LARGE DEVIATIONS FOR COUNTABLE TO ONE MARKOV SYSTEMS

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ABSTRACT. In this note, we announce new results on large deviation properties for countable to one Markov systems estabished in [35] (http://www.sapporo-u.ac.jp/yuri/) Our results are applicable to higher-dimensional number theoretical transformations.

§1 Introduction

Let $T : X \to X$ be a non-invertible map of a compact metric space X which is not necessarily continuous and piecewise C^0-invertible with respect to a countable generating Markov partition $Q = \{X_i\}_{i\in I}$ of X. The main purpose of this article is to establish large deviation estimates for such countable to one piecewise invertible Markov systems (T, X, Q). In particular, we shall be concerned with large deviation properties of weak Gibbs measures μ for non-Hölder potentials ϕ. We shall clarify a class of functions f in which we can describe the (Helmholtz) free energy function associated to μ in terms of the topological pressure defined in §1 (Theorem 2.1). As we will see in §2, our class of functions is larger than $C(X)$ so that our results on large deviations are generalizations of those in the standard context which are applicable to hyperbolic systems with equilibrium states for Hölder potentials (c.f. [2], [3], [7], [8], [9], [10], [12], [21]). We apply our results to countable Markov maps which arise from number theory and exihibit common phenomena in transition to turbulence (the so-called *Intermittency*). Since these countable to one maps are not expansive, the dual variational principle may not hold even if the variational principle for the pressure holds. For this reason, we observe naturally different stages of phase transitions which were not treated yet in previous works. It was proved in [26,27,31] that intermittent systems typically admit weak Gibbs ergodic equilibrium states absolutely continuous with respect to weak Gibbs smooth measures. These weak Gibbs measures are non-Gibbsian states in the sense of Bowen and both non-Gibbsianness and non-uniqueness of equilibrium states are caused by an appearance of indifferent periodic orbits with respect to potentials ([31-34]). We shall formulate different stages of indifferency and relate our new characterization of

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phase transitions to these indifferent periodic points. In §3, we shall study the level 1 upper large deviations by restricting our attention to the multifractal version of large deviation laws (Theorem 3.1). In [3,8,15-17], a connection between the multifractal formalism and the theory of large deviation was established for subshifts of finite type and expanding conformal dynamics. We recall that the theory of multifractals is based on Kolmogorov's work ([11]) on completely developed turbulence and Mandelbrot's observation for intermittent turbulence in [14]. Taking this physical background into consideration, in §4 we shall associate non-differentiability of the Hausdorff dimension of level sets with phase transitions for intermittent systems. In Appendix, we collect definitions and present previous results for piecewise C^0-invertible transitive FRS Markov systems.

§2 Large deviations for weak Gibbs measures

Let (T, X, Q) be a piecewise C^0-invertible transitive FRS Markov system.

Definition. Let $\mu \in M(X)$ and $f : X \to \mathbb{R}$. The *(Helmholtz) free energy function* for a random process $(\sum_{h=0}^{n-1} fT^h, \mu)_{n\geq 1}$ is defined by

$$\mathcal{H}_\mu(f) := \lim_{n\to\infty} \frac{1}{n} \log \int_X \exp[\sum_{h=0}^{n-1} fT^h(x)]d\mu(x)$$

whenever it exists.

In the standard framework of the theory of large deviations, $\mathcal{H}_\mu$ on $C(X)$ can be characterized by $\mathcal{H}_\mu(g) = \sup_{\nu\in M(X)}\{\nu(g) - \mathcal{H}_\mu{}^*(\nu)\}$ for each $g \in C(X)$, where $\mathcal{H}_\mu{}^*(\nu) := \sup_{f\in C(X)}\{\nu(f) - \mathcal{H}_\mu(f)\}$ is the convex conjugate of $\mathcal{H}_\mu$ (c.f. [9], [21]). Define $\mathcal{W}(T) := \{f : X \to \mathbb{R} | f$ satisfies the WBV property and $P_{\text{top}}(T, f) < \infty\}$. The next result show that the limit $\mathcal{H}_\mu(f)$ exists for any $f \in \mathcal{W}(T) \cup C(X)$.

Theorem 2.1. *Let (T, X, Q) be a transitive FRS Markov system. If μ is a weak Gibbs measure for $\phi \in \mathcal{W}(T)$ with $-P_{\text{top}}(T, \phi)$, then $\forall f \in \mathcal{W}(T) \cup C(X)$*

$$\mathcal{H}_\mu(f) = P_{\text{top}}(T, \phi + f) - P_{\text{top}}(T, \phi).$$

Since $P_{\text{top}}(T, .)$ satisfies convexity, $\mathcal{H}_\mu : \mathcal{W}(T) \cup C(X) \to \mathbb{R}$ is a convex function. We define

$$\mathcal{W}_1(T) := \{f \in \mathcal{W}(T) | Var_n(f) \to 0 (n \to \infty)\},$$

where $Var_n(f) := \sup_{X_{i_1\ldots i_n}} \sup_{x,y\in X_{i_1\ldots i_n}} \{|f(x) - f(y)|\}$. Since

$$\sup_{X_{i_1\ldots i_n}} \sup_{x,y\in X_{i_1\ldots i_n}} \exp[\sum_{h=0}^{n-1}\{fT^h(x) - fT^h(y)\}] \leq \exp[\sum_{h=1}^{n} Var_h(f)],$$

