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MAASS SPACES OF SIEGEL MODULAR FORMS OF DEDREE 2N AND THE IMAGE OF IKEDA LIFTSING (JOINT WORK WITH W.KOHNEN)

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§1 Notation and preliminaries. The main important problems in the theory of lifting of automorphic forms are the construction and the characterization of kernel and image of this correspondence. In this lecture, we shall discuss the latter problem in the case of Ikeda lifting of Siegel modular forms of even degree. Our result is a joint work with W. Kohnen. The details shall be appeared in Compositio Math. (see [6]).

We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For an associative ring R with identity element we denote by $M_{m,n}(R)$ the set of $m \times n$ matrices entries in R. We set $M_n(R) = M_{n,n}(R)$ and $R^n = M_{1,n}(R)$. Let $\mathcal{S}_m(R)$ be the set of symmetric matrices $T = (t_{i,j})$ of degree m satisfying $2t_{i,j} \in R$ and $t_{i,i} \in R$. For $T \in \mathcal{S}_m(R)$ and $\lambda \in M_{m,n}(R)$, we put $T[\lambda] = \lambda' T \lambda$ where λ' is the transpose of λ. If A and B are square matrices over R, we often write $A \oplus B$ for the diagonal block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Put $SL_n(R) = \{g \in M_n(R) | \det g = 1\}$ and $GL_n(R) = \{g \in M_n(R) | \det g \in R^\times\}$, where $R^\times$ denotes the group of all invertible elements of R.

For a positive integer m, $\mathfrak{H}_m$ denotes the Siegel upper-half plane of degree m. For $z \in \mathbb{C}$, we set $e[z] = \exp(2\pi i z)$. For $z \in \mathbb{C}$, we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \leqq \pi/2$ and put $z^{\kappa/2} = (\sqrt{z})^\kappa$ for every $\kappa \in \mathbb{Z}$.

For positive integers n and k, let $S_k(\Gamma_n)$ be the space of all Siegel cusp forms $F(Z) = \sum_{T \in \mathcal{S}_n^+(\mathbb{Z})} A(T) e[\mathrm{tr}(TZ)]$ of weight k with respect to the full Siegel modular group $\Gamma_n := Sp_n(\mathbb{Z}) \subset GL_{2n}(\mathbb{Z})$ of genus n such that

$$F((AZ+B)(CZ+D)^{-1}) = (\det(CZ+D))^k F(Z) \text{ for every } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n,$$

where $\mathcal{S}_n^+(\mathbb{Z}) = \{T \in \mathcal{S}_n(\mathbb{Z}) | T > 0\}$.

Furthermore, we denote by $S^+_{k+\frac{1}{2}}$ the space of cusp forms $f(\tau)$ of weight $k + \frac{1}{2}$ and of level 4 such that $f(\tau) =$

$$\sum_{m>1, (-1)^k m \equiv 0,1(4)} a(m) e[m\tau], \quad f(\frac{a\tau+b}{c\tau+d}) = (\left(\frac{-1}{d}\right)^{-\frac{1}{2}} \left(\frac{c}{d}\right) (c\tau+d)^{\frac{1}{2}})^{2k+1} f(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \pmod 4 \right\}$, where $\left(\frac{*}{*}\right)$ means the quadratic residue symbol given in [7]. We may refer [4] to the Kohnen plus space.

The following striking theorem is proved by Ikeda.

Theorem 1.1(T.Ikeda). *Suppose that g is a Hecke eigenform in $S^+_{k+\frac{1}{2}}$ and let n and k be positive integers with $n \equiv k \pmod 2$. Then there exists a Hecke eigenform $F(Z) \in S_{k+n}(\Gamma_{2n})$ such that the Fourier coefficients of F are explicitly determined and its standard zeta function is equal to*

$$\zeta(s) \prod_{j=1}^{2n} L(f, s+k+n-j),$$

where f is the normalized Hecke eigen form in $S_{2k}(\Gamma_1)$ which is the image of g under the Shimura correspondence and $L(f,s)$ is the Hecke L-function of f.

Remark. When $n = 1$, F is the Saito-Kurokawa lifting of g (cf. [2], [9]).

