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Orlicz-Morrey spaces and some integral operators

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1. Introduction

For the Hardy-Littlewood maximal operator M, Calderón-Zygmund operator T and fractional integral operator I_α, $0<\alpha<n$, it is well known that

$$M: L^1(\mathbb{R}^n) \to L^1_{\mathrm{weak}}(\mathbb{R}^n), \quad M: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1<p\le\infty,$$
$$T: L^1(\mathbb{R}^n) \to L^1_{\mathrm{weak}}(\mathbb{R}^n), \quad T: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad 1<p<\infty,$$
$$I_\alpha: L^1(\mathbb{R}^n) \to L^{n/(n-\alpha)}_{\mathrm{weak}}(\mathbb{R}^n), \quad I_\alpha: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n), \quad \begin{cases} 1<p<q<\infty, \\ -n/p+\alpha=-n/q, \end{cases}$$

These boundedness extended to several function spaces which are generalizations of L^p-spaces, for example, Orlicz spaces, Morrey spaces, Lorentz spaces, Herz spaces, etc.

For boundedness of M on Orlicz spaces $L^\Phi(\mathbb{R}^n)$, Kita [10] (1997) proved a necessary and sufficient condition on Φ and Ψ for

$$M: L^\Phi(\mathbb{R}^n) \to L^\Psi(\mathbb{R}^n).$$

Cianchi [3] (1999) proved necessary and sufficient conditions on Φ and Ψ for

$$M,T,I_\alpha: L^\Phi(\mathbb{R}^n) \to L^\Psi_{\mathrm{weak}}(\mathbb{R}^n),$$
$$M,T,I_\alpha: L^\Phi(\mathbb{R}^n) \to L^\Psi(\mathbb{R}^n).$$

For boundedness of I_α on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, see Peetre (Spanne) [18] (1969) Adams [1], 1975 Chiarenza and Frasca [2] (1987), Nakai (1995). Chiarenza and Frasca showed

$$M,T: L^{p,\lambda}(\mathbb{R}^n) \to L^{p,\lambda}(\mathbb{R}^n),$$
$$I_\alpha: L^{p,\lambda}(\mathbb{R}^n) \to L^{q,\lambda}(\mathbb{R}^n).$$

The author studied boundedness of genaralized fractional integral operators on Orlicz-Morrey spaces in [17]. Orlicz-Morrey spaces are useful to estimate genaralized fractional integral operators. In this paper we investigate boundedness and weak boundedness of the Hardy-Littlewood maximal operator, singular integtral operators and generalized fractional integral operators. We refine on the results in [17].

2. Defnitions

A function $\theta : (0,+\infty) \to (0,+\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\theta(r) \le C\theta(s) \quad (\theta(r) \ge C\theta(s)) \quad \text{for} \quad r \le s.$$

A function $\theta : (0,+\infty) \to (0,+\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\theta(r)}{\theta(s)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{r}{s} \le 2.$$

For functions $\theta, \kappa : (0,+\infty) \to (0,+\infty)$, we denote $\theta(r) \sim \kappa(r)$ ($\theta(r) \approx \kappa(r)$) if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r) \quad (\theta(C^{-1}r) \le \kappa(r) \le \theta(Cr)) \quad \text{for} \quad r > 0.$$

Let $\mathcal{Y}$ be the set of all convex functions $\Phi : [0,+\infty) \to [0,+\infty)$ such that $\Phi(r) > 0$ for $r > 0$. If $\Phi \in \mathcal{Y}$, then Φ is increasing, absolutely continuous and bijective.

For a measurable set $\Omega \subset \mathbb{R}^n$, we denote the characteristic function of Ω by χ_Ω and the Lebsgue measure of Ω by $|\Omega|$. For a measuralbe set $\Omega \subset \mathbb{R}^n$, a measuralbe function f and $t > 0$, let

$$m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,t)$.

Definition 2.1 (Orlicz space). For $\Phi \in \mathcal{Y}$ let

$$L^\Phi(\mathbb{R}^n) = \left\{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^\Phi} < +\infty\right\},$$
$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\},$$
$$L^\Phi_{\text{weak}}(\mathbb{R}^n) = \left\{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^\Phi_{\text{weak}}} < +\infty\right\},$$
$$\|f\|_{L^\Phi_{\text{weak}}} = \inf\left\{\lambda > 0 : \sup_{t>0} \Phi(t)\, m\left(\frac{f}{\lambda}, t\right) \le 1\right\}.$$

If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^\Phi_{\text{weak}}(\mathbb{R}^n) = L^p_{\text{weak}}(\mathbb{R}^n)$.