$Var_n(f) \to 0 (n \to \infty)$ is suffices for f to satisfy the WBV property. Now we define a generalized *Legendre transform* of the convex function $\mathcal{H}_\mu$ on $\mathcal{W}_1(X) \cup C(X)$ as follows :

$$\mathcal{H}_\mu{}^*(\nu) := \sup_{f\in\mathcal{W}_1(T)\cup C(X)} \{\nu(f) - \mathcal{H}_\mu(f)\} \ (\nu \in M(X)).$$

Then we have $\mathcal{H}_\mu{}^*(\nu) \geq 0$ $(\forall\nu \in M(X))$ as $\mathcal{H}_\mu{}^*(\nu) \geq -\mathcal{H}_\mu(0) = 0$ and we can establish a weak duality between $\mathcal{H}_\mu$ and $\mathcal{H}_\mu{}^*$.

Proposition 2.1.

$$\mathcal{H}_\mu(g) \geq \sup_{\nu \in M(X)} \{\nu(g) - \mathcal{H}_\mu{}^*(\nu)\}.$$

Although a complete duality between $\mathcal{H}_\mu$ and $\mathcal{H}_\mu{}^*$ may not hold in our setting, Theorem 2.1 allows one to establish the following upper large deviation estimates.

Theorem 2.2(The level 2 upper large deviation inequality). *Let (T, X, Q) be a transitive FRS Markov system. Let μ be a weak Gibbs measure for $\phi \in \mathcal{W}_1(T)$ with $-P_{\text{top}}(T, \phi)$. Then $\forall K \subset M(X)$ a compact subset,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu(\{x \in X | \frac{1}{n} \sum_{h=0}^{n-1} \delta_{T^h x} \in K\}) \leq - \inf_{\nu \in K} \mathcal{H}_\mu{}^*(\nu).$$

Definition. If $\nu \in M(X)$ satisfies $\mathcal{H}_\mu(f) = \nu(f) - \mathcal{H}_\mu{}^*(\nu)$ for $f \in \mathcal{W}_1(T) \cup C(X)$, then ν is called an *equilibrium state for f with respect to μ*. $\hat{E}_\mu(f)$ denotes the set of all equilibrium states for f with respect to μ.

By Theorem 2.1 we can write

$$(1) : \mathcal{H}_\mu{}^*(\nu) = (P_{\text{top}}(T, \phi) - \nu(\phi)) - \inf_{g \in \mathcal{W}_1(T) \cup C(X)} \{P_{\text{top}}(T, \phi + g) - \nu(\phi + g)\},$$

which allows us to see that the equality

$$P_{\text{top}}(T, \phi + f) - \nu(\phi + f) = \inf_{g \in \mathcal{W}_1(T) \cup C(X)} \{P_{\text{top}}(T, \phi + g) - \nu(\phi + g)\}$$

is equivalent for ν being an equilibrium state for $f \in \mathcal{W}_1(T) \cup C(X)$ with respect to μ. Recalling non-negativity of $\mathcal{H}_\mu{}^*(\nu)$ allows us to see that $\mathcal{H}_\mu{}^*(\nu)$ measures the distance of an orbitary $\nu \in M(X)$ from the set $\hat{E}_\mu(0)$.

Definition. $\nu \in M(X)$ is a *tangent functional* of $P_{\text{top}}(T, .)$ at $\phi \in \mathcal{W}_1(T)$ if

$$P_{\text{top}}(T, \phi + f) - P_{\text{top}}(T, \phi) \geq \nu(f) \ (\forall f \in \mathcal{W}_1(T) \cup C(X)).$$

The set of all tangent functionals of $P_{\text{top}}(T, .)$ at ϕ is denoted by $D_P(\phi)$ (c.f. [7]).

Theorem 2.3 (Exponential decreasing property). *Let (T, X, Q) be a transitive FRS Markov system. Suppose that μ is a weak Gibbs measure for $\phi \in \mathcal{W}_1(T)$ with $-P_{\text{top}}(T, \phi)$. If $K(\subset M(X))$ is a compact subset with $D_P(\phi) \cap K = \emptyset$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu(\{x \in X | \frac{1}{n} \sum_{h=0}^{n-1} \delta_{T^h x} \in K\}) < 0.$$

Corollary 2.1. *If $U \subset M(X)$ is an open neighbourhood of $D_P(\phi)$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu(\{x \in X | \frac{1}{n} \sum_{h=0}^{n-1} \delta_{T^h x} \notin U\}) < 0.$$

Theorem 2.4. *Suppose that (T, X, Q) is a transitive FRS Markov system and $\phi \in \mathcal{W}_1(T) \cup C(X)$. Assume further that $\bigcup_{intX_i=\emptyset} X_i$ consists of periodic orbits. Then $P_{\mathrm{top}}(T, \phi) \geq h_m(T) + m(\phi)$ for all $m \in M_T(X)$ with $m(\phi) > -\infty$.*

Corollary 2.2(Variational Principle). *If ϕ admits an indifferent periodic point, then $P_{\mathrm{top}}(T, \phi) = \sup\{h_m(T) + m(\phi) | m \in M_T(X)$ with $m(\phi) > -\infty\}$.*

Definition. We say that $m \in M_T(X)$ is an *equilibrium state for* $\phi \in \mathcal{W}_1(T)$ if $h_m(T) + m(\phi) = P_{\mathrm{top}}(T, \phi)$ holds. $E_T(\phi)$ denotes the set of all equilibrium states for $\phi \in \mathcal{W}_1(T)$.

