数理解析研究所講究録 1404 巻 2004 年 12-20

Semiconjugacies for skew products of interval maps

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September 15, 2004

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Abstract

Distribution functions of non-atomic Gibbs measures on the unit interval define natural semiconjugacies between maps on $[0,1]$. Using this method we extend a result of Milnor and Thurston in [3] about the semiconjugacy of unimodal maps to skew products with maps of the interval as fiber maps.

1 Introduction

In this note we use the existence of Gibbs measures for a discrete time dynamical system to define a semiconjugacy between the system and a piecewise linear map. In particular, we discuss the analogue of this construction in the case of skew products $(X\times Y,\tau,(T_x)_{x\in X})$ where $\tau : X\to X$, $T_x : Y\to Y$ $(x\in X)$ and

$$T(x,y)=(\tau(x),T_x(y)).$$

In the latter case the notion of a Gibbs measure can be generalized to that of Gibbs families whose existence and uniqueness was discussed in [1]. Also recall that a dynamical system $T : Z\to Z$ is called semiconjugate to the system

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‡This research was supported by the Deutsche Forschungsgemeinschaft.

$T' : Z' \to Z'$ if there is a continuous surjective map $\Pi : Z \to Z'$ such that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{T} & Z \\ \Pi \downarrow & & \downarrow \Pi \\ Z' & \xrightarrow{T'} & Z' \end{array}$$

commutes, and we call $\Pi : X \times Y \to X' \times Y'$ a semiconjugacy between the skew products $(X \times Y, T)$ and $(X' \times Y', T')$ if Π semiconjugates the dynamical systems and if Π maps fibers to fibers, i.e. if $\Pi(\{x\} \times Y) \subset \{x'\} \times Y'$ for some $x' \in X'$.

Consider the special case of a skew product where $Y = [0,1]$ and where each T_x is a piecewise continuous and monotone map of the interval Y with positive relative topological entropy $h(T_x)$. Certain fiberwise expanding transformations T will be shown to be semiconjugate to a skew product where each fiber map is a continuous piecewise monotone map of the interval with slope $\exp h(T_x)$.

Note that this result parallels the case of a map of the interval since the theory of skew products and their Gibbs families reduces to this case if X consists of a single point. For unimodal maps we rediscover a result of Milnor and Thurston in [3], where it has been shown in Theorem 7.4, that every unimodal map, for which the number of monotonicity intervals of T^n increases exponentially fast, is semiconjugate to a unimodal map with constant common slopes on each of the monotonicity branches. The proof given here is different.

The idea of the proof relies on the following simple fact. If μ is a distribution on the unit interval, then its distribution function is monotone, surjective and even continuous if μ has no atoms. Hence the Milnor-Thurston result is a statement of a piecewise scaling property of a distribution function without any atom. Such distributions are obtained as non-atomic Gibbs measures, in particular, as measures of maximal entropy.

In order to be more precise, let $T : Z \to Z$ be a dynamical system and $\varphi : Z \to \mathbb{R}$ be a function. Recall that a measure m is called a *Gibbs measure* for φ if the Jacobian $d\mu \circ T / d\mu$ is defined μ - a.e. and is given by

$$\frac{d\mu \circ T}{d\mu} = e^{\varphi}.$$

The following standard chain of arguments gives the existence of a Gibbs measure in the case of an open and expanding map T acting on a compact space Z and a continuous function φ. By these assumptions, the map T has locally a constant number of preimages, which implies that T acts on continuous functions by its Perron-Frobenius operator

$$V_\varphi f(y) = \sum_{T(y')=y} f(y') e^{\varphi(y')}.$$

Furthermore, its dual operator acts continuously on the space of signed measures on Z. Therefore, by the Schauder-Tychonoff theorem there exists an eigenvalue

$\lambda > 0$ and a measure μ such that $d\mu \circ T/d\mu = \lambda \exp(-\varphi)$, where $d\mu \circ T/d\mu$ refers to the Jacobian. In other words, μ is a Gibbs measure for the potential $\varphi + \log \lambda$.