If $\Phi(r) \le \Psi(Cr)$, then $L^\Phi(\mathbb{R}^n) \supset L^\Psi(\mathbb{R}^n)$ and $\|f\|_{L^\Phi} \le C\|f\|_{L^\Psi}$. If $\Phi \approx \Psi$, then $L^\Phi(\mathbb{R}^n) = L^\Psi(\mathbb{R}^n)$ and $\|f\|_{L^\Phi} \sim \|f\|_{L^\Psi}$.

Let $B(a,r)$ be the ball $\{x \in \mathbb{R}^n : |x-a| < r\}$ with center a and of radius $r > 0$.

Definition 2.2 (Morrey space). For $1 \le p < \infty$ and $0 \le \lambda \le n$, let

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}} < +\infty\},$$

$$\|f\|_{L^{p,\lambda}} = \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda}\int_B |f(x)|^p\,dx\right)^{1/p},$$

$$L^{p,\lambda}_{\mathrm{weak}}(\mathbb{R}^n) = \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}_{\mathrm{weak}}} < +\infty\right\},$$

$$\|f\|_{L^{p,\lambda}_{\mathrm{weak}}} = \sup_{B=B(a,r)}\sup_{t>0} \frac{t^p m(B,f,t)}{r^\lambda}.$$

If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,\lambda}_{\mathrm{weak}}(\mathbb{R}^n) = L^p_{\mathrm{weak}}(\mathbb{R}^n)$. If $\lambda = n$, then $L^{p,\lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Let $\mathcal{G}$ be the set of all functions $\phi : (0,+\infty) \to (0,+\infty)$ such that ϕ is almost decreasing and $\phi(r)r^n$ is almost increasing. If $\phi \in \mathcal{G}$ then ϕ satisfies doubling condition.

Definition 2.3. For $1 \le p < \infty$ and a function $\phi \in \mathcal{G}$,

$$L^{(p,\phi)}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(p,\phi)}} < +\infty\},$$

$$\|f\|_{L^{(p,\phi)}} = \sup_{B=B(a,r)} \left(\frac{1}{|B|\phi(r)}\int_B |f(x)|^p\,dx\right)^{1/p},$$

$$L^{(p,\phi)}_{\mathrm{weak}}(\mathbb{R}^n) = \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(p,\phi)}_{\mathrm{weak}}} < +\infty\right\},$$

$$\|f\|_{L^{(p,\phi)}_{\mathrm{weak}}} = \sup_{B=B(a,r)}\sup_{t>0} \frac{t^p m(B,f,t)}{|B|\phi(r)}.$$

If $\phi(r) = r^{-n+\lambda}$, $0 \le \lambda \le n$, then $L^{(p,\phi)}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ and $L^{(p,\phi)}_{\mathrm{weak}}(\mathbb{R}^n) = L^{p,\lambda}_{\mathrm{weak}}(\mathbb{R}^n)$.

If $p \le q$, then $L^{(p,\phi)}(\mathbb{R}^n) \supset L^{(q,\phi^{q/p})}(\mathbb{R}^n)$ and $\|f\|_{L^{(p,\phi)}} \le \|f\|_{L^{(q,\phi^{q/p})}}$. If $\phi(r) \le C\psi(r)$, then $L^{(p,\phi)}(\mathbb{R}^n) \subset L^{(p,\psi)}(\mathbb{R}^n)$ and $\|f\|_{L^{(p,\phi)}} \ge C^{-1/p}\|f\|_{L^{(p,\psi)}}$. If $\phi \sim \psi$, then $L^{(p,\phi)}(\mathbb{R}^n) = L^{(p,\psi)}(\mathbb{R}^n)$ and $\|f\|_{L^{(p,\phi)}} \sim \|f\|_{L^{(p,\psi)}}$.

For $\Phi \in \mathcal{Y}$, a function $\phi \in \mathcal{G}$ and a ball $B = B(a,r)$, let

$$\|f\|_{\Phi,\phi,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|\phi(r)}\int_B \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\},$$

$$\|f\|_{\Phi,\phi,B,\mathrm{weak}} = \inf\left\{\lambda > 0 : \sup_{t>0}\frac{\Phi(t)\,m(B,f/\lambda,t)}{|B|\phi(r)} \le 1\right\}.$$

We note that

$$\sup_{t>0} \Phi(t)m(\Omega, f, t) = \sup_{t>0} t\, m(\Omega, f, \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega, \Phi(|f|), t).$$

Definition 2.4 (Orlicz-Morrey space). For $\Phi \in \mathcal{Y}$ and a function $\phi \in \mathcal{G}$, let

$$L^{(\Phi,\phi)}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(\Phi,\phi)}} < +\infty \right\},$$
$$\|f\|_{L^{(\Phi,\phi)}} = \sup_B \|f\|_{\Phi,\phi,B},$$
$$L^{(\Phi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} < +\infty \right\},$$
$$\|f\|_{L^{(\Phi,\phi)}_{\mathrm{weak}}} = \sup_B \|f\|_{\Phi,\phi,B,\mathrm{weak}}.$$

If $\Phi \approx \Psi$ and $\phi \sim \psi$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{(\Psi,\psi)}(\mathbb{R}^n)$.

