数理解析研究所講究録 1401 巻 2004 年 152-160

On a removable isolated singularity theorem for the stationary Navier-Stokes equations

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1 Introduction

The purpose of this note is to provide a removable isolated singularity theorem for smooth solutions of the Navier-Stokes equations

$$-\Delta u+\operatorname{div}(u\otimes u)+\nabla p=f \qquad \text{and} \qquad \operatorname{div} u=0 \qquad \text{(NS)},$$

where Ω is a nonempty open subset of $\mathbf{R}^{n}$ with $n\geq 3$. Here $u=(u^{1},u^{2},\cdots,u^{n})$ and p denote the unknown velocity and pressure fields of a stationary viscous incompressible fluid driven by an external force f. We also denote by $\operatorname{div}(u\otimes u)$ the vector field whose j-th component is $\operatorname{div}(uu^{j})=\sum_{i=1}^{n}\frac{\partial}{\partial x_{i}}(u^{i}u^{j})$.

Our main result reads

Theorem 1 *Let (u,p) be a C^{∞}-solution of the Navier-Stokes equations* (NS) *in $B_{R}\setminus\{0\}$. Suppose that*

$$f\in C^{\infty}(B_{R})$$

and

$$u\in L^{n}(B_{R}) \qquad \textit{or} \qquad |u(x)|=o\left(|x|^{-1}\right) \tag{1}$$

as $x\rightarrow 0$. Then (u,p) can be defined at 0 *so that it is a C^{∞}-solution of* (NS) *in B_{R}.*

Theorem 1 improves the previous results by Dyer and Edmunds [2], Shapiro [9, 10] and by Choe and Kim [1]. Moreover, for the three-dimensional case (n=3), Theorem 1 is best possible due to singular solutions constructed by Tian

*supported by Japan Society for the Promotion of Science under JSPS Postdoctoral Fellowship For Foreign Researchers.

and Xin [12]. For any real number c with $|c| > 1$, let us define $u = (u^1, u^2, u^3)$ and p by

$$u^1(x) = 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \quad u^2(x) = 2\frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2},$$
$$u^3(x) = 2\frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2} \quad \text{and} \quad p(x) = 4\frac{cx_1 - |x|}{|x|(c|x| - x_1)^2}.$$

Then a straightforward calculation shows that (u, p) is a C^∞-solution of (NS) in $B_1 \setminus \{0\}$ with $f = 0$, $|u(x)| = O(|x|^{-1})$ as $x \to 0$ but the singularity at 0 is irremovable.

Our proof of Theorem 1 is based on Shapiro's removable singularity result and our new regularity result for distribution solutions of (NS). In [10], Shapiro proved

Theorem 2 (Shapiro [10]) *Suppose that*

1. $u \in L^\beta_{loc}(B_R)$ *for some* $\beta > 2$, $p \in L^1_{loc}(B_R \setminus \{0\})$, $f \in L^1_{loc}(B_R)$,
2. (u, p) *is a distribution solution of* (NS) *in* $B_R \setminus \{0\}$
3. *and* $\left(r^{-n} \int_{B_r} |u|^\beta \, dx\right)^{1/\beta} = o(r^{-(n-1)/2})$ *as* $r \to 0$.

Then $p \in L^1_{loc}(B_R)$ *and* (u, p) *is a distribution solution of* (NS) *in* B_R.

To state our regularity result, let us introduce the definition of the weak $L^n(\Omega)$-norm:

$$||u||_{L^n_w(\Omega)} = \sup_{\sigma>0} \sigma \, |\{x \in \Omega : |u(x)| > \sigma\}|^{\frac{1}{n}}.$$

Then since

$$||u||_{L^n_w(B_r)} \leq ||u||_{L^n(B_r)} \quad \text{and} \quad ||\, |x|^{-1} \,||_{L^n_w(\mathbf{R}^n)} = C(n) < \infty,$$

we easily show that if u satisfies the condition (1), then

$$||u||_{L^n_w(B_r)} \to 0 \qquad \text{as} \qquad r \to 0.$$

Therefore, in view of Theorem 2, Theorem 1 is an immediate consequence of the following regularity result.