In the case of skew products we use the notion of Gibbs families on skew products as a generalization of Gibbs measures. Recall that a family $\{\mu_x : x \in X\}$ of probability measures on Y is called a *Gibbs family* for a measurable function $\varphi : X \times Y \to \mathbb{R}$ if there exists a positive measurable function (called gauge function) $A : X \to \mathbb{R}$, such that, for each $x \in X$, the Jacobian of μ_x is given by

$$\frac{d\mu_{T(x)} \circ T_x}{d\mu_x} = A(x) \exp(-\varphi). \tag{1}$$

Using the existence of Gibbs families we extend the result of semiconjugacies for maps of the interval to certain skew products where the maps T_x $(x \in X)$ are maps of the interval.

2 Semiconjugacies for skew-products

In this section we prove our result about semiconjugacies. We begin with the case of piecewise monotone map T of a totally ordered Polish space X. Recall that X is totally ordered if there exists an order relation '$\preceq$' such that for each $x, y \in X$ either $x \preceq y$ or $y \preceq x$ and $x \preceq y \preceq x$ implies that $x = y$. This gives rise to a further relation '$\prec$', where $x \prec y$ if $x \preceq y$ and $x \neq y$. With this setting, the notion of closed and open intervals can be easily extended to the space X and these intervals will be denoted by $[a, b]$ and (a, b), respectively. The topology on X is assumed to be generated by the open intervals or in other words, the topology on X is the order topology.

The map T is referred to be piecewise continuous and monotone if there exists a finite partition α of X into intervals such that for each $a \in \alpha$ the restriction $T|_a$ is continuous and monotone. Let m be a non–atomic and nonsingular probability measure on X and let $\Pi : X \to [0, 1] \subset \mathbb{R}$ and $S : [0, 1] \to [0, 1]$ be defined as follows.

$$\begin{aligned} \Pi : X &\to [0,1], \quad x \mapsto m(\{z \in X \mid z \preceq x\}), \\ S : [0,1] &\to [0,1], \quad y \mapsto \Pi(Tx) \text{ where } x \in \Pi^{-1}(\{y\}). \end{aligned}$$

Note that, since m has no atoms and is nonsingular, the map Π is onto, and S is well defined. Moreover, we have that $S \circ \Pi = \Pi \circ T$ and S is piecewise continuous and monotone on $\Pi(a)$ for each $a \in \alpha$. Furthermore, we obtain the following immediate result.

Proposition 2.1. *The map Π is continuous and semiconjugates T and S. Furthermore, S is continuous and monotone on the interior $(\Pi(a))^\circ$ of $\Pi(a)$) for each $a \in \alpha$, and $\lambda = m \circ \Pi^{-1}$ where λ refers to the Lebesgue measure. Moreover, Π is a homeomorphism if and only if $m((a, b]) \neq 0$ for all $a, b \in X, a \prec b$.*

In case that m is a Gibbs measure for the potential φ the following proposition gives the relation between the derivative DS of S and φ.

Proposition 2.2. *Let m be a Gibbs measure for the potential φ. Assume that y_0 belongs to the interior $(\Pi(a))^o$ of $\Pi(a)$ for some $a \in \alpha$ and that $\exp(\varphi)$ is constant on $\Pi^{-1}(\{y_0\}$ and continuous in $\partial\Pi^{-1}(\{y_0\})$. Then S is differentiable in y_0 and, for $x \in \Pi^{-1}(\{y_0\})$,*

$$DS(y_0) = \begin{cases} e^{\varphi(x)} & : \ T|_a \text{ is increasing} \\ -e^{\varphi(x)} & : \ T|_a \text{ is decreasing.} \end{cases}$$

Proof. Assume without loss of generality that S is monotone increasing on $a \in \alpha$. For $y, y_0 \in \Pi(a), y > y_0$ and $x, x_0 \in X$ such that $\Pi(x) = y$ and $\Pi(x_0) = y_0$ we have that

$$\frac{S(y) - S(y_0)}{y - y_0} = \frac{m([T(x_0), T(x)))}{m([x_0, x))}.$$

If $\exp(\varphi)$ is constant on $\Pi^{-1}(\{y_0\})$ and is continuous in $\partial\Pi^{-1}(\{y_0\})$ the limit as $y \to y_0$ is independent of the choice of the representatives of y_0 in X. Hence,

$$\lim_{y \to y_0} \frac{S(y) - S(y_0)}{y - y_0} = \frac{dm \circ T}{dm}(x_0) \ = \ e^{\varphi(x_0)}.$$