If $\Phi(r) = r^p$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{(p,\phi)}(\mathbb{R}^n)$ and $L^{(\Phi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n) = L^{(p,\phi)}_{\mathrm{weak}}(\mathbb{R}^n)$. If $\phi(r) = r^{-n}$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{\Phi}(\mathbb{R}^n)$ and $L^{(\Phi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n) = L^{\Phi}_{\mathrm{weak}}(\mathbb{R}^n)$. If $\phi(r) \equiv 1$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$. If $\Phi(r) = r^p$ and $\phi(r) = r^{\lambda-n}$, then $L^{(\Phi,\phi)}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$.

A function $\Phi \in \mathcal{Y}$ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \quad r \ge 0,$$

for some $C > 0$.

A function $\Phi \in \mathcal{Y}$ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \quad r \ge 0,$$

for some $k > 1$.

Let $\mathcal{Y}^+$ be the set of all $\Phi \in \mathcal{Y}$ with $\int_0^1 \Phi(t)t^{-2}\,dt < +\infty$. For $\Phi \in \mathcal{Y}^+$, let

$$\Phi^+(r) = r\int_0^r \frac{\Phi(t)}{t^2}\,dt, \quad r \ge 0.$$

Then $\Phi^+ \in \mathcal{Y}$ and $\Phi(r) \le \Phi^+(2r)$. If Φ satisfies the ∇_2-condition, then $\Phi \in \mathcal{Y}^+$ and $\Phi^+ \approx \Phi$.

Example 2.1. For $\epsilon > 0$ and $\delta \ge 0$, let $\Phi \in \mathcal{Y}$ with

$$\Phi(r) = \begin{cases} r(\log(1/r))^{-\epsilon-1} & \text{for small } r > 0, \\ r(\log r)^{\delta} & \text{for large } r > 0. \end{cases}$$

Then $\Phi \in \mathcal{Y}^+$ and

$$\Phi^+(r) \approx \begin{cases} r(\log(1/r))^{-\epsilon} & \text{for small } r > 0, \\ r(\log r)^{\delta+1} & \text{for large } r > 0. \end{cases}$$

Example 2.2. For $1 < p < \infty$, $\epsilon \in \mathbb{R}$ and $\delta \in \mathbb{R}$, let $\Phi \in \mathcal{Y}$ with

$$\Phi(r) = \begin{cases} r^p(\log(1/r))^{-\epsilon} & \text{for small } r > 0, \\ r^p(\log r)^\delta & \text{for large } r > 0. \end{cases}$$

Then $\Phi \in \nabla_2$ and $\Phi^+ \approx \Phi$.

For a function $\theta : (0, +\infty) \to (0, +\infty)$, let

$$\theta^*(r) = \int_0^r \frac{\theta(t)}{t}\, dt, \quad \theta_*(r) = \int_r^{+\infty} \frac{\theta(t)}{t}\, dt.$$

If θ satisfies the doubling condition, then $\theta(r) \le C\theta^*(r)$ and $\theta(r) \le C\theta_*(r)$. If $\theta(r)r^{-\epsilon}$ is almost increasing for some $\epsilon > 0$, then $\theta^*(r) \le C\theta(r)$. If $\theta(r)r^\epsilon$ is almost dereasing for some $\epsilon > 0$, then $\theta_*(r) \le C\theta(r)$.

If $\phi \in \mathcal{G}$ and $\int_0^1 \phi(t)\, dt < +\infty$, then $\phi^* \in \mathcal{G}$. If $\phi \in \mathcal{G}$ and $\int_1^{+\infty} \phi(t)\, dt < +\infty$, then $\phi_* \in \mathcal{G}$.

3. Main results

The Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B containing x.

The following are our main results.

Theorem 3.1. *Let $\phi \in \mathcal{G}$ and $\Phi \in \mathcal{Y}$. Then*

$$M : L^{(\Phi,\phi)}(\mathbb{R}^n) \to L^{(\Phi,\phi)}_{\mathrm{weak}}(\mathbb{R}^n).$$

If $\Phi \in \mathcal{Y}^+$, then

$$M : L^{(\Phi^+,\phi)}(\mathbb{R}^n) \to L^{(\Phi,\phi)}(\mathbb{R}^n).$$

If $\Phi \in \nabla_2$, then

$$M : L^{(\Phi,\phi)}(\mathbb{R}^n) \to L^{(\Phi,\phi)}(\mathbb{R}^n).$$

Next let T be a singular integral operator with a kernel $K(x, y)$ which satisfies

$$|K(x,y)| \le \frac{C}{|x-y|^n}, \quad x, y \in \mathbb{R}^n, x \ne y, \tag{3.1}$$

and, for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad \text{a.e. } x \notin \operatorname{supp} f. \tag{3.2}$$