As $S^+_{k+\frac{1}{2}}$ has a basis consisting of Hecke eigen forms, by Ikeda theorem, we can formulate the Ikeda lifting as a linear mapping

$$I_{k,n} : S^+_{k+\frac{1}{2}} \to S_{k+n}(\Gamma_{2n}).$$

To find out an analogue of Maass spaces of degree two in higher degree which is the image of $I_{k,n}$, Kohnen [5] expressed explicitly the Fourier coefficients of $I_{k,n}(g)$ in terms of those of $g \in S^+_{k+\frac{1}{2}}$.

Theorem 1.2(W.Kohnen). *Suppose that $n \equiv k \pmod 2$. Let $g(\tau) = \sum_{m>1,(-1)^k m \equiv 0,1(4)} c(m)e[m\tau]$ be an element of $S^+_{k+\frac{1}{2}}$. Then the Fourier coefficient of $I_{k,n}(g)$ at $T \in S^+_{2n}(\mathbb{Z})$ is given by*

$$\sum_{a|f_T} a^{k-1}\phi(a;T)c(|D_T|/a^2),$$

where $D_T = (-1)^n \det(2T) = D_{T,0} f_T^2$ with $f_T \in \mathbb{Z}\ (>0)$ and $D_{T,0}$ is the discriminant of $\mathbb{Q}(\sqrt{D_T})$ and $\phi(a;T)$ $(a|f_T)$ is a certain multiplicative function of a $(a|f_T)$ determined by T.

It is an interesting problem to determine whether the image of Ikeda lifting is characterized in terms of Fourier coefficients or not, which is purposed by [5].

Conjecture. *Under the same assumption as in the Theorem 1.2, the space*

$I_{k,n}(S^+_{k+\frac{1}{2}})$ *coincides with the following space*

$$M_{k+n}(\Gamma_{2n}) = \Big\{F(Z) = \sum_{T\in S_n^+(\mathbb{Z})} A(T)e[tr(TZ)] \in S_{k+n}(\Gamma_{2n})\Big|$$

$$A(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)\tilde{c}(|D_T|/a^2) \text{ with certain complex numbers}$$

$$\tilde{c}(m)\ \ (m>1, (-1)^k m \equiv 0, 1 (mod\ 4)) \text{ for every } T \in S_n^+(\mathbb{Z})\cdots(*)\Big\}.$$

§2 Our results. We may deduce that the Kohnen's conjecture is ture under the some assumption. Our main result is the following.

Theorem 2.1. *Suppose that* $n \equiv 0, 1 \pmod 4$ *and* $n \equiv k \pmod 2$. *Then* $I_{k,n}(S^+_{k+\frac{1}{2}})$ *is equal to* $M_{k+n}(\Gamma_{2n})$.

To state our second result, recall [1, chap. 15, sect. 8.2, table 15.5] that for each $g \in \mathbb{Z}\ (>0)$ there exists exactly one genus of integral, even, symmetric matrices S of size g with determinant equal to 2. A matrix in this genus is positive definite if and only if $g \equiv \pm 1 \pmod 8$, and in this case as a representative we can take

$$S_0 = \begin{cases} E_8^{\oplus\frac{g-1}{8}} \oplus 2 & \text{if } g \equiv 1 \pmod 8, \\ E_8^{\oplus\frac{g-7}{8}} \oplus E_7 & \text{if } g \equiv -1 \pmod 8, \end{cases}$$

where (by abuse of language) E_8 and E_7 denote the Gram matrices of the E_8- and E_7-root lattices, respectively. Explicitly, recall that

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and E_7 is the upper $(7,7)$-submatrix of E_8.

For $m \in \mathbb{Z}\ (>0)$ with $(-1)^n m \equiv 0, 1 \pmod 4$, define a rational, half-integral, symmetric, positive definite matrix T_m of size $2n$ by

$$T_m = \begin{cases} \begin{pmatrix} \frac{1}{2}S_0 & 0 \\ 0 & \frac{m}{4} \end{pmatrix} & \text{if } m \equiv 0 \pmod 4, \\ \begin{pmatrix} \frac{1}{2}S_0 & \frac{1}{2}e_{2n-1} \\ \frac{1}{2}e'_{2n-1} & \frac{m+2+(-1)^n}{4} \end{pmatrix} & \text{if } m \equiv (-1)^n \pmod 4, \end{cases}$$

where $e_{2n-1} = (0, \ldots, 0, 1)' \in M_{2n-1,1}(\mathbb{Z})$ is the usual standard column vector.