$\square$

Note that the latter condition for the existence of DS can be reformulated as follows. If the assignment $y \mapsto exp(\varphi(\hat{x})$, where $y \in (\Pi(a))^o$ and $\hat{x} \in \Pi^{-1}\{y\}$, is independent of the choice of $\hat{x}$ and extends to a continuous function in y then $DS(y)$ exists. Furthermore, there is a straightforward generalization of these results to skew products of the following class. Let X be a topological space, Y be a totally ordered space as above and $T : X \times Y \to X \times Y, (x, y) \mapsto (\tau(x), T_x(y))$ where each fiber map is monotone and continuous on each atom of the partition α_x of Y. Moreover, assume that $\{\mu_x \mid x \in X\}$ is a family of non–atomic, nonsingular Borel probability measures on Y such that $x \mapsto \mu_x$ is weak* continuous. We then have, for

$$\begin{aligned} \Pi_x : Y &\to [0,1], \quad y \mapsto (x, \mu_x(\{z \mid z \preceq y\})) \\ S : X \times [0,1] &\to X \times [0,1], \quad (x, y) \mapsto (\tau(x), \Pi_{\tau(x)}(T_x(\hat{y}))) \\ & \quad\ \text{where } \hat{y} \in \Pi_x^{-1}(\{x\}). \end{aligned}$$

Proposition 2.3. *The map $\Pi : X \times Y = X \times [0,1], (x, y) \mapsto (x, \Pi_x(y))$ semi-conjugates the skew products T and S, and S_x is continuous and monotone on $(\Pi_x(a))^o$ for each atom $a \in \alpha_x$. The map Π is a homeomorphism if and only if $\mu_x((a, b]) \neq 0$ for all $x \in X$, $a, b \in Y$, $a \prec b$.*

Let $\{\mu_x : x \in X\}$ be a weak continuous Gibbs family for the continuous potential φ and continuous gauge function $A : X \to \mathbb{R}$ having no atom on each* *fiber. We then have for $x \in X$ and $y \in (\Pi_x(a))^\circ$ for $a \in \alpha_x$, such that the assignment $y \mapsto \exp(\varphi(\hat{y}))$ is independent of the choice of $\hat{y} \in \Pi_x^{-1}(\{y\})$ and continuous in y,*

$$DS_x(y) = \begin{cases} e^{\varphi(x,\hat{y})} & : \ T_x|_a \text{ is increasing} \\ -e^{\varphi(x,\hat{y})} & : \ T_x|_a \text{ is decreasing.} \end{cases}$$

Proof. Since the assertions concerning the fiber maps follow by Propositions 2.1 and 2.2 it is left to show that $(x,y) \mapsto (x, \Pi_x(y))$ is continuous. So assume that $((x_n, y_n))$ is a sequence in $X \times Y$ converging to (x,y). Since μ_{x_n} has no atoms for each $n \in \mathbb{N}$, $\lim_{m\to\infty} \Pi_{x_n}(y_m) = \Pi_{x_n}(y)$. Furthermore, the weak* continuity of $x \to \mu_x$ gives that $\lim_{n\to\infty} \mu_{x_n}(\{z \mid z \preceq y_m\}) = \mu_x(\{z \mid z \preceq y_m\})$ for all $m \in \mathbb{N}$. This essentially gives the assertion. □

Note that sufficient conditions for the existence of weak* continuous Gibbs families can be deduced from [1]. A skew product $T : X \times Y \to X \times Y$, where X and Y are compact metric spaces with metrics d_X and d_Y, respectively, is called *fiber expanding*, if the fiber maps $T_x : \{x\} \times Y \to \{\tau(x)\} \times Y$ are uniformly expanding in Ruelle's sense. This means that there exists $a > 0$ and $\rho \in (0,1)$ such that for $x \in X$ and $u, v' \in Y$ and $d_Y(T_x(u), v') < 2a$, then there exists a unique $v \in Y$ such that $T_x(v) = v'$ and $d_Y(u,v) < 2a$. Furthermore, we have that

$$d_Y(u,v) \le \rho d_Y(T_x(u), T_x(v)).$$

The system $(X \times Y, T)$ is called *topologically exact* along fibers if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that, for any $(x,y) \in X \times Y$ and $n \ge N$, we have that

$$T_x^n(B(y,\varepsilon)) = Y,$$

where $B(y,\varepsilon) \subset Y$ denotes the ball of radius ε centered at the point y and where $T_x^n = T_{\tau^{n-1}(x)} \circ T_x^{n-1}$ for $n \ge 1$. Under these conditions Gibbs families do exists (see [1]).

