ZETA FUNCTIONS FOR THE RENEWAL SHIFT

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ABSTRACT. We exhibit a topological Markov shift on a countable alphabet with the property that for every sequence of complex numbers $c_n$ such that $\limsup_{n\to\infty} \sqrt{|c_n|} < \infty$ there exists a weight function $A : X \to \mathbb{C}$ which depends only on the first two coordinates such that the corresponding weighted dynamical zeta function satisfies $\frac{1}{\zeta_A(z)} = 1 + \sum_{i\geq 1} a_i z^i$.

1. INTRODUCTION

Let $S$ be a countable set and $A = (a_{ij})_{S \times S}$ a matrix of zeroes and ones. $S$ is called the set of states. $A$ is called a topological transition matrix if $\forall a \in S \exists i, j \ (a_{ai} = t_{ja} = 1)$. If this is the case then one defines the (one sided) countable Markov shift generated by $A$ to be

$$X = \Sigma_A^+ = \{ x \in S^{\mathbb{N} \cup \{0\}} : \forall i \ t_{x_ix_{i+1}} = 1 \}.$$

We endow this set with the metric $d(x, y) := (\frac{1}{2})^{\min\{n : x_n \neq y_n\}}$, and equip it with the action of the left shift map:

$$T : \Sigma_A^+ \to \Sigma_A^+, \quad (Tx)_i = x_{i+1}.$$

Let $Fix(T^n) := \{ x \in \Sigma_A^+ : T^n x = x \}$.

Let $A : X \to \mathbb{C}$ be some function, called a weight function. The generalized dynamical zeta function, for the weight function $A$ is

$$\zeta_A(z) = \exp \sum_{n=1}^{\infty} \sum_{x \in Fix(T^n)} \prod_{k=0}^{n-1} A(T^k x).$$

These functions were introduced (in a more general context) by Ruelle [8],[9], as a generalization of certain generating functions which were considered by Artin and Mazur [1].

If $|S| < \infty$ and $A$ is regular enough (e.g., when $\log A$ is Hölder continuous), then $\zeta_A$ is holomorphic in a neighborhood of zero and its first pole is in $e^{-P}$, where $P$ is the topological pressure of $\log A$ (see [9]). A series of studies have focused on meromorphic extensions of $\zeta_A$ to larger domains (see for example [8], [5], [7], [4]).

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We show here that if $|S| = \infty$ then no such results are possible, even if one restricts one attention to locally constant potentials. We do this by exhibiting a specific topological Markov shift with the following property: Every function $f$ such that $f(0) = 1$, which is holomorphic in a neighborhood of zero, can be represented a dynamical zeta function for a suitable weight function $A : X \to \mathbb{C}$ which depends only on the first two coordinates.

This topological Markov shift is the shift with set of states $\mathbb{N}$ and transition matrix

$$
R = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
$$

We call this shift the renewal shift because of its obvious connection to renewal theory (see [2]). We prove:

**Theorem 1.** Let $X$ be the renewal shift and $\{c_n\}_{n=1}^{\infty}$ a sequence of complex numbers such that $\lim_{n \to \infty} \sqrt[n]{|c_n|} < \infty$. There exists a function $A : X \to \mathbb{C}$ which depends only on the first two coordinates, for which in the neighborhood of zero

$$
\frac{1}{\zeta_A(z)} = 1 + \sum_{i=1}^{\infty} c_i z^i.
$$

In particular, any type of singular behavior can occur away from zero. This should be contrasted with the case $|S| < \infty$, for which every zeta function with a weight function of the form $A(x) = A(x_0, x_1)$ is rational [6]. We remark that the dynamical zeta functions without meromorphic extensions have been constructed before [3].

2. PROOF OF THEOREM 1

Set

$$
c_i^* = \begin{cases}
    c_i & c_i \neq 0 \\
    1 & c_i = 0
\end{cases}
$$

and

$$
\alpha_1 = c_1^* ; \quad \alpha_i = c_i^*/c_{i-1}^* \\
\beta_1 = -c_1 ; \quad \beta_i = -c_i/c_{i-1}^*.
$$
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Let $A = (a_{ij})_{N \times N}$ be the matrix given by

$$A = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\
\alpha_1 & 0 & 0 & 0 & \cdots \\
0 & \alpha_2 & 0 & 0 & \cdots \\
0 & 0 & \alpha_3 & 0 & \cdots \\
0 & 0 & 0 & \alpha_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
$$

(1)

Let $A_n$ be the upper left $n \times n$ block. Set $r = \left( \lim_{n \to \infty} \sqrt[|n|]{c_n} \right)^{-1}$. This number is positive or infinite, by the assumptions of the theorem.