Theorem 3.2. *Let $\Phi \in \Delta_2$, $\phi, \psi \in \mathcal{G}$ and $\int_1^{+\infty} \Phi^{-1}(\phi(t))/t\,dt < +\infty$. Assume that there exists a constant $A > 0$ such that*

$$\Phi\left((\Phi^{-1} \circ \phi)_*(r)\right) \le A\psi(r), \quad r > 0.$$

(i) *If $T : L^{\Phi}(\mathbb{R}^n) \to L^{\Phi}_{\text{weak}}(\mathbb{R}^n)$, then*

$$T : L^{(\Phi,\phi)}(\mathbb{R}^n) \to L^{(\Phi,\psi)}_{\text{weak}}(\mathbb{R}^n).$$

(ii) *If $T : L^{\Phi}(\mathbb{R}^n) \to L^{\Phi}(\mathbb{R}^n)$, then*

$$T : L^{(\Phi,\phi)}(\mathbb{R}^n) \to L^{(\Phi,\psi)}(\mathbb{R}^n).$$

Remark 3.1. Since $\Phi^{-1} \circ \phi$ satisfies doubling condition, $\phi(r) = \Phi(\Phi^{-1} \circ \phi(r)) \le \Phi(C(\Phi^{-1} \circ \phi)_*(r)) \le C'\psi(r)$ for all $r > 0$.

For a function $\rho : (0, +\infty) \to (0, +\infty)$, let

$$I_\rho f(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n}\,dy.$$

We consider the following conditions on ρ:

$$\int_0^1 \frac{\rho(t)}{t}\,dt < +\infty, \tag{3.3}$$

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le A_1 \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{3.4}$$

$$\frac{\rho(r)}{r^n} \le A_2 \frac{\rho(s)}{s^n} \quad \text{for} \quad s \le r. \tag{3.5}$$

If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the fractional integral denoted by I_α.

Theorem 3.3. *Let $\phi \in \mathcal{G}$, $\Phi, \Psi \in \mathcal{Y}$. Assume that ϕ is bijective. If there exists a constant $A > 0$ such that*

$$\Psi\left(\frac{\Phi^{-1} \circ \phi(r)\rho^*(r) + ((\Phi^{-1} \circ \phi)\rho)_*(r)}{A}\right) \le \phi(r), \quad r > 0,$$

then

$$I_\rho : L^{(\Phi,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}_{\text{weak}}(\mathbb{R}^n).$$

If $\Phi \in \mathcal{Y}^+$ and there exists a constant $A > 0$ such that

$$\Psi\left(\frac{(\Phi^+)^{-1} \circ \phi(r)\rho^*(r) + (((\Phi^+)^{-1} \circ \phi)\rho)_*(r)}{A}\right) \le \phi(r), \quad r > 0,$$

then

$$I_\rho : L^{(\Phi^+,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}(\mathbb{R}^n).$$

Moreover, if $\Phi \in \nabla_2$, *then*

$$I_\rho : L^{(\Phi,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}(\mathbb{R}^n).$$

Example 3.1. Let $0 < \alpha < n$, $1 < p < q < \infty$, $-1/p + \alpha/(n-\lambda) = -1/q$,

$$\rho(r) = r^\alpha,$$
$$\Phi(r) = r^p, \quad \Psi(r) = r^q,$$
$$\phi(r) = r^{\lambda-n}.$$

Then $\Phi \in \nabla_2$ and

$$\Phi^{-1} \circ \phi(r)\rho^*(r) + ((\Phi^{-1} \circ \phi)\rho)_*(r) \sim r^{(\lambda-n)/p+\alpha} = r^{(\lambda-n)/q}.$$

We have

$$I_\alpha : L^{p,\lambda}(\mathbb{R}^n) \to L^{q,\lambda}(\mathbb{R}^n).$$

This is the result of Adams [1] (1975).

Example 3.2. Let $\ell : (0,+\infty) \to (0,+\infty)$ satisfy the doubling condition and

$$\ell(r) = \begin{cases} (\log 1/r)^{-1} & \text{for small } r > 0, \\ \log r & \text{for large } r > 0. \end{cases}$$

For $\beta > 0$, let

$$\rho(r) = \begin{cases} (\log 1/r)^{-\beta-1} & \text{for small } r > 0, \\ (\log r)^{\beta-1} & \text{for large } r > 0. \end{cases}$$