As a consequence of Theorem 2.1, we may deduce the following.

Corollary ("Maass relations"). *Under the same assumptions as in the Theorem 2.1 and* $F(Z) = \sum_{T \in S_n^+(\mathbb{Z})} A(T) e[tr(TZ)] \in S_{k+n}(\Gamma_{2n})$, *the following assertions are equivalent:*

i) $F \in I_{k,n}(S^+_{k+\frac{1}{2}})$;

ii) for all T, *one has*

$$A(T) = \sum_{a | f_T} a^{k-1} \phi(a;T) A(T_{|D_T|/a^2}).$$

§3 Outline of our proof. The key point of our proof is to verify that the complex number $\tilde{c}(m)$ appeared in the relation (*) of Fourier coefficients of $F \in M_{k+n}(\Gamma_{2n})$ is equal to the Fourier coefficients of an element of $S^+_{k+\frac{1}{2}}$. We proceed to details. To perform it, we need to look at a Fourier-Jacobi coefficient of F. Let $F(Z) = \sum_{T \in S_n^+(\mathbb{Z})} A(T) e[\mathrm{tr}(TZ)]$ be an elment of $S_{k+n}(\Gamma_{2n})$. For convenience, let us put $\tilde{T_0} = \frac{1}{2} S_0$. For $(\tau, z) \in \mathfrak{H}_1 \times M_{2n-1,1}(\mathbb{C})$ and $\tilde{Z} \in \mathfrak{H}_{2n-1}$, consider the following Fourier expansion of F.

$$F(\begin{pmatrix} \tilde{Z} & z \\ z' & \tau \end{pmatrix}) = \sum_{\tilde{T} \in S^+_{2n-1}(\mathbb{Z})} \phi_{\tilde{T}}(\tau, z) e[tr(\tilde{T}\tilde{z})],$$

where

$$\phi_{\tilde{T}}(\tau, z) = \sum_{T = \begin{pmatrix} \tilde{T} & r/2 \\ r'/2 & N \end{pmatrix} \in S^+_{2n}(\mathbb{Z})} A(\begin{pmatrix} \tilde{T} & r/2 \\ r'/2 & N \end{pmatrix}) e[r'z + N\tau].$$

Then $\phi_{\tilde{T}}(\tau, z)$ is Jacobi cusp forms of index $\tilde{T}$ and weight $k+n$. We know that $\phi_{\tilde{T}_0}$ has an expansion in terms of Jacobi theta functions

$$\phi_{\tilde{T}_0}(\tau, z) = \sum_{\lambda \in \Lambda} h_\lambda(\tau) \theta_\lambda(\tau, z),$$

where $\Lambda = S_0^{-1} M_{2n-1,1}(\mathbb{Z}) / M_{2n-1,1}(\mathbb{Z})$ and where for $\lambda \in \Lambda$ one sets

$$h_\lambda(\tau) = \sum_{N \in \mathbb{Z}, N - \tilde{T}_0[\lambda] > 0} A(\begin{pmatrix} \tilde{T}_0 & \tilde{T}_0 \lambda \\ \lambda' \tilde{T}_0 & N \end{pmatrix}) e[(N - \tilde{T}_0[\lambda])\tau] \ \ (\tau \in \mathfrak{H}_1)$$

and

$$\theta_\lambda(\tau, z) = \sum_{r \in M_{2n-1,1}(\mathbb{Z})} e[(\tilde{T}_0[r+\lambda]\tau + 2(r+\lambda)'\tilde{T}_0 z)] \ \ (\tau \in \mathfrak{H}_1,\ z \in M_{2n-1,1}(\mathbb{C})).$$