Since ergodicity of $\mu \in M_T(X)$ implies $\lim_{n\to\infty} \frac{1}{n}\sum_{h=0}^{n-1} \delta_{T^h x} = \mu$ in the weak topology on $M(X)$ for μ-a.e.$x \in X$, if μ is an ergodic weak Gibbs measure for $\phi \in \mathcal{W}_1(T)$ with $h_\mu(T) < \infty$ and $\phi \in L^1(\mu)$, then $\frac{1}{n}\sum_{h=0}^{n-1} \delta_{T^h x}$ converges to $\mu \in E_T(\phi)$ weakly μ-a.e.$x \in X$. However, for each $n \geq 1$, $\frac{1}{n}\sum_{h=0}^{n-1} \delta_{T^h x}$ may be still far from μ. In general, ergodicity of μ may not be necessarily for the weak convergence of the empirical measures $\frac{1}{n}\sum_{h=0}^{n-1} \delta_{T^h x}$ to $\mu \in E_T(\phi)$. Theorem 2.3 insists that the decreasing rates of $\mu(\{x \in X | \frac{1}{n}\sum_{h=0}^{n-1} \delta_{T^h x} \notin U\})$ is exponential for any open neighbourhood U of $\hat{E}_\mu(0) = D_P(\phi)$ and the rate is determined by the distance of U^c from $D_P(\phi)$.

Definition. For $\nu \in M(X)$, define $\hat{h}_\nu(T) := \inf_{g \in \mathcal{W}_1(T) \cup C(X)} \{P_{\mathrm{top}}(T, g) - \nu(g)\}$ which is called a *generalized entropy* of ν.

Definition. We say that $\nu \in M(X)$ is a *generalized equilibrium state for* ϕ if $P_{\mathrm{top}}(T, \phi) - \nu(\phi) = \hat{h}_\nu(T)$ holds. $\hat{E}(\phi)$ denotes th set of all generalized equilibrium states for ϕ and $\hat{E}_T(\phi)$ denotes the set of all T-invariant generalized equilibrium states for ϕ.

Definition. We say that $P_{\mathrm{top}}(T, .) : \mathcal{W}_1(T) \to \mathbb{R}$ is differentiable at $\phi \in \mathcal{W}_1(T)$ if the limit ;

$$\lim_{t\to 0} \frac{P_{\mathrm{top}}(T, \phi + tf) - P_{\mathrm{top}}(T, \phi)}{t}$$

exists for all $f \in C(X)$.

Since co-existence of tangent functionals implies failure of differentiability of the pressure function (see [33]), if $D_P(\phi) \neq \emptyset$ and $P_{\mathrm{top}}(T, .)$ is differentiable at ϕ, then $D_P(\phi)$ consists of a single element ν and $\lim_{t\to 0} \frac{\mathcal{H}_\mu(tf)}{t} = \nu(f)$ for all $f \in C(X)$.

Theorem 2.5. *Let (T, X, Q) be a transitive FRS Markov system. Suppose that $\bigcup_{intX_i=\emptyset} X_i$ consists of periodic orbits. Let μ be a T-invariant weak Gibbs measure for $\phi \in \mathcal{W}_1(T)$ with $-P_{\mathrm{top}}(T, \phi)$ which satisfies $h_\mu(T) < \infty$ and $\phi \in L^1(\mu)$. If $P_{\mathrm{top}}(T, .)$ is differentiable at ϕ, then we have the followings.*

(i) *$\mu(\{x \in X | \frac{1}{n}\sum_{h=0}^{n-1} \delta_{T^h x} \notin U\})$ decays exponentially as $n \to \infty$ for any open neighbourhood of $E_T(\phi)$.*
(ii) *(The level 1 upper large deviations.) $\forall \epsilon > 0$ and $\forall f \in \mathcal{W}_1(T) \cup C(X)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(\{x \in X | \frac{1}{n}\sum_{h=0}^{n-1} f \circ T^h(x) > \mu(f) + \epsilon\} < 0.$$

In general, differentiability of $P_{\text{top}}(T,.)$ easily fails, even if uniqueness of equilibrium states holds. Indeed, we have four stages of phase transitions as follows :

(1) $\sharp E_T(\phi) > 1$ (stage 1 phase transition)
(2) $\sharp E_T(\phi) = 1$ and $\sharp \hat{E}_T(\phi) > 1$ (stage 2 phase transition)
(3) $\sharp \hat{E}_T(\phi) = 1$ and $\sharp \hat{E}(\phi) > 1$ (stage 3 phase transition)
(4) $\sharp \hat{E}(\phi) = 1$ and $\sharp D_P(\phi) > 1$ (stage 4 phase transition)

All (1)-(4) are sufficient for a lack of differentiability of the pressure function which may give a crucial difficulty in establishing lower large deviations bounds. As we have already mentioned above, we will see in §4 that the stage 1 phase transition happens for the important potentials $-\log|\det DT|$ in the class of intermittent maps. More specifically, we recall that if x_0 is an indifferent periodic point with respect to ϕ, then $\frac{1}{q}\sum_{h=0}^{q-1}\delta_{T^h x_0} \in E_T(\phi)$. Since $-\log|\det DT|$ typically admits an indifferent periodic point, the Dirac measure supported on the periodic orbit is one equilibrium state for $-\log|\det DT|$. On the other hand, it is known that a T-invariant absolutely continuous weak Gibbs equilibrium state exists ([27]). In order to clarify occurance of various types of phase transitions in the context of nonhyperbolic systems, we consider x_0 as an indifferent periodic point at stage 1 and generalize indifferency as follows.