Then ρ satisfies (3.3)–(3.5) and

$$\rho^*(r) = \int_0^r \frac{\rho(t)}{t}\, dt \sim \ell^\beta(r).$$

Let

$$\Phi(r) = r^p \ (1 \le p < \infty), \ \Psi(r) = r^p \ell^{p\beta}(r),$$

and $\phi(r) = r^{\lambda-n}$ $(0 \le \lambda < n)$. Then we have

$$I_\rho : L^{1,\lambda}(\mathbb{R}^n) = L^{(1,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}_{\text{weak}}(\mathbb{R}^n),$$
$$I_\rho : L^{p,\lambda}(\mathbb{R}^n) = L^{(p,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}(\mathbb{R}^n),$$
$$1 < p < \infty.$$

Example 3.3. Let ℓ and ρ be as in Example 3.2. For $p > 0$, let

$$e_p(r) = \begin{cases} 1/\exp(1/r^p) & \text{for small } r > 0, \\ \exp(r^p) & \text{for large } r > 0. \end{cases}$$

Let

$$\Phi(r) = e_p(r), \quad \Psi(r) = e_q(r),$$
$$-1/p + \beta = -1/q < 0,$$
$$\phi(r) = r^{-\mu}, \ 0 < \mu \le n.$$

Then we have

$$I_\rho : L^{(\Phi,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}(\mathbb{R}^n).$$

Example 3.4. Let ℓ and ρ be as in Example 3.2. For $\epsilon > 0$, $\delta \ge 0$ and $\beta > 0$, let

$$\Phi(r) = \begin{cases} r(\log(1/r))^{-\epsilon-1} & \text{for small } r > 0, \\ r(\log r)^{\delta} & \text{for large } r > 0, \end{cases}$$
$$\Psi(r) = \begin{cases} r(\log(1/r))^{-\epsilon-\beta} & \text{for small } r > 0, \\ r(\log r)^{\delta+1+\beta} & \text{for large } r > 0, \end{cases}$$
$$\phi(r) = r^{-\mu}, \ 0 < \mu \le n.$$

Then

$$\Phi^+(r) \approx \begin{cases} r(\log(1/r))^{-\epsilon} & \text{for small } r > 0, \\ r(\log r)^{\delta+1} & \text{for large } r > 0. \end{cases}$$

Then we have

$$I_\rho : L^{(\Phi^+,\phi)}(\mathbb{R}^n) \to L^{(\Psi,\phi)}(\mathbb{R}^n).$$

4. Proofs

Lemma 4.1. *For $B = B(a,r)$,*

$$\int_B f(x)g(x)\,dx \le 2|B|\phi(r)\|f\|_{\Phi,B,\phi}\|g\|_{\tilde{\Phi},B,\phi}.$$

Proof. By Hölder's inequality for the Orlicz spaces $L^{\Phi}(B, dx/(|B|\phi(r)))$ and $L^{\tilde{\Phi}}(B, dx/(|B|\phi(r)))$, we have

$$\begin{aligned}\int_B f(x)g(x)\frac{dx}{|B|\phi(r)} &\le 2\|f\|_{L^{\Phi}(B,dx/(|B|\phi(r)))}\|g\|_{L^{\tilde{\Phi}}(B,dx/(|B|\phi(r)))} \\ &= 2\|f\|_{\Phi,B,\phi}\|g\|_{\tilde{\Phi},B,\phi}. \quad \square\end{aligned}$$

Lemma 4.2. *If a Young function Φ is bijective and $B = B(a,r)$, then $\|1\|_{\tilde{\Phi},B,\phi} \le \Phi^{-1}(\phi(r))/\phi(r)$.*

Proof. By $\Phi(t)s/t - \Phi(s) \le \Phi(t)$ for $s < t$ and $\Phi(t)s/t - \Phi(s) \le 0$ for $s \ge t$, we have $\widetilde{\Phi}(\Phi(t)/t) \le \Phi(t)$. Let $\Phi(t) = \phi(r)$ and $\lambda = t/\Phi(t) = \Phi^{-1}(\phi(r))/\phi(r)$. Then

$$\frac{1}{|B|}\int_B \widetilde{\Phi}\left(\frac{1}{\lambda}\right) dx = \widetilde{\Phi}\left(\frac{1}{\lambda}\right) \le \Phi(t) = \phi(r). \quad \square$$

Lemma 4.3. *If $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and $\operatorname{supp} f \cap B(a, 2r) = \emptyset$. Then*

$$Mf(x) \le C\Phi^{-1}(\phi(r))\|f\|_{L^{(\Phi,\phi)}} \quad \textit{for} \quad x \in B(a,r).$$

Proof. For all balls $B = B(b,s)$, if $s \le r/2$ then $\int_B |f(x)|\,dx = 0$, and, if $s > r/2$, then

$$\begin{aligned}\int_B |f(x)|\,dx &\le 2\phi(s)\|f\|_{\Phi,\phi,B}\|1\|_{\widetilde{\Phi},\phi,B} \\ &\le 2\phi(s)\|f\|_{L^{(\Phi,\phi)}}\Phi^{-1}(\phi(s))/\phi(s) \\ &\le 2\Phi^{-1}(\phi(s))\|f\|_{L^{(\Phi,\phi)}} \\ &\le C\Phi^{-1}(\phi(r))\|f\|_{L^{(\Phi,\phi)}}. \quad \square\end{aligned}$$