The (weak*) continuity of the Gibbs family depends on properties of the map

$$i : X \times Y \to \{(x,(z,y)) \in X^2 \times Y : \ z = \tau(x)\}$$

defined by $i((x,y)) = (x, T((x,y)))$. In order that a Gibbs family is weak* continuous it is sufficient that i is a local homeomorphism.

3 Applications

Let $S : [0,1] \to [0,1]$ be a piecewise monotone and continuous map. By this we mean that there are finitely many points $0 = p_0 < p_1 < ... < p_s = 1$ partitioning the unit interval, so that for each $k \in \{0, 1 \dots s-1\}$, $S|_{(p_k, p_{k+1})}$ can be extended to a monotone and continuous map on $J_k = [p_k, p_{k+1}]$. We first recall the Hofbauer-Keller construction in [2]. Dividing each point p_k and all its forward and backward iterates p into two points $p+ = \lim_{x\downarrow p} x$ and $p- = \lim_{\uparrow x} x$, one constructs a compact extension $(X, \widetilde{S})$ of $([0,1], S)$, such that $\widetilde{S}$ is an open map and the natural projection $\pi : X \to [0,1]$ is one-to-one except in countably many points. Hence for every continuous potential $\varphi : [0,1] \to \mathbb{R}$ there is a Gibbs measure $\widetilde{m}$ on X so that

$$\int_X \widetilde{V} f(\pi(z))\widetilde{m}(dz) = \lambda \int_X f(\pi(z))\widetilde{m}(dz).$$

If $\widetilde{m}$ has no atoms, then $m = \widetilde{m} \circ \pi$ defines a Gibbs measure on $[0,1]$ for the potential φ.

Proposition 3.1. *Let $S : [0,1] \to [0,1]$ be a continuous and piecewise monotone map with positive topological entropy $h(S)$. Then there exists a non-atomic Gibbs measure for the potential $\varphi = 0$ and $\lambda = e^{h(S)}$*

Proof. Let $(X, \widetilde{S})$ denote the extension of $([0,1], S)$ as above. Let $\widetilde{m}$ denote the Gibbs measure for $\varphi \circ \pi$ on X. It is well known that for piecewise continuous maps of the interval topological entropy equals the asymptotic growth rate of the number of inverse branches of S^n. By inspecting the construction in [2] one can easily show that $\log \lambda$ is also equal to this asymptotic growth rate with respect to $\widetilde{S}^n$, which implies that $\lambda > 1$ by assumption. Let $x \in X$. Then $\widetilde{m}(\{\widetilde{S}^n(x)\}) = \lambda^n \widetilde{m}(\{x\})$. In case x is non-periodic we have $\widetilde{m}(\{\widetilde{S}^n(x)) \to \infty$ unless $\widetilde{m}(\{x\}) = 0$, and in case $\widetilde{S}^n(x) = x$ for some $n \geq 1$ we get $\lambda = 1$ unless $\widetilde{m}(\{x\}) = 0$. It follows that $\widetilde{m}$ has no atoms, whence π is a measure theoretic isomorphism and $m = \widetilde{m} \circ \pi$ is a non-atomic Gibbs measure with $\lambda = \exp[h(S)]$. □

Applying Propositions 2.1 and 2.2 in this situation immediately gives the following result which is the advertised generalization of the result in [3].

Theorem 1. *Let $S : [0,1] \to [0,1]$ be a piecewise monotone and continuous transformation of the unit interval. Assume that*

$$\limsup_{n\to\infty} \frac{1}{n} \log c_n = h(S) = M > 0,$$

where c_n denotes the number of monotone branches of S^n. Then there exists a Gibbs measure m for the constant potential with no atoms, and

$$h(x) = m([0,x]) \qquad 0 \leq x \leq 1$$

defines a semiconjugacy between S and a piecewise linear and continuous map T of the interval with slope e^M.