Theorem 3 *For each integer* $m \geq 0$, *let* q *be a real number such that*

$$q \in (1, \infty) \ \text{if} \ m = 0 \quad \text{and} \quad q \in (1, \infty) \cap [n/4, \infty) \ \text{if} \ m \geq 1.$$

Then there exists a small constant $\varepsilon = \varepsilon(n,q) > 0$ with the following property. If $(u,p) \in L^2_{loc}(\Omega) \times L^1_{loc}(\Omega)$ is a distribution solution of (NS) *in Ω with $f \in W^{m,q}_{loc}(\Omega)$ and if u satisfies*

$$||u||_{L^n_w(\Omega)} \leq \varepsilon,$$

then

$$u \in W^{m+2,q}_{loc}(\Omega) \quad and \quad p \in W^{m+1,q}_{loc}(\Omega).$$

As an easy corollary of Theorem 3, we also obtain the following interior regularity theorem for the Navier-Stokes equations (NS).

Corollary 4 *Let $(u,p) \in L^n_{loc}(\Omega) \times L^1_{loc}(\Omega)$ be a distribution solution of* (NS) *in Ω. Suppose that*

$$f \in W^{m,q}_{loc}(\Omega)$$

for some integer m and real number q such that

$$m = 0 \ \ and \ \ q \in (1,\infty) \quad or \quad m \geq 1 \ \ and \ \ q \in (1,\infty) \cap [n/4,\infty).$$

Then

$$u \in W^{m+2,q}_{loc}(\Omega) \quad and \quad p \in W^{m+1,q}_{loc}(\Omega).$$

Corollary 4 improves an interior regularity result in a book [3] by Galdi as well as Shapiro's one in [9]. It was shown in [3, Section VIII.5] that if $(u,p) \in L^n_{loc}(\Omega) \cap W^{1,2}_{loc}(\Omega) \times L^2_{loc}(\Omega)$ is a weak solution of (NS) in Ω and if $f \in W^{m,q}_{loc}(\Omega)$ for some (m,q) such that $q \in [2n/(n+2),\infty)$ if $m = 0$ and $q \in [n/2,\infty)$ if $m \geq 1$, then $u \in W^{m+2,q}_{loc}(\Omega)$ and $p \in W^{m+1,q}_{loc}(\Omega)$.

Theorem 3 and its proof are inspired by our recent works [5, 7] on the interior regularity of weak solutions with small $L^\infty(0,T;L^3_w(\Omega))$-norm of the non-stationary Navier-Stokes equations in three dimensions. The remaining part of the note is devoted to giving a sketch of the proof of Theorem 3. For a more complete proof, pleas refer to our original paper [6].

2 A sketchy proof of Theorem 3

Let us first consider the following boundary value problem for the perturbed Stokes equations

$$\begin{cases} -\Delta v + \operatorname{div}(u \otimes v) + \nabla p = f & \text{in } B \\ \operatorname{div} v = g & \text{in } B \\ v = 0 & \text{on } \partial B, \end{cases} \tag{2}$$

where u is a known divergence-free vector field in $L^n_w(B)$ and $B = B_1, B_2$ or B_3.

The following lemma is of basic importance to derive estimates for the convective term in (2).

Lemma 5 *If $v \in L^n_w(B)$ and $w \in W^{1,q}(B)$ with $1 < q < n$, then*

$$v \cdot w \in L^q(B) \quad \textit{and} \quad ||v \cdot w||_{L^q(B)} \le C||v||_{L^n_w(B)}||w||_{W^{1,q}(B)}.$$

Here and after C denotes a positive constant depending only on n and q.

