(s)} := \int_{E_{\mathfrak{p}}^\times} (h.\phi)(t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix})\xi(t)|t|_{\mathfrak{p}}^s \frac{\mathrm{d}t}{t}$$

with $\phi \in \mathcal{S}(E_{\mathfrak{p}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \oplus E_{\mathfrak{p}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$. Choose ϕ suitably. □

<Problems>

Several problems are remained. First, there are other tamely ramified π_p, i.e. subquotient of $I_\nu(\pm\frac{1}{2})$ and $I_\nu(0)$. Is it possible to calculate local Rankin-Selberg integral $\mathcal{Z}_\infty(s; W_\psi^\pi, \Phi_\xi^{(s)})$? There is a related result of Watanabe [Wat]. Second, it is also interesting to study ramified local factors $\mathcal{Z}_v(s; W_\eta^\pi, W_{(\xi\eta)^*}^\ominus, \Phi_{\sigma\xi}^{(s)})$ of Shimura type integral.

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