Remark 3.2. *The map $T : [0,1] \to [0,1]$ in Theorem 1 is defined as follows:*

Let $p_0 = 0 < p_1 < \ldots < p_r = 1$ denote the coarsest partition so that S is monotone on each of the intervals $J_k = [p_k, p_{k+1}]$. Let $a_k = h(p_k)$. In case that S is non-decreasing on $[p_0, p_1]$, for $a_k \le y \le a_{k+1}$

$$T(y) = h(S(p_0)) + e^M \left(2 \sum_{j=1}^{k} (-1)^{j+1} a_j + (-1)^k y \right). \tag{2}$$

Similarly, if S is non-increasing on $[p_0, p_1]$, for $a_k \le y \le a_{k+1}$

$$T(y) = h(S(p_0)) - e^M \left(2 \sum_{j=1}^{k} (-1)^{j+1} a_j - (-1)^k y \right). \tag{3}$$

If S is unimodal with turning point $p_1 = c$ and $T(0) = T(1) = 0$, then

$$T(y) = \begin{cases} e^M y & \text{if } y \le 1/2 \\ e^M (1-y) & \text{if } y \ge 1/2. \end{cases}$$

It is also immediately clear that h is a conjugacy if the Gibbs measure m is positive on non-empty open intervals. This occurs for example, if the map T is piecewise expanding.

We give a short proof of (2) and (3). For $x \in [p_k, p_{k+1})$ and $S(x) \ge S(p_k)$ one has

$$\begin{aligned} h(S(x)) &= m([0, S(x)]) = m([0, S(p_k)]) + m(S(p_k, x]) \\ &= h(S(p_k)) + e^M m((p_k, x]) = h(S(p_k)) + e^M (h(x) - h(p_k)). \end{aligned}$$

Similarly, for $x \in [p_k, p_{k+1})$ and $S(x) \le S(p_k)$ one has

$$\begin{aligned} h(S(x)) &= m([0, S(x)]) = m([0, S(p_k)]) - m(S(p_k, x]) \\ &= h(S(p_k)) - e^M m((p_k, x]) = h(S(p_k)) - e^M (h(x) - h(p_k)). \end{aligned}$$

By induction one shows in case that S is non-decreasing on the first interval

$$h(S(p_k)) = h(S(p_0)) + 2e^M \sum_{j=1}^{k-1} (-1)^{j+1} a_j + e^M (-1)^{k+1} a_k,$$

and similarly if S is non-increasing on the first interval. If T is defined as in Remark 3.2, we get $h \circ S = T \circ h$.

Suppose T is semiconjugate to the piecewise linear map S with slope λ and with semiconjugacy h. Clearly, h defines a probability measure m on $[0,1]$ and satisfies

$$h(T(x)) = m([0, T(x)]) = h(T(p_k)) \pm \lambda m([p_k, x])$$

for $x \in [p_k, p_{k+1}]$. This implies that m is a Gibbs measure. If this Gibbs measure is unique, there is only one semiconjugacy to a piecewise linear map S with constant slope.

In case of skew products, the existence of a Gibbs family is equivalent to the existence of an eigenspace for some relative version of the transfer operator. Namely, for a skew product $(X \times Y, T)$ and a Borel measurable function $\varphi : X \times Y \to \mathbb{R}$ the family $\{\mu_x \mid x \in X\}$ is a Gibbs family (cf. section 1) for φ if and only if there exists a Borel measurable function $A_\varphi : X \to \mathbb{R}$ such that for $x \in X$ and $f \in L_1(\mu_x)$ we have that

$$\int V_x f(y)\mu_{\tau(x)}(dy) = A_\varphi(x) \int f(y)\mu_x(dy),$$

where $V_x f(y) := \sum_{T_x(y')=y} f(y')e^{\varphi(y')}$ denotes the relative transfer operator.

We conclude describing two setups when Proposition 2.3 can be applied.