Proof. Note that $L^q(B) = L^{q,q}(B)$ and $L^n_w(B) = L^{n,\infty}(B)$. Hence it follows from Hölder and Sobolev inequalities in Lorenz spaces (see Proposition 2.1 and Proposition 2.2 in [8]) that

$$\begin{aligned} ||v \cdot w||_{L^q(B)} = ||v \cdot w||_{L^{q,q}(B)} &\le C||v||_{L^{n,\infty}(B)}||w||_{L^{\frac{nq}{n-q},q}(B)} \\ &= C||v||_{L^n_w(B)}||w||_{W^{1,q}(B)}. \end{aligned}$$

□

In view of Lemma 5, we have

$$\begin{aligned} \int_B |u \otimes v : \nabla\Phi| \, dx &\le C||v||_{L^q(B)}|| \, |u||\nabla\Phi| \, ||_{L^{q'}(B)} \\ &\le C||v||_{L^q(B)}||u||_{L^n_w(B)}||\Phi||_{W^{2,q'}(B)} \end{aligned} \tag{3}$$

whenever

$$v \in L^q(B), \quad \Phi \in W^{2,q'}(B) \quad \text{and} \quad 1 < q' = \frac{q}{q-1} < n.$$

Hence if $\frac{n}{n-1} < q < \infty$, then weak solutions in $L^q(B)$ to the problem (2) can be defined as follows.

Definition 6 *A vector field $v \in L^q(B)$ with $\frac{n}{n-1} < q < \infty$ is called a q-weak solution or simply a weak solution to the problem* (2), *provided that*

$$-\int_B \{v \cdot \Delta\Phi + u \otimes v : \nabla\Phi\} \, dx = \, < f, \Phi > \tag{4}$$

and

$$-\int_B v \cdot \nabla\varphi \, dx = \, < g, \varphi > \tag{5}$$

for all $\Phi \in C^\infty(\overline{B})$ and $\varphi \in C^\infty(\overline{B})$ such that $\operatorname{div}\Phi = 0$ in B and $\Phi = 0$ on ∂B. Here f and g are sufficiently regular distributions so that the right hand sides of (4) *and* (5) *are well-defined.*

The uniqueness of q-weak solutions to the problem (2) can be proved under the assumption that $||u||_{L^n_w(B)}$ is sufficiently small.

Lemma 7 *For each $q \in (\frac{n}{n-1}, \infty)$, there exists a small positive number $\varepsilon_1 = \varepsilon_1(n,q)$ such that if u satisfies*

$$||u||_{L^{\frac{n}{w}}(B)} \le \varepsilon_1,$$

then q-weak solutions to the problem (2) are unique.

Proof. We prove the lemma by an elementary duality argument. Let v be a weak solution to (2) with $f = 0$ and $g = 0$ so that

$$\int_B \{v \cdot \Delta\Phi + u \otimes v : \nabla\Phi\}\, dx = 0 \quad \text{and} \quad \int_B v \cdot \nabla\varphi\, dx = 0 \tag{6}$$

for all $\Phi \in C^\infty(\overline{B})$ and $\varphi \in C^\infty(\overline{B})$ such that $\operatorname{div}\Phi = 0$ in B and $\Phi = 0$ on ∂B.

Let $w \in C^\infty(\overline{B})$ be fixed. Then in view of a classical theory (see [3] for instance), the Stokes problem

$$-\Delta\Phi + \nabla\varphi = w, \quad \operatorname{div}\Phi = 0 \quad \text{in} \quad B \qquad \text{and} \qquad \Phi = 0 \quad \text{on} \quad \partial B$$

has a unique solution (Φ, φ) such that

$$\Phi \in C^\infty(\overline{B}), \quad \varphi \in C^\infty(\overline{B}) \quad \text{and} \quad ||\Phi||_{W^{2,q'}(B)} \le C||w||_{L^{q'}(B)}.$$

Hence by virtue of (6) and (3), we have

$$\begin{aligned}\int_B v \cdot w\, dx = \int_B v \cdot (-\Delta\Phi + \nabla\varphi)\, dx &= \int_B u \otimes v : \nabla\Phi\, dx \\ &\le C||v||_{L^q(B)}||u||_{L^{\frac{n}{w}}(B)}||\Phi||_{W^{2,q'}(B)} \\ &\le C_1||v||_{L^q(B)}||u||_{L^{\frac{n}{w}}(B)}||w||_{L^{q'}(B)}.\end{aligned}$$

Since $w \in C^\infty(\overline{B})$ is arbitrary and $C^\infty(\overline{B})$ is dense in $L^{q'}(B)$, it follows that

$$||v||_{L^q(B)} \le C_1||u||_{L^{\frac{n}{w}}(B)}||v||_{L^q(B)}.$$

Therefore, taking $\varepsilon_1 = 1/2C_1$, we conclude that if $||u||_{L^{\frac{n}{w}}(B)} \le \varepsilon_1$, then $||v||_{L^q(B)} = 0$. This completes the proof of Lemma 7. □

We can also prove the existence of weak solutions in $W^{1,q}(B)$ and $W^{2,q}(B)$.