Definition. A periodic point x_0 with period q which is repelling for all $g \in \mathcal{W}_1(T)$ is called *indifferent* with respect to ϕ at stage 2 (resp. stage3) if $\frac{1}{q}\sum_{h=0}^{q-1}\delta_{T^h x_0} \in \hat{E}_T(\phi)/E_T(\phi)$ (resp. $\frac{1}{q}\sum_{h=0}^{q-1}\delta_{T^h x_0} \in D_P(\phi)/\hat{E}_T(\phi)$).

Moreover, we introduce the following quantities which measure the distance of an arbitrary $\nu \in M(X)$ (or $\nu \in M_T(X)$) from the sets $E_T(\phi), \hat{E}(\phi), D_P(\phi)$ respectively.

$$d_\phi^{(1)}(\nu) := \{P_{\text{top}}(T,\phi) - \nu(\phi)\} - h_\nu(T) \ (\nu \in M_T(X)).$$
$$d_\phi^{(2)}(\nu) := \{P_{\text{top}}(T,\phi) - \nu(\phi)\} - \hat{h}_\nu(T) \ (\nu \in M(X)).$$
$$d_\phi^{(3)}(\nu) := \mathcal{H}_\mu{}^*(\nu) \ (\nu \in M(X)).$$

Then we see that a generalized indifferent periodic point x_0 with period q with respect to ϕ at stage $i (i = 1,2,3)$ are characterized by

$$d_\phi^{(i-1)}\Big(\frac{1}{q}\sum_{h=0}^{q-1}\delta_{T^h x_0}\Big) > 0 \ \ and \ \ d_\phi^{(i)}\Big(\frac{1}{q}\sum_{h=0}^{q-1}\delta_{T^h x_0}\Big) = 0.$$

A lack of differentiability of $P_{\text{top}}(T,.)$ is caused, for example, by more than one indifferent periodic orbits at verious stages. In case when $P_{\text{top}}(T,0) < \infty$, we have the following result.

Theorem 2.6 (Exponential decreasing property). *Let (T,X,Q) be a transitive FRS Markov system with $P_{\text{top}}(T,0) < \infty$. Suppose that $\bigcup_{intX_i=\emptyset} X_i$ consists of periodic orbits. If $\phi \in \mathcal{W}_1(T)$ is a bounded function and μ is a T-invariant weak Gibbs measure for ϕ, then $\nu \notin \hat{E}(\phi)$ iff $\mathcal{H}_\mu{}^*(\nu) > 0$. In particullar, for any open neighbourhood U of $\hat{E}(\phi)$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu(\{x \in X | \frac{1}{n}\sum_{h=0}^{n-1}\delta_{T^h x} \notin U\}) < 0.$$

Moreover, if $\sup_{x\in X}\phi(x)\le P_{\rm top}(T,\phi)$, *then the above inequality holds for any open neighbourhood* U *of* $\hat{E}_T(\phi)$.

We recall that (DVP) is valid when the entropy map is upper semi-continuous. For any continuous expansive action of a compact metric space, the entropy map is upper semi-continuous. Moreover, differentiability of $P_{\rm top}(T,.)$ is valid at Hölder potentials ϕ in the context of hyperbolic systems (e.g., aperiodic SFT, uniformly expanding maps), where the differentiability is equivalent to uniqueness of equilibrium states ($\sharp E_T(\phi)=1$). If $\sharp Q<\infty$, then T is expansive because of the property (03) so that (DVP) is valid. Hence by Theorem 2.6 we have exponential decreasing property for any compact subset K of $M(X)$ with $K\cap E_T(\phi)=\emptyset$. On the other hand, in case when $\sharp Q=\infty$, T may fail expansiveness even if it is piecewisely expanding. Such phenomena are easily found for many number theoretical transformations (like Gauss-type transformations). However, we can still establish differentiability of $P_{\rm top}(T,.)$ at ϕ when $\{\phi\circ\psi_i\}_{i\in I}$ is equi-Hölder continuous and uniqueness of equilibrium states for ϕ follows from this property. Without differentiability of $P_{\rm top}(T,.)$, we may have $\inf_{\nu\in K}d^{(3)}_{\phi}(\nu)=0$ for a compact subset K with $K\cap E_T(\phi)=\emptyset$ as $E_T(\phi)\subset D_P(\phi)$ and we may observe the verious types of phase transitions in the above. For establishing (level 2) lower large deviations bounds, $\sharp\hat{E}_\mu(f)=1$ for all $f\in C(X)$ which are finite linear combinations of functions of a countable subset in $C(X)$ is suffices in the usual setting ([9]). Indeed, it is well-known that this condition is satisfied for the unique equilibrium state μ for Hölder potentials in case of hyperbolic systems. We relate $E_\mu(f)$ to $E_T(\phi+f)$ as follows.

Proposition 2.2. *Assume that* $P_{\rm top}(T,0)<\infty$ *and* $\sup_{x\in X}|\phi(x)|<\infty$. *If* $m\in M_T(X)$ *satisfies* $h_m(T)=\hat{h}_m(T)$, *then* $m\in\hat{E}_T(\phi+f)$ *iff* $m\in\hat{E}_\mu(f)$.