**Lemma 1.** The following limit holds and is uniform on compacts in $D_r := \{ z : |z| < r \}$:

$$\lim_{n \to \infty} \det(1-zA_n) = 1 + \sum_{i=1}^{\infty} c_i z^i$$

**Proof.**

$$\det(1-zA_n) = \left| \begin{array}{cccccc}
1 - \beta_1 z & -\beta_2 z & \cdots & -\beta_{n-1} z & -\beta_n z \\
-\alpha_1 z & 1 & 0 & \cdots & 0 \\
0 & -\alpha_2 z & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & -\alpha_{n-1} z & 1 \\
\end{array} \right|$$

$$= (1 - \beta_1 z) \left| \begin{array}{cccccc}
0 & -\alpha_3 z & \cdots & \vdots & 0 \\
\vdots & \vdots & \ddots & 1 & \vdots \\
0 & 0 & \cdots & -\alpha_{n-1} z & 1 \\
-\alpha_1 z & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{array} \right|$$

$$-(-\beta_2 z) \left| \begin{array}{cccccc}
-\alpha_2 z & \cdots & \vdots & 0 & + \cdots \\
\vdots & 1 & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & -\alpha_{n-1} z & 1 \\
-\alpha_1 z & 1 & 0 & \cdots & 0 \\
0 & -\alpha_2 z & 1 & 0 & 0 \\
\end{array} \right|$$

$$+(-1)^{n+1}(-\beta_n z) \left| \begin{array}{cccccc}
-\alpha_1 z & 1 & 0 & \cdots & 0 \\
0 & -\alpha_2 z & 1 & 0 & 0 \\
0 & -\alpha_3 z & \cdots & \vdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\alpha_{n-1} z \\
\end{array} \right|$$
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\[
1 - \beta_1 z - \beta_2 \alpha_1 z^2 - \ldots - \beta_n \alpha_1 \ldots \alpha_{n-1} z^n \\
= 1 + c_1 z + \frac{c_2}{c_1} z^2 + \ldots + \frac{c_n}{c_{n-1}} \ldots \frac{c_{n-2}}{c_1} \ldots \alpha_{n-1} z^n \\
= 1 + c_1 z + \ldots + c_n z^n \xrightarrow{n \to \infty} 1 + \sum_{i=1}^{\infty} c_i z^i.
\]

This convergence is uniform on compacts in \( D_r \), because \( r \) is the radius of convergence of this power series.

\[ \square \]

Lemma 2. \( E := \{ \lambda \in \mathbb{C} : \exists n \ det(\lambda I - A_n) = 0 \} \) is a bounded subset of \( \mathbb{C} \).

Proof. Else, \( \exists n_k \to \infty \) and \(|\lambda_{n_k}| \to \infty \), such that \( \det(\lambda_{n_k} I - A_{n_k}) = 0 \). Without loss of generality, assume that \( \forall k |\lambda_{n_k}| \geq \frac{2}{r} \) (if \( r = \infty \) assume that \(|\lambda_{n_k}| \geq 1 \).

According to the previous lemma, the following limit exists and is uniform on compacts in \( D_r = \{ z : |z| < r \} \):

\[ (3) \\
f(z) = \lim_{n \to \infty} \det(1 - zA_n)\]

Note that \( f(0) = 1 \), and that \( f \) is continuous in 0. In particular, since \( \lambda_{n_k}^{-1} \to 0 \) and \( \lambda_{n_k} \in D_r \)

\[ |f(0) - f(\lambda_{n_k}^{-1})| \xrightarrow{k \to \infty} 0. \]

By the uniform convergence of (3) in \( \overline{D}_{r/2} \) (or in \( \overline{D}_1 \) if \( r = \infty \)) we have that

\[ |f(\lambda_{n_k}^{-1}) - \det(1 - \lambda_{n_k}^{-1} A_{n_k})| \xrightarrow{k \to \infty} 0 \]

Hence, since \( \forall k \ det(1 - \lambda_{n_k}^{-1} A_{n_k}) = 0 \),

\[ |f(0) - 0| \leq |f(0) - f(\lambda_{n_k}^{-1})| + |f(\lambda_{n_k}^{-1}) - \det(1 - \lambda_{n_k}^{-1} A_{n_k})| \xrightarrow{k \to \infty} 0 \]

which implies that \( 1 = f(0) = 0 \), a contradiction. \( \square \)

We are now ready to prove the theorem. Let \( A : X \to \mathbb{C} \) be given by

\[ A(x_0, x_1, \ldots) = A_{x_0 x_1} \]

where \( A \) is given by (1).