Lemma 8 *For each $q \in (1,n)$, there exists a small positive constant $\varepsilon_2 = \varepsilon_2(n,q)$ such that if u satisfies*

$$||u||_{L^{\frac{n}{w}}(B)} \le \varepsilon_2,$$

then for every

$$f \in W^{-1,q}(B) \quad \textit{and} \quad g \in L^q(B) \quad \textit{with} \quad \int_B g\, dx = 0,$$

there exists a unique weak solution v in $W_0^{1,q}(B)$ to the problem (2).

Remark 9 This solution v is actually a $nq/(n-q)$-weak solution in the sense of Definition 6 since $W_0^{1,q}(B) \subset L^{nq/(n-q)}(B)$ and $\frac{n}{n-1} < \frac{nq}{n-q} < \infty$.

Proof. By virtue of Lemma 5, we have

$$||u \otimes v||_{L^q(B)} \leq C||u||_{L^n_w(B)}||v||_{W^{1,q}(B)} \quad \text{for all} \quad v \in W^{1,q}(B).$$

Hence it follows from the classical theory of the Stokes equations (see [3]) that for each $v \in W_0^{1,q}(B)$, there exists a unique weak solution $\overline{v} = Lv \in W_0^{1,q}(B)$ to the problem

$$\begin{cases} -\Delta\overline{v} + \nabla\overline{p} = f - \operatorname{div}(u \otimes v) & \text{in} \quad B \\ \operatorname{div}\overline{v} = g & \text{in} \quad B \\ \overline{v} = 0 & \text{on} \quad \partial B, \end{cases}$$

which satisfies the estimate

$$||\overline{v}||_{W^{1,q}(B)} \leq C\left(||f||_{W^{-1,q}(B)} + ||g||_{L^q(B)} + ||u \otimes v||_{L^q(B)}\right).$$

Moreover, the operator L on $W_0^{1,q}(B)$ satisfies

$$\begin{aligned} ||Lv_1 - Lv_2||_{W^{1,q}(B)} &\leq C||u \otimes (v_1 - v_2)||_{L^q(B)} \\ &\leq C_2||u||_{L^n_w(B)}||v_1 - v_2||_{W^{1,q}(B)} \end{aligned}$$

for all $v_1,\ v_2 \in W_0^{1,q}(B)$. Therefore, taking $\varepsilon_2 = 1/(2C_2)$, we conclude that if $||u||_{L^n_w(B)} \leq \varepsilon_2$, then L is a contraction on $W_0^{1,q}(B)$ and so have a unique fixed point. This proves Lemma 8. □

Lemma 10 *For each $q \in (1, n)$, there exists a small positive constant $\varepsilon_3 = \varepsilon_3(n, q)$ such that if u satisfies*

$$||u||_{L^n_w(B)} \leq \varepsilon_3,$$

then for every

$$f \in L^q(B) \quad \textit{and} \quad g \in W^{1,q}(B) \quad \textit{with} \quad \int_B g\, dx = 0,$$

there exists a unique weak solution v in $W_0^{1,q}(B) \cap W^{2,q}(B)$ to the problem (2).

Proof. Similar to the proof of Lemma 8. □

Now Theorem 3 can be deduced from the following result by a standard scaling argument and induction on m.