Even if $\sharp\hat{E}_\mu(\phi+f)=1$ for all Hölder functions f, again a lack of the (DVP) may cause crucial difficulty in establishing the level 2 lower large deviations bounds.

§3 Multifractal large deviation laws

Let (T,X,Q) be a piecewise C^0-invertible transitive FRS Markov system. Let $\phi\in\mathcal{W}(T)$ be a negative function which can be unbounded ($\inf_{x\in X}\phi(x)\ge-\infty$). Choose a nonpositive function $f\in\mathcal{W}(T)$ such that for each $q\ge0$, a zero $t(q)$ of the next generalized Bowen's equation : $P_{\rm top}(T,qf+t(q)\phi)=0$ is uniquely determined (see §4). We can show that $t(q)$ is a strictly convex function because of the (strictly) covexity of $P_{\rm top}(T,.):\mathcal{W}(T)\to\mathbb{R}$. For establishing multifractal version of (level 1) upper large deviation inequality, we restrict our attention to piecewise conformal (countable to one) transitive FRS Markov systems (T,X,Q) with $X\subset\mathbb{R}^D$ and potentials $\phi=-\log||DT||$. Choose an observable function $f\in\mathcal{W}(T)$ and for each $R>0$, we define

$$n_R(x):=\inf\{n\in\mathbb{N}|\max\{\sum_{h=0}^{n-1}\phi T^h(x),\sum_{h=0}^{n-1}fT^h(x)\}\le-R\}\le\infty.$$

If both ϕ and f are strictly negative (i.e., $sup_{x\in X}\phi(x)<0, sup_{x\in X}f(x)<0$), then $n_R(x)<\infty(\forall x\in X)$. We consider a generalized (Helmholtz) free energy function for the random process $(\sum_{h=0}^{n_R(.)-1}qfT^h,\mu)_{R>0}$,

$$\hat{\mathcal{H}}_\mu(qf):=\limsup_{R\to\infty}\frac{1}{R}\log\int_X\exp[\sum_{h=0}^{n_R(x)-1}qf(T^hx)]d\mu(x).$$

Theorem 3.1. *Let* $\phi = -\log||DT||, f \in \mathcal{W}(T)$ *be strictly negative functions satisfying* $\sup_{R>0} \frac{N_R}{R} < \infty$. *Suppose that for* $q \geq 0$, $\exists_1 t(q)$ *a decreasing function such that* $P_{\text{top}}(T, qf + t(q)\phi) = 0$. *Assume further that*

(C1) $\exists m_1$ *a weak Gibbs measure for* $qf + t(q)\phi$ *(with 0).*
(C2) $\exists m_2$ *a weak Gibbs measure for* $q(-f) + t(q)\phi$ *(with 0).*

If μ *is a weak Gibbs measure for* $t(0)\phi$ *and* ν *is a weak Gibbs measure for* f, *then* $\forall \alpha > 0$

$$\limsup_{R\to\infty} \frac{1}{R} \log \mu(\{x \in X | \frac{\log \nu(X_{i_1 \dots i_{n_R(x)}}(x))}{\log diam X_{i_1 \dots i_{n_R(x)}}(x)} \geq \alpha\})$$

$$\leq \limsup_{R\to\infty} \frac{1}{R} \log \mu(\{x \in X | \frac{\sum_{h=0}^{n_R(x)-1} (-f)(T^h x)}{R} \geq \alpha\})$$

$$\leq \inf_{q\leq 0} \{q\alpha + \hat{\mathcal{H}}_\mu(qf)\} \leq -t^*(\alpha) - t(0),$$

where $t^*(\alpha) := \sup_{q\geq 0}\{q\alpha - t(q)\}$ *is the Legendre transform of* $t(q)$.

Corollary 3.1. *If we replace the weak Gibbs property imposed on* μ, ν, m_1, m_2 *by the Gibbs property, then we can remove the assumption* $\sup_{R>0} \frac{N_R}{R} < \infty$.

Remark If $\inf_{x\in X} \phi(x), \inf_{x\in X} f(x) > -\infty$, then $\sup_{R>0} \frac{N_R}{R} < \infty$. On the other hand, as we will see in §4, for typical derived systems arising from Intermittent maps both $\inf_{x\in X} \phi(x) = -\infty$ and $\sup_{R>0} \frac{\sup_{x\in X} \inf\{n\in\mathbb{N} | \sum_{h=0}^{n-1} \phi T^h(x) \leq -R\}}{R} < \infty$ are satisfied for $-\log||DT||$.

As we will see in the next section, our examples of nonhyperbolic piecewise conformal systems admit jump systems which satisfy the following two properties.

(a) $P_{\text{top}}(T, -\log|\det DT|) = 0$ so that $t(0) = D$.
(b) $\exists J \subset \mathbb{R}^+$ such that $\forall q \in J, \exists_1 t(q) > 0$ with $P_{\text{top}}(qf - t(q)\log||DT||) = 0$, $\exists_1 \mu_q$ a T-invariant Gibbs measure for $qf - t(q)\log||DT||$ and for $\beta(q) := \frac{\int_X f d\mu_q}{-\int_X \log||DT||d\mu_q} (> 0)$,

$$dim_H \Lambda_{\beta(q)} = q\beta(q) + t(q),$$

where

$$\Lambda_{\beta(q)} = \{x \in X | \frac{\sum_{h=0}^{n-1} fT^h(x)}{-\sum_{h=0}^{n-1} \log||DT(T^h(x))||} \to \beta(q)(n \to \infty)\}.$$