Theorem 4.4 (see [3, 10, 12]). *Let $\Phi \in \mathcal{Y}$. Then*

$$\sup_{t>0} \Phi(t)m(f,t) \le c_1 \int_{\mathbb{R}^n} \Phi(c_1|f(x)|)\,dx.$$

If $\Phi \in \mathcal{Y}^+$, then

$$\int_{\mathbb{R}^n} \Phi(Mf(x))\,dx \le c_1 \int_{\mathbb{R}^n} \Phi^+(c_1|f(x)|)\,dx.$$

If $\Phi \in \nabla_2$, then

$$\int_{\mathbb{R}^n} \Phi(Mf(x))\,dx \le c_1 \int_{\mathbb{R}^n} \Phi(c_1|f(x)|)\,dx.$$

Proof of Theorem 3.1. For all balls $B = B(a,r)$, let $f = f_1 + f_2$, $f_1 = f\chi_{2B}$. For f_1, applying Theorem 4.4, we have

$$\sup_{t>0} \Phi(t)m(B, f_1/\lambda, t) \le c_1 \int_{\mathbb{R}^n} \Phi(c_1|f_1(x)|/\lambda)\,dx, \quad \text{if } \Phi \in \mathcal{Y},$$

$$\int_B \Phi(Mf_1(x)/\lambda)\,dx \le c_1 \int_{\mathbb{R}^n} \Phi^+(c_1|f_1(x)|/\lambda)\,dx, \quad \text{if } \Phi \in \mathcal{Y}^+.$$

We may assume that $c_1 \ge 1$. Let $c_* \ge 1$ be a constant such that $\phi(r) \le c_*\phi(s)$ for $r \ge s$. Let $\lambda = 2^n c_* c_1^2 \|f\|_{L^{(\Phi,\phi)}}$ in the cases $\Phi \in \mathcal{Y}$. Then we have

$$\begin{aligned}c_1\int_{\mathbb{R}^n} \Phi(c_1|f_1(x)|/\lambda)\,dx &= c_1\int_{2B} \Phi(c_1|f(x)|/\lambda)\,dx \\ &\le 2^{-n}{c_*}^{-1}\int_{2B} \Phi(|f(x)|/\|f\|_{L^{(\Phi,\phi)}})\,dx \le 2^{-n}{c_*}^{-1}|2B|\phi(2r) \le |B|\phi(r).\end{aligned}$$

Let $\lambda = 2^n c_* c_1^2 \|f\|_{L^{(\Phi^+,\phi)}}$ in the case $\Phi \in \mathcal{Y}^+$. Then we have in the same wey

$$c_1 \int_{\mathbb{R}^n} \Phi^+(c_1|f_1(x)|/\lambda)\,dx \le |B|\phi(r).$$

For f_2, applying Lemma 4.3, we have, for $x \in B$,

$$Mf_2(x) \le c_3 \Phi^{-1}(\phi(r))\|f\|_{L^{(\Phi,\phi)}}, \quad \text{if } \Phi \in \mathcal{Y},$$
$$Mf_2(x) \le c_3 \Phi^{-1}(\phi(r))\|f\|_{L^{(\Phi^+,\phi)}}, \quad \text{if } \Phi \in \mathcal{Y}^+.$$

Let $\lambda = c_3\|f\|_{L^{(\Phi,\phi)}}$ in the case $\Phi \in \mathcal{Y}$, or $\lambda = c_3\|f\|_{L^{(\Phi^+,\phi)}}$ in the case $\Phi \in \mathcal{Y}^+$. Then we have

$$\sup_{t>0} \Phi(t) m(B, f_2/\lambda, t) \le \int_B \Phi(Mf_2(x)/\lambda)\,dx \le |B|\phi(r).$$

This shows the norm inequations. □

Let $\Phi \in \mathcal{Y}$ and T is a operator on $L^\Phi(\mathbb{R}^n)$ which satisfies $|T(cf)| = |cTf|$. If there exists a constant $c > 0$ such that

$$\text{(4.1)} \qquad \int_{\mathbb{R}^n} \Phi(|Tf(x)|)\,dx \le c\int_{\mathbb{R}^n} \Phi(c|f(x)|)\,dx, \quad f \in L^\Phi(\mathbb{R}^n),$$

then there exists a constant $C > 0$ such that

$$\text{(4.2)} \qquad \|Tf\|_{L^\Phi} \le C\|f\|_{L^\Phi}, \quad f \in L^\Phi(\mathbb{R}^n), \quad \text{i.e.} \quad T: L^\Phi(\mathbb{R}^n) \to L^\Phi(\mathbb{R}^n).$$