We note that $|\Lambda| = 2$ and that representatives can be chosen as λ_0, λ_1 where λ_0 is the zero vector and $\lambda_1 = S_0^{-1} e_{2n-1}$. We claim that the function

$$h(\tau) := h_0(4\tau) + h_1(4\tau) \ \ (\tau \in \mathfrak{H}_1)$$

is in $S^+_{k+\frac{1}{2}}$, where $h_i(\tau) = h_{\lambda_i}(\tau)$ $(i = 0, 1)$. Indeed, employing transformation formulas of $\phi_{\tilde{T}_0}(\tau, z)$ and $\theta_\lambda(\tau, z)$, we find that

$$\begin{pmatrix} h_0(\tau+1) \\ h_1(\tau+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon_n i \end{pmatrix} \begin{pmatrix} h_0(\tau) \\ h_1(\tau) \end{pmatrix},$$

$$\begin{pmatrix} h_0(-\frac{1}{\tau}) \\ h_1(-\frac{1}{\tau}) \end{pmatrix} = \frac{1+\epsilon_n i}{2} \tau^{k+\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} h_0(\tau) \\ h_1(\tau) \end{pmatrix}$$

where $\epsilon_n = (-1)^{n+1}$ (*cf.* [8]). By virtue of this formula, we deduce that

$$h(\tau+1) = h(\tau), \; h(\frac{\tau}{4\tau+1}) = (4\tau+1)^{k+\frac{1}{2}} h(\tau).$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ generate $\Gamma_0(4)$, we conclude that $h(\tau)$ behaves like a modular form of weight $k + \frac{1}{2}$ and level 4. From the above transformation formulas and the definition of $h(\tau)$ one sees that $h(\tau)$ has the Fourier expansion $\sum_{(-1)^k m \equiv 0,1 \pmod 4} c'(m) e[m\tau]$ and it is cuspidal then imply that $h(\tau)$ in fact is contained in $S^+_{k+\frac{1}{2}}$. Define a linear mapping $\Psi_{k,n} : S_{n+k}(\Gamma_{2n}) \longrightarrow S^+_{k+\frac{1}{2}}$ by

$$\Psi_{k,n}(F) = h \quad \text{for every } F \in S_{n+k}(\Gamma_{2n}).$$

Now we impose the condition that $F \in M_{n+k}(\Gamma_{2n})$. By a formal calculation, using only the definition of $\phi(a; \tilde{T})$ in exactly the same way as in the proof of [5, Prop. 2, p. 801], we may deduce that

$$c'(m) = A(\tilde{T}) = \sum_{a | f_{\tilde{T}}} a^{k-1} \phi(a; \tilde{T}) \tilde{c}(|D_{\tilde{T}}|/a^2) = \tilde{c}(m),$$

where $\tilde{T}$ is the matrix given in $h_{\lambda_i}(\tau)$ and $\tilde{c}(m)$ is the complex number appeared in the Fourier expansion of F. Therefore the restriction mapping $\Psi_{k,n} | M_{k+n}(\Gamma_{2n}) : M_{k+n}(\Gamma_{2n}) \longrightarrow S^+_{k+\frac{1}{2}}$ is injective and it gives the converse mapping of $I_{k,n}$. This proves our assertions.

References

[1] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, Grundl.d.math. Wiss. no. 290, Springer: New York Berlin Heidelberg, 1988.

[2] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progr. Math. Birkhauser Boston, Inc, Boston, Mass. **55** (1985).

[3] T. Ikeda, *On the lifting of elliptic modular forms to Siegel modular cusp forms of degree 2n*, Ann. of Math. **154** (2001), 641–681.

[4] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249–266.

[5] W. Kohnen, *Lifting modular forms of half integral weight to Siegel modular forms of even genus*, Math. Ann **322** (2002), 787–809.

[6] W. Kohnen and H. Kojima, *A Maass space in higher genus*, to appear in Compositio Math.

[7] G. Shimura, *On modular forms of half-integral weight*, Ann. of Math. **97** (1973), 440–481.

[8] G. Shimura, *On certain reciprocity laws for theta functions and modular forms*, Acta Math. **141** (1978), 35–71.

[9] D. Zagier, *Sur la conjecture de Saito-Kurokawa (d'apres H. Maass). In: Sem. de Theorie des Nombres, Paris: 1979-1980, Sem. Delange-Pisot-Poitou(ed. M.-J. Bertin), pp. 371-394. Progress in Math. vol. 12, Birkhauser:Boston, 1981.*

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