(C.f. [30])

Corollary 3.2. *Under (a) and (b),* $\forall \alpha \in \{\beta' \in \mathbb{R}^+ | \exists q' \in J$ *such that* $\beta(q') = \beta'\}$ $\exists q \in J$ *such that*

$$\limsup_{R\to\infty} \frac{1}{R} \log \mu(\{x \in X | \frac{\log \nu(X_{i_1 \dots i_{n_R(x)}}(x))}{\log diam X_{i_1 \dots i_{n_R(x)}}(x)} \geq \alpha\})$$

$$\leq dim_H \Lambda_\alpha - t(0) - 2q\alpha = dim_H \Lambda_\alpha - D - 2q\alpha < 0.$$

§4 Applications to Intermittent systems

Let (T, X, Q) be a transitive FRS Markov system. Let $B_1 \subset X$ be a union of cylinders $X_i \in Q$ with $cl(intTX_i) = X$. Define the stopping time over B_1, $R : X \to \mathbb{N} \cup \{\infty\}$ by $R(x) = \inf\{n \geq 0 : T^n x \in B_1\} + 1$ and for each $n > 1$, define inductively $B_n := \{x \in X | R(x) = n\}$. Now we define Schweiger's jump transformation ([23]) $T^* : \bigcup_{n=1}^{\infty} B_n \to X$ by $T^* x = T^{R(x)} x$. We denote $X^* := X \backslash (\bigcup_{m=0}^{\infty} T^{*-m}(\bigcap_{n \geq 0}\{R(x) > n\}))$ and

$$I^* := \bigcup_{n \geq 1} \{(i_1 \ldots i_n) \in I^n : X_{i_1 \ldots i_n} \subseteq B_n\}.$$

Then it is easy to see that $(T^*, X^*, Q^* = \{X_{\underline{i}}\}_{\underline{i} \in I^*})$ is a piecewise C^0-invertible Bernoulli system. Assume further that (T, X, Q) is a piecewise conformal system. For $f \in \mathcal{W}_1(T)$ with $P_{\text{top}}(T, f) = 0$ and $q \geq 0$ we shall consider the following equations :

$$(2) : P_{\text{top}}(T^*, qf^* - t\log ||DT^*||) = 0$$

$$(3) : P_{\text{top}}(T, qf - t\log ||DT||) = 0.$$

Definition. We say that a piecewise C^1-invertible Markov system is *locally uniformly expanding with respect to* B_1 if T^* associated to B_1 is uniformly expanding (i.e., $\sup_{\underline{i} \in I^*} \sup_{x \in X^*} ||D\psi_{\underline{i}}(x)|| < 1$).

Definition. We say that a potential $\phi : X \to \mathbb{R}$ satisfies *local bounded distortion (LBD) with respect to* B_1 if $\forall \underline{i} = (i_1 \ldots i_{|\underline{i}|}) \in I^*, \exists\, 0 < L_\phi(\underline{i}) < \infty$ satisfying

$$|\phi(\psi_{\underline{i}}(x)) - \phi(\psi_{\underline{i}}(y))| \leq L_\phi(\underline{i}) d(x, y)^\theta$$

and

$$L_\phi(\infty) := \sup_{\underline{i} \in I^*} \sum_{j=0}^{|\underline{i}|-1} L_\phi(i_{j+1} \ldots i_{|\underline{i}|}) < \infty.$$

Under the locally uniformly expanding property with respect to B_1, we see that $-\log ||DT^*||$ is a negative function. Choose f a nonpositive function satisfying LBD with respect to B_1. By Lemma 12 in [29], if $P_{\text{top}}(T, f^*) \geq 0$ and $||\mathcal{L}_{qf^*} 1|| < \infty (\forall 0 \leq q \leq 1)$ then we can determine $t(q) \geq 0$ satisfying the equation (2). Futhermore, by Lemma 7 in [31] $P_{\text{top}}(T^*, T^*, qf^* - t(q)\log ||DT^*||) = 0$ forces to $P_{\text{top}}(T, qf - t(q)\log ||DT||) = 0$. We claim that $t(q)$ is not necessarily a unique solution of (3). It was proved in [30] that the properties (a) and (b) are valid for (T^*, X^*, Q^*) and for f^*. Moreover, $t(q)$ is analytic on J and $\beta(q) = -t'(q)$ holds. The next result allows us to specify the first order phase transition point at which $t(q)$ is non-differentiable.

Theorem 4.1. *Suppose that* $q\phi - t(q)\log ||DT|| \in \mathcal{W}_1(T)$. *Assume further that there exist two different tangent functionals of* $P_{\text{top}}(\sigma, .)$ *at* $\{q\phi \circ \pi - t(q)\log ||DT \circ \pi||\}$, $\overline{\mu}_1, \overline{\mu}_2$ *satisfying* $0 \neq \overline{\mu}_i(\log ||DT \circ \pi||) < \infty, \phi \circ \pi \in L_1(\overline{\mu}_i)(i = 1, 2)$ *and*

$$\frac{\overline{\mu}_1(\phi \circ \pi)}{\overline{\mu}_1(\log ||DT \circ \pi||)} \neq \frac{\overline{\mu}_2(\phi \circ \pi)}{\overline{\mu}_2(\log ||DT \circ \pi||)}.$$

Then q is the first order phase transition point.

All our results are applicable to the next two-dimensional number theoretical transformations. We refer the reader to [35] for further details.