If there exists a constant $c > 0$ such that

$$\text{(4.3)} \qquad \sup_{t>0} \Phi(t) m(Tf, t) \le c\int_{\mathbb{R}^n} \Phi(c|f(x)|)\,dx, \quad f \in L^\Phi(\mathbb{R}^n),$$

then there exists a constant $C > 0$ such that

$$\text{(4.4)} \qquad \|Tf\|_{L^\Phi_{\mathrm{weak}}} \le C\|f\|_{L^\Phi}, \quad f \in L^\Phi(\mathbb{R}^n), \quad \text{i.e.} \quad T: L^\Phi(\mathbb{R}^n) \to L^\Phi_{\mathrm{weak}}(\mathbb{R}^n).$$

If $\Phi \in \Delta_2$, then (4.1) and (4.3) are equivalent to (4.2) and (4.4), respectively. If $T: L^1(\mathbb{R}^n) \to L^1_{\mathrm{weak}}(\mathbb{R}^n)$ and $T: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$, then (4.1) holds for $\Phi \in \Delta_2 \cap \nabla_2$ and (4.3) holds for $\Phi \in \Delta_2$.

Proof of Theorem 3.2. For $f \in L^{(\Phi,\phi)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we choose a ball B such that $x \in B$, and let

$$Tf(x) = Tf_1(x) + \int_{\mathbb{R}^n} K(x,y) f_2(y)\,dy, \quad f = f_1 + f_2, \; f_1 = f\chi_{2B}.$$

We show that $Tf(x)$ is finite a.e.x and independent of the choice of B. Since $f_1 \in L^\Phi(\mathbb{R}^n)$, Tf_1 is in $L^\Phi(\mathbb{R}^n)$ or $L^\Phi_{\mathrm{weak}}(\mathbb{R}^n)$. Let $B = B(a,r)$ and $B_k = B(a, 2^k r)$,

$k = 1, 2, \cdots$. Then

$$\int_{\mathbb{R}^n} |K(x,y)f_2(y)|\,dy = \sum_{k=2}^{\infty} \int_{B_k \setminus B_{k-1}} |K(x,y)f_2(y)|\,dy$$
$$\leq \sum_{k=2}^{\infty} \frac{C}{|B_k|} \int_{B_k} |f_2(y)|\,dy.$$

By Lemmas 4.1 and 4.2 we have

$$\frac{1}{|B_k|} \int_{B_k} |f_2(y)|\,dy \leq 2\phi(2^k r)\|f\|_{\Phi,\phi,B_k} \|1\|_{\tilde{\Phi},\phi,B_k}$$
$$\leq 2\phi(2^k r)\|f\|_{L^{(\Phi,\phi)}} \Phi^{-1}(\phi(2^k r))/\phi(2^k r)$$
$$= 2\Phi^{-1}(\phi(2^k r))\|f\|_{L^{(\Phi,\phi)}}.$$

Hence

$$\int_{\mathbb{R}^n} |K(x,y)f_2(y)|\,dy \leq C \sum_{k=2}^{\infty} \Phi^{-1}(\phi(2^k r))\|f\|_{L^{(\Phi,\phi)}} \tag{4.5}$$
$$\leq C \int_r^{\infty} \Phi^{-1}(\phi(t))t^{-1}dt \|f\|_{L^{(\Phi,\phi)}}.$$

If $x \in B \subset B'$ and $f = f_1 + f_2 = f_1' + f_2'$, $f_1 = f\chi_{2B}$, $f_1' = f\chi_{2B'}$, then

$$Tf(x) = Tf_1'(x) + \int_{\mathbb{R}^n} K(x,y)f_2'(y)\,dy$$
$$= T(f_1 + f_1' - f_1)(x) + \int_{\mathbb{R}^n} K(x,y)f_2'(y)\,dy$$
$$= T(f_1)(x) + T(f_1' - f_1)(x) + \int_{\mathbb{R}^n} K(x,y)f_2'(y)\,dy$$
$$= T(f_1)(x) + \int_{\mathbb{R}^n} K(x,y)(f_1'(y) - f_1(y))\,dy + \int_{\mathbb{R}^n} K(x,y)f_2'(y)\,dy$$
$$= Tf_1(x) + \int_{\mathbb{R}^n} K(x,y)f_2(y)\,dy.$$

Now we show the boundedness. For every ball $B = B(a,r)$, let $f = f_1 + f_2$, $f_1 = f\chi_{2B}$. For $x \in B$ we write

$$Tf(x) = Tf_1(x) + Tf_2(x), \quad Tf_2(x) = \int_{\mathbb{R}^n} K(x,y)f_2(y)\,dy.$$