Proposition 11 *Assume that* $\Omega = B_3$ *and* $q \in (1,n)$. *Then there exists a small positive constant* $\varepsilon = \varepsilon(n,q)$ *with the following property.*

If u *satisfies* $||u||_{L^n_w(B_3)} \leq \varepsilon$ *and if* $(v,p) \in L^n_w(B_3) \times L^1(B_3)$ *is a distribution solution of*

$$\begin{cases} -\Delta v + \operatorname{div}(u \otimes v) + \nabla p = f & in \quad \Omega \\ \operatorname{div} v = 0 & in \quad \Omega \end{cases} \tag{7}$$

with $f \in L^q(B_3)$, *then*

$$v \in W^{2,q}(B_1) \quad and \quad p \in W^{1,q}(B_1).$$

Proof. It is easy to show that $L^n_w(B_3) \subset L^{n-\delta}(B_3)$ for any $\delta > 0$. This fact together with Sobolev inequality yields

$$\nabla v - u \otimes v \in W^{-1,n-\delta}(B_3) + L^{\frac{n}{2}(1-\frac{\delta}{2n-\delta})}(B_3) \subset W^{-1,n-\delta}(B_3)$$

for any $\delta > 0$ and so $\nabla p = f + \operatorname{div}(\nabla v - u \otimes v) \in W^{-2,q}(B_3)$ because $1 < q < n$. Hence it follows that $p \in W^{-1,q}(B_3)$.

Let us choose a cut-off function $\varphi \in C^\infty_c(B_3)$ such that $\varphi = 1$ in B_2 and $\varphi = 0$ in $B_3 \setminus B_{5/2}$. Then it is easy to show that $\overline{v} = \varphi v \in L^2(B_3) \cap L^q(B_3)$ is a 2-weak solution (in the sense of Definition 6) to the following problem

$$\begin{cases} -\Delta \overline{v} + \operatorname{div}(u \otimes \overline{v}) + \nabla \overline{p} = \overline{f} & \text{in} \quad B_3 \\ \operatorname{div} \overline{v} = g & \text{in} \quad B_3 \\ \overline{v} = 0 & \text{on} \quad \partial B_3, \end{cases} \tag{8}$$

where

$$\overline{p} = \varphi p \in W^{-1,q}(B_3), \quad g = \nabla \varphi \cdot v \in L^q(B_3)$$

and

$$\overline{f} = \varphi f + \nabla \varphi \cdot (u \otimes v - 2\nabla v + pI) - (\Delta \varphi) v \in W^{-1,q}(B_3).$$

We now assume that u satisfies

$$||u||_{L^n_w(B_3)} \leq \varepsilon_2(n,q). \tag{9}$$

Then by virtue of Lemma 8, there exists a unique solution $w \in W^{1,q}_0(B_3)$ to the problem (8). Note that

$$w \in L^{\frac{nq}{n-q}}(B_3) \qquad \text{and} \qquad \frac{n}{n-1} < \frac{nq}{n-q} < \infty.$$

Hence by virtue of Lemma 7, we deduce that

$$\overline{v} = w \in W^{1,q}(B_3) \quad \text{and so} \quad v \in W^{1,q}(B_2),$$

provided that

$$||u||_{L^n_w(B_3)} \leq \varepsilon_1(n, q_1), \qquad \text{where} \quad q_1 = \min\left(2, \frac{nq}{n-q}\right). \tag{10}$$

Moreover, it follows from Lemma 5 that

$$\nabla p = f + \operatorname{div}(\nabla v - u \otimes v) \in W^{-1,q}(B_2),$$
$$p \in L^q(B_2), \quad \overline{f} \in L^q(B_2) \quad \text{and} \quad g \in W^{1,q}(B_2).$$

On the other hand, we observe that if we choose $\varphi \in C_c^\infty(B_3)$ so that $\varphi = 1$ in B_1 and $\varphi = 0$ in $B_3 \setminus B_{3/2}$, then $\overline{v} = \varphi v \in W^{1,q}(B_2)$ is a q_1-weak solution to the problem (8) with B_3 replaced by B_2.

Therefore, assuming in addition to (9) and (10) that

$$||u||_{L^n_w(B_3)} \leq \varepsilon_3(n, q).$$

we conclude from Lemma 10 and Lemma 7 that

$$\overline{v} \in W^{2,q}(B_2) \quad \text{and so} \quad v \in W^{2,q}(B_1),$$

which implies then that $p \in W^{1,q}(B_1)$. This completes the proof of Proposition 11. □

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