Example 1 (A complex continued fraction [23,24,31]*).* We can define a complex continued fraction transformation $T : X \to X$ on the diamond shaped region $X = \{z = x_1\alpha + x_2\overline{\alpha} : \ -1/2 \le x_1, z_2 \le 1/2\}$, where $\alpha = 1+i$, by $T(z) = 1/z - [1/z]_1$. Here $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$, where z is written in the form $z = x_1\alpha + x_2\overline{\alpha}$, $[x] = \max\{n \in \mathbb{Z} | n \le x\}(x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} | n < x\}(x \in \mathbb{Z} - \mathbb{N})$. This transformation has an indifferent periodic orbit $\{1, -1\}$ of period 2 and two indifferent fixed points at i and $-i$. For each $n\alpha + m\overline{\alpha} \in I := \{m\alpha + n\overline{\alpha} : (m,n) \in \mathbb{Z}^2 - (0,0)\}$, we define $X_{n\alpha+m\overline{\alpha}} = \{z \in X : [1/z]_1 = n\alpha + m\overline{\alpha}\}$. Then we have a countable Markov partition $Q = \{X_a\}_{a \in I}$ of X and (T, X, Q) is a transitive FRS Markov system with $P_{\text{top}}(T, 0) = \infty$.

Example 2 (Brun's map [19,26,27,31]*).* Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_2 \le x_1 \le 1\}$, and let

$$X_i = \{(x_1, x_2) \in X : x_i + x_1 \ge 1 \ge x_{i+1} + x_1\}$$

for $i = 0, 1, 2$ where we put $x_0 = 1$ and $x_3 = 0$. T is defined by

$$T(x_1, x_2) = \left(\tfrac{x_1}{1-x_1}, \tfrac{x_2}{1-x_1}\right) \text{ on } X_0,$$
$$T(x_1, x_2) = \left(\tfrac{1}{x_1} - 1, \tfrac{x_2}{x_1}\right) \text{ on } X_1,$$
$$T(x_1, x_2) = \left(\tfrac{x_2}{x_1}, \tfrac{1}{x_1} - 1\right) \text{ on } X_2.$$

This map admits an indifferent fixed point $(0,0)$ (i.e., $|\det DT(0,0)| = 1$). We can easily see that $TX_i = X (i = 0,1,2)$, i.e., $Q = \{X_i\}_{i=0}^2$ is a Bernoulli partition.

Example 3 (Inhomogeneous Diophantine approximations). We define $X = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1, -y \le x < -y + 1\}$ and $T : X \to X$ by

$$T(x,y) = \left(\frac{1}{x} - \left[\frac{1-y}{x}\right] + \left[-\frac{y}{x}\right], -\left[-\frac{y}{x}\right] - \frac{y}{x}\right),$$

where $[x] = \max\{n \in \mathbb{Z} | n \le x\}(x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} | n < x\}(x \in \mathbb{Z} \backslash \mathbb{N})$. This map admits indifferent periodic points $(1,0)$ and $(-1,1)$ with period 2, i.e., $|\det DT^2(1,0)| = |\det DT^2(-1,1)| = 1$. Let $a(x,y) = [\frac{(1-y)}{x}] - [\frac{-y}{x}]$ and $b(x,y) = -[-\frac{y}{x}]$. We can introduce an index set

$$I = \left\{\binom{a}{b} : a, b \in \mathbb{Z}, a > b > 0, \text{ or } a < b < 0\right\}$$

and a partition $Q := \left\{X_{\binom{a}{b}} : \binom{a}{b} \in I\right\}$, where $X_{(a,b)} = \{(x,y) \in X : a(x,y) = a, b(x,y) = b\}$. Then we can directly verify all conditions (01)-(03) so that (T, X, Q) is a transitive FRS Markov system with $P_{\text{top}}(T, 0) = \infty$.

Appendix

Let X be a compact metric space with metric d and let $T : X \to X$ be a noninvertible map which is not necessarily continuous. Let $Q = \{X_i\}_{i \in I}$ be a countable disjoint partition $Q = \{X_i\}_{i \in I}$ of X such that $\bigcup_{i \in I} intX_i$ is dense in X and satisfy the following properties.

(01) For each $i \in I$ with $intX_i \ne \emptyset, T|_{intX_i} : intX_i \to T(intX_i)$ is a homeomorphism and $(T|_{intX_i})^{-1}$ extends to a homeomorphism ψ_i on $cl(T(intX_i))$.

(02) $T(\bigcup_{intX_i = \emptyset} X_i) \subset \bigcup_{intX_i = \emptyset} X_i$.

For $\underline{i} = (i_1 \dots i_n) \in I^n$ with $int(X_{i_1} \cap T^{-1}X_{i_2} \cap \dots T^{-(n-1)}X_{i_n}) \neq \emptyset$, we define $X_{\underline{i}} := X_{i_1} \cap T^{-1}X_{i_2} \cap \dots T^{-(n-1)}X_{i_n}$ which is called a cylinder of rank n and write $|\underline{i}| = n$. By (01), $T^n|_{intX_{i_1\dots i_n}} : intX_{i_1\dots i_n} \to T^n(int(X_{i_1\dots i_n}))$ is a homeomorphism and $(T^n|_{intX_{i_1\dots i_n}})^{-1}$ extends to a homeomorphism $\psi_{i_1} \circ \psi_{i_2} \circ \dots \circ \psi_{i_n} = \psi_{i_1\dots i_n} : cl(T^n(intX_{\underline{i}})) \to cl(intX_{\underline{i}})$. We assume further the next generator condition :

(03) $\sigma(n) := \sup\{diamX_{i_1\dots i_n} | X_{i_1\dots i_n} \in \bigvee_{j=0}^{n-1} T^{-j}(Q)\} \to 0 \ (n \to \infty)$.