Then, in the case (i), by (4.3) we have

$$\Phi(t)m\left(B, \frac{Tf_1}{C\|f\|_{L^{(\Phi,\phi)}}}, t\right) \leq \Phi(t)m\left(\frac{Tf_1}{C\|f\|_{L^{(\Phi,\phi)}}}, t\right) \leq C\int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\|f\|_{L^{(\Phi,\phi)}}}\right) dx,$$

and, in the case (ii), by (4.1) we have

$$\int_B \Phi\left(\frac{|Tf_1(x)|}{C\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le \int_{\mathbb{R}^n} \Phi\left(\frac{|Tf_1(x)|}{C\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\|f\|_{L^{(\Phi,\phi)}}}\right) dx.$$

By the almost decreasingness of ϕ and Remark 3.1 we have

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f_1(x)|}{\|f\|_{L^{(\Phi,\phi)}}}\right) dx = \int_{2B} \Phi\left(\frac{|f(x)|}{\|f\|_{L^{(\Phi,\phi)}}}\right) dx$$
$$\le |2B|\phi(2r) \le C|B|\phi(r) \le C|B|\psi(r),$$

Hence

$$\|Tf_1\|_{\Phi,\psi,B,\mathrm{weak}} \le C\|f\|_{L^{(\Phi,\phi)}}, \quad \text{in the case (i)}$$
$$\|Tf_1\|_{\Phi,\psi,B} \le C\|f\|_{L^{(\Phi,\phi)}}, \quad \text{in the case (ii)}.$$

By (4.5) we have

$$\int_B \Phi\left(\frac{|Tf_2(x)|}{C\|f\|_{L^{(\Phi,\phi)}}}\right) dx \le \int_B \Phi\left(\int_r^\infty \Phi^{-1}(\phi(t))t^{-1}dt\right) dx. \le |B|\psi(r),$$

i.e.

$$\|Tf_2\|_{\Phi,\psi,B,\mathrm{weak}} \le \|Tf_2\|_{\Phi,\psi,B} \le C\|f\|_{L^{(\Phi,\phi)}}. \quad \square$$

Proof of Theorem 3.3. Using the pointwise estimate in the proof of Theorem 2.2 in [17];

$$\Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_{L^{(\Phi,\phi)}}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi,\phi)}}}\right),$$

and Theorem 3.1 we have the norm inequalities. □

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. 42 (1975), 765–778.

[2] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. (7) 7 (1987), 273–279.

[3] A. Cianchi, *Strong and weak type inequalities for some classical operators in Orlicz spaces*, J. London Math. Soc. (2) 60 (1999), no. 1, 187–202.

[4] Eridani, H. Gunawan and E. Nakai, *On generalized fractional integral operators*, to appear in Sci. Math. Jpn.

[5] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115.

[6] J.García-Cuerva and J.L.Rubio de Francia, Weighted norm inequalities and related topics, North-Holland Publishing Co., Amsterdam, 1985.

[7] I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, Weight theory for integral transforms on spaces of homogeneous type, Longman, Harlow, 1998.

[8] H. Gunawan, *A note on the generalized fractional integral operators*, J. Indonesian Math. Soc. (MIHMI) 9 (1) (2003), 39–43.

[9] A. Y. Karlovich and L. Maligranda, *On the interpolation constant for Orlicz spaces*, Proc. Amer. Math. Soc. 129 (2001), 2727–2739.

[10] H. Kita, *On Hardy-Littlewood maximal functions in Orlicz spaces*, Math. Nachr. 183 (1997), 135–155.

[11] H. Kita, *On maximal functions in Orlicz spaces*, Proc. Amer. Math. Soc. 124 (1996), 3019–3025.

[12] V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces, World Scientific Publishing Co., Inc., River Edge, NJ, 1991.

[13] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. 166 (1994), 95–103.

[14] E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math. 5 (2001), 587–602.

[15] E. Nakai, *On generalized fractional integrals in the Orlicz spaces on spaces of homogeneous type*, Sci. Math. Jpn. 54 (2001), 473–487.

[16] E. Nakai, *On generalized fractional integrals on the weak Orlicz spaces,* BMO_ϕ, *the Morrey spaces and the Campanato spaces*, Function spaces, interpolation theory and related topics (Lund, 2000), 389–401, de Gruyter, Berlin, 2002.

[17] E. Nakai, *Generalized fractional integrals on Orlicz-Morrey spaces*, Proceedings of Interpolation Symposium on Banach and Function Spaces 2003, to appear.

[18] J. Peetre, *On the theory of* $\mathcal{L}_{p,\lambda}$ *spaces*, J. Functional Analysis 4 (1969), 71–87.

[19] B. Rubin, *Fractional integrals and potentials*, Addison Wesley Longman Limited, Essex, 1996.

[20] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, NJ, 1970.