Example 1. Let $(X \times [0,1], T)$ be a skew product where $\tau : X \to X$ is bounded–to–one and each fiber map T_x is a piecewise continuous and monotone map of the interval $Y = [0,1]$. Like in the case of an interval map as above we split each point in the partition $p_0(x) < p_1(x) < \ldots < p_{s(x)}(x)$ for the fiber map T_x over $x \in X$ into two points, as well as their grand orbits. This procedure does not give a continuous extension in general, but we assume here it does. The extended system is then a fibered system (no longer a skew product in general), denoted by $(\widetilde{Y}, \widetilde{T})$. Taking the order topology we may assume w.l.o.g. that for each $x \in X$ the map T_x is open. If this Hofbauer-Keller extension is fiberwise expanding and exact along fibers we can proceed by taking $\varphi : X \times [0,1] \to \mathbb{R}$ to be constant, hence its lift $\widetilde{\varphi} : \widetilde{Y} \to \mathbb{R}$ is Hölder continuous in the order space topology. Hence by [1], if $i : \widetilde{Y} \to X \times \widetilde{Y}$, $i(\tilde{y}) = (\pi(y), \widetilde{T}(\tilde{y}))$ is a local homeomorphism, where $\pi : \widetilde{Y} \to X$ denotes the canonical projection, the semiconjugacy of T exists according to Proposition 2.3.

Example 2. If $T : X \times Y \to X \times Y$ is an open map and bounded-to-one, the operator $V_x : C(\{x\} \times Y) \to C(\{\tau(x)\} \times Y)$ acts on continuous functions for each $x \in X$. Moreover, we consider the map

$$V^* : C(X, C^*(Y)) \to C(X, C^*(Y))$$

defined by

$$\int f dV^* d\mu_x = \int V_x f(\tau(x), \cdot) d\mu_{\tau(x)},$$

where $\mu \in C(X, C^*(Y))$ and $f \in C(Y)$.

For $\mu \in C(X, C^*(Y))$ define

$$(L\mu)_x = V^*\mu_x / V^*\mu_x(Y),$$

and note that it is continuous since

$$\begin{aligned}\|V^*\mu\|_\infty &= \sup_{x\in X}\sup_{f\in C(Y),\|f\|_\infty=1}\|\int fV_x^* d\mu_x\|\\ &\le \|\mu\|_\infty \sup_{x\in X}\|V_x\|\|f\|_\infty.\end{aligned}$$

Define $\mathcal{M}$ to be the set of all $\mu = (\mu_x)_{x\in X} \in C(X, C^*(Y))$ such that for all $f \in C(Y)$ with $\|f\|_\infty \le 1$ the map $x \mapsto \int f d\mu_x$ is Hölder continuous with Hölder exponent s and Hölder constant bounded by some M (independently of f).

Proposition 3.3. *Let $(X\times Y, T)$ be a skew product with open map T and assume that L leaves $\mathcal{M}$ invariant. For every continuous potential $\varphi : X \times Y \to \mathbb{R}$ there exists a Gibbs family $\{\mu_x : x \in X\}$. Moreover, for this family the map $x \to \mu_x$ is continuous in the weak* topology.*

Proof. As it easily can be seen the set M is convex. Assume that $(\mu^n)_{n\in\mathbb{N}}$ is a sequence in M converging pointwise to μ. By the triangle inequality, for any $f \in C(Y)$ with $\|f\|_\infty \le 1$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for $n \ge n_0$

$$\begin{aligned}&|\int f d\mu_x - \int f d\mu_y|\\ \le\ &|\int f d\mu_x - \int f d\mu_x^n| + |\int f d\mu_x^n - \int f d\mu_y^n| + |\int f d\mu_y^n - \int f d\mu_y|\\ \le\ &Md(x,y)^s + 2\epsilon.\end{aligned}$$

Clearly $\mu \in M$, whence the set M is compact. The proposition follows from the Schauder-Tychonoff fixed point theorem.

References

[1] M. Denker, M. Gordin: Gibbs measures for fibered systems. Advances in Mathematics **48** (1999), no. 2, 161–192.

[2] F. Hofbauer, G. Keller: Ergodic properties of invariant measures for piecewise monotone transformations. Math. Z. **180** (1982), no. 1, 119–140.

[3] J. Milnor, W. Thurston: On iterated maps of the interval. In: Dynamical Systems, College Park, 1986–1987, Lecture Notes in Mathematics **1342**, 465–563, Springer Verlag, Berlin, Heidelberg, New York 1988.