We say that $(T, X, Q = \{X_i\}_{i\in I})$ is a piecewise C^0-invertible *Markov system*, if $int(cl(intX_i) \cap cl(intTX_j)) \neq \emptyset$ implies $cl(intTX_j) \supset cl(intX_i)$. By the condition (03), (T, X, Q) provides a countable Markov shift (Σ, σ) such that there exists a uniformly continuous map $\pi : \Sigma \to X$ defined by

$$\pi(i_1 i_2 \dots) = \bigcap_{j=0}^{\infty} cl(T^{-j}(intX_{i_{j+1}})) \ (\neq \emptyset)$$

which satisfies $\pi \circ \sigma = T \circ \pi$. We should note that, in general $\pi(\Sigma)$ is a proper subset of X. The next condition gives a nice countable states symbolic dynamics similar to sofic shifts (cf. [25]):

(Finite Range Structure). $\mathcal{U} = \{int(T^n X_{i_1\dots i_n}) : \forall X_{i_1\dots i_n}, \forall n > 0\}$ consists of finitely many open subsets $U_1 \dots U_N$ of X.

Definition. We say that $\phi : X \to \mathbb{R}$ is a potential of *weak bounded variation* (WBV) if there exists a sequence of positive numbers $\{C_n\}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n = 0$ and $\forall n \geq 1, \forall X_{i_1\dots i_n} \in \bigvee_{j=0}^{n-1} T^{-j}Q$,

$$\frac{\sup_{x\in X_{i_1\dots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x))}{\inf_{x\in X_{i_1\dots i_n}} \exp(\sum_{j=0}^{n-1} \phi(T^j x))} \leq C_n.$$

Let (T, X, Q) be a piecewise C^0-invertible Markov system with FRS and satisfy the next condition which is automatically satisfied by Bernoulli systems:

(Transitivity). $intX = \cup_{k=1}^N U_k$ and $\forall l \in \{1, 2, \dots N\}, \exists 0 < s_l < \infty$ such that for each $k \in \{1, 2, \dots N\}$, U_k contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank s_l such that $T^{s_l}(intX^{(k,l)}(s_l)) = U_l$.

Then for $\phi : X \to \mathbb{R}$ a potential of WBV we can define the partition function

$$Z_n(\phi) := \sum_{\underline{i}:|\underline{i}|=n, int(TX_{i_n})\supset intX_{i_1}} \exp[\sum_{h=0}^{n-1} \phi T^h(x(\underline{i}))],$$

where $x(\underline{i})$ is the unique point satisfying $\psi_{\underline{i}} x(\underline{i}) = x(\underline{i}) \in cl(intX_{\underline{i}})$. By Theorem 1 in [31] we know that there exists the limit

$$P_{\text{top}}(T, \phi) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi) \in (-\infty, \infty].$$

Moreover, for any $\hat{\phi} : X \to \mathbb{R}$ with $\phi|_{\bigcup_{i\in I} cl(intX_i)} = \hat{\phi}|_{\bigcup_{i\in I} cl(intX_i)}$, we see that $P_{\text{top}}(T, \phi) = P_{\text{top}}(T, \hat{\phi})$. For this reason, WOLG we assume

(04) $\bigcup_{i\in I, intX_i\neq\emptyset} cl(intX_i) \cap \bigcup_{i\in I, intX_i=\emptyset} X_i = \emptyset$.

and for ϕ of WBV with $P_{\rm top}(T,\phi) < \infty$,

(05) $\sup_{x \in \bigcup_{i \in I, int X_i = \emptyset} X_i} \phi(x) \leq P_{\rm top}(T,\phi)$.

Then we see immediately that $\forall x_0 \in X$ with $T^q x_0 = x_0$,

$$P_{\rm top}(T,\phi) \geq \frac{1}{q} \sum_{h=0}^{q-1} \phi T^h(x_0)$$

(Lemma 10 in [31]). Let $\mathcal{F}$ be the σ-algebra of Borel sets of the compact space X. $M(X)$ denotes the set of all probability measures on $(X,\mathcal{F})$ and $M_T(X) (\subset M(X))$ denotes the set of all T-invariant probability measures.

Definition. x_0 is called an *indifferent periodic point* with period q with respect to ϕ if $P_{\rm top}(T,\phi) = \frac{1}{q}\sum_{h=0}^{q-1} \phi T^h(x_0)$. If x_0 is not indifferent, then we call x_0 a *repelling periodic point.*

The following definition appeared in [25] gives a weak notion of Bowen's Gibbs measure in the category of piecewise C^0-invertible systems.

Definition ([25-34]). A probability measure ν on $(X,\mathcal{F})$ is called a *weak Gibbs measure* for a function ϕ with a constant P if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that ν-a.e.x,

$$K_n^{-1} \leq \frac{\nu(X_{i_1\ldots i_n}(x))}{\exp[\sum_{j=0}^{n-1}\phi T^j(x) + nP]} \leq K_n,$$

where $X_{i_1\ldots i_n}(x)$ denotes the cylinder containing x.

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