Set

\[ Z_n = \sum_{x \in Fix(T^n)} \prod_{k=0}^{n-1} A(T^k x) \]

Then

\[ \log \zeta_A = \sum_{n=1}^{\infty} \frac{Z_n}{n} \cdot Z_n. \]
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By the definition of $A$

$$Z_n = \sum_{x \in Fix(T^n)} A_{x_0 x_1} A_{x_1 x_2} \cdots A_{x_n x_0}.$$

$\forall x_0, \ldots, x_{n-1} \in \mathbb{N}$ if $A_{x_0 x_1} A_{x_1 x_2} \cdots A_{x_n x_0} > 0$ then

$$(x_0, x_1, \ldots, x_{n-1}; x_0, x_1, \ldots, x_{n-1} \ldots)$$

belongs to $\Sigma^+_A$ and constitutes a periodic point of order $n$. Thus

$$Z_n = \sum_{x \in Fix(T^n)} \prod_{i=0}^{n-1} A(T^i x) = \sum_{x_1 \cdots x_n} A_{x_0 x_1} \cdots A_{x_n x_0}.$$

By the definition of the renewal shift, if $(x_0, x_1, \ldots, x_{n-1}, x_0)$ is admissible then $\forall i x_i \leq n$ (if $m$ appears, so must $m-1, m-2, \ldots, 1$. Since there are at most $n$ different symbols $x_i$, $m$ must be smaller than $n$). Thus,

$$\forall n \leq N : Z_n = \sum_{x_0 \cdots x_{n-1} = 1}^n A_{x_0 x_1} \cdots A_{x_n x_0} =$$

$$= \sum_{x_0 \cdots x_{n-1} = 1}^N A_{x_0 x_1} \cdots A_{x_n x_0} = \text{tr}(A^n_N).$$

This shows that

$$\left| \sum_{n=1}^\infty \frac{z^n}{n} \cdot Z_n - \sum_{k=1}^\infty \frac{z^n}{n} \cdot \text{tr}(A^n_N) \right| \leq \sum_{n>N} \left| \frac{z^n}{n} \cdot Z_n \right| + \sum_{n>N} \left| \frac{z^n}{n} \cdot \text{tr}(A^n_N) \right|.$$

We estimate these tails. According to the previous lemma, $E = \{ \lambda \in \mathbb{C} : \exists n \det(\lambda I - A_n) = 0 \}$ is bounded. Let $\lambda = \text{sup}\{|z| : z \in E\}$. Let $\lambda_1(k), \ldots, \lambda_k(k)$ the eigenvalues of $A_k$, written with multiplicities. Then $|\lambda_i(k)| \leq \lambda$. Using the fact that every matrix can be triangulated, it is easy to verify that

$$|\text{tr}(A^n_k)| = |\lambda_1(k)^n + \ldots + \lambda_k(k)^n| \leq k \lambda^n$$

Thus, for every $|z| < \lambda^{-1}$,

$$\left| \sum_{n>N} \frac{z^n}{n} \cdot Z_n \right| = \left| \sum_{n>N} \frac{z^n}{n} \cdot \text{tr}(A^n_N) \right| \leq \sum_{n>N} \left| \frac{z^n}{n} \cdot n \lambda^n \right| = \sum_{n>N} |z \cdot \lambda|^n \xrightarrow{N \to \infty} 0.$$
and
\[ \left| \sum_{n>N} \frac{z^n}{n} \cdot tr(A_N^n) \right| \leq \sum_{n>N} |z \cdot \lambda|^n \xrightarrow{N \to \infty} 0. \]
Thus, \( \forall |z| < \lambda^{-1} \)
\[ \left| \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot Z_n - \sum_{k=1}^{\infty} \frac{z^n}{n} \cdot tr(A_N^n) \right| \]
\[ \leq \left| \sum_{n>N} \frac{z^n}{n} \cdot Z_n \right| + \left| \sum_{n>N} \frac{z^n}{n} \cdot tr(A_N^n) \right| \xrightarrow{N \to \infty} 0. \]
Using the Taylor expansion of \( z \mapsto \log(1 - z) \) and the identities
\[ tr(A_N^n) = \lambda_1 (N)^n + \ldots + \lambda_N (N)^n \]
and
\[ \det(1 - zA_N) = (1 - z\lambda_1 (N)) \cdot \ldots \cdot (1 - z\lambda_N (N)) \]
it is not difficult to show that if \( |z| < \lambda^{-1} \) then
\[ - \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot tr(A_N^n) = \ln \det(1 - zA_N) \]
Thus, the following limit holds in \( D_{\lambda^{-1}} \)
\[ \ln \det(1 - zA_N) \xrightarrow{N \to \infty} - \log \zeta_A(z). \]
But by (2) if \( |z| < r \) then
\[ \det(1 - zA_N) \xrightarrow{N \to \infty} 1 + \sum_{i=1}^{\infty} c_i z^i \]
Hence, for \( |z| < \min\{r, \lambda^{-1}\} \) we have
\[ \frac{1}{\zeta_A(z)} = 1 + \sum_{i=1}^{\infty} c_i z^i \]
as required. \[ \square \]

REFERENCES
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