Semiconjugacies for skew products of interval maps

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Abstract

Distribution functions of non-atomic Gibbs measures on the unit interval define natural semiconjugacies between maps on $[0, 1]$. Using this method we extend a result of Milnor and Thurston in [3] about the semi-conjugacy of unimodal maps to skew products with maps of the interval as fiber maps.

1 Introduction

In this note we use the existence of Gibbs measures for a discrete time dynamical system to define a semiconjugacy between the system and a piecewise linear map. In particular, we discuss the analogue of this construction in the case of skew products $(X \times Y, \tau, (T_x)_{x \in X})$ where $\tau : X \to X$, $T_x : Y \to Y$ ($x \in X$) and

$$T(x, y) = (\tau(x), T_x(y)).$$

In the latter case the notion of a Gibbs measure can be generalized to that of Gibbs families whose existence and uniqueness was discussed in [1]. Also recall that a dynamical system $T : Z \to Z$ is called semiconjugate to the system

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$T' : Z' \to Z'$ if there is a continuous surjective map $\Pi : Z \to Z'$ such that the diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{T} & Z \\
\Pi & \downarrow & \Pi \\
Z' & \xrightarrow{T'} & Z'
\end{array}
\]
commutes, and we call $\Pi : X \times Y \to X' \times Y'$ a semiconjugacy between the skew products $(X \times Y, T)$ and $(X' \times Y', T')$ if $\Pi$ semiconjugates the dynamical systems and if $\Pi$ maps fibers to fibers, i.e. if $\Pi([x] \times Y) \subset [x'] \times Y'$ for some $x' \in X'$.

Consider the special case of a skew product where $Y = [0,1]$ and where each $T_x$ is a piecewise continuous and monotone map of the interval $Y$ with positive relative topological entropy $h(T_x)$. Certain fiberwise expanding transformations $T$ will be shown to be semiconjugate to a skew product where each fiber map is a continuous piecewise monotone map of the interval with slope $\exp h(T_x)$.

Note that this result parallels the case of a map of the interval since the theory of skew products and their Gibbs families reduces to this case if $X$ consists of a single point. For unimodal maps we rediscover a result of Milnor and Thurston in [3], where it has been shown in Theorem 7.4, that every unimodal map, for which the number of monotonicity intervals of $T^n$ increases exponentially fast, is semiconjugate to a unimodal map with constant common slopes on each of the monotonicity branches. The proof given here is different.

The idea of the proof relies on the following simple fact. If $\mu$ is a distribution on the unit interval, then its distribution function is monotone, surjective and even continuous if $\mu$ has no atoms. Hence the Milnor-Thurston result is a statement of a piecewise scaling property of a distribution function without any atom. Such distributions are obtained as non-atomic Gibbs measures, in particular, as measures of maximal entropy.

In order to be more precise, let $T : Z \to Z$ be a dynamical system and $\varphi : Z \to \mathbb{R}$ be a function. Recall that a measure $m$ is called a Gibbs measure for $\varphi$ if the Jacobian $d\mu \circ T/d\mu$ is defined $\mu$ - a.e. and is given by
\[
\frac{d\mu \circ T}{d\mu} = e^{\varphi}.
\]

The following standard chain of arguments gives the existence of a Gibbs measure in the case of an open and expanding map $T$ acting on a compact space $Z$ and a continuous function $\varphi$. By these assumptions, the map $T$ has locally a constant number of preimages, which implies that $T$ acts on continuous functions by its Perron-Frobenius operator
\[
V_\varphi f(y) = \sum_{T(y') = y} f(y') e^{\varphi(y')}.
\]

Furthermore, its dual operator acts continuously on the space of signed measures on $Z$. Therefore, by the Schauder-Tychonoff theorem there exists an eigenvalue
\( \lambda > 0 \) and a measure \( \mu \) such that \( d\mu \circ T/d\mu = \lambda \exp(-\varphi) \), where \( d\mu \circ T/d\mu \) refers to the Jacobian. In other words, \( \mu \) is a Gibbs measure for the potential \( \varphi + \log \lambda \).

In the case of skew products we use the notion of Gibbs families on skew products as a generalization of Gibbs measures. Recall that a family \( \{\mu_x : x \in X\} \) of probability measures on \( Y \) is called a Gibbs family for a measurable function \( \varphi : X \times Y \to \mathbb{R} \) if there exists a positive measurable function (called gauge function) \( A : X \to \mathbb{R} \), such that, for each \( x \in X \), the Jacobian of \( \mu_x \) is given by

\[
\frac{d\mu_{\tau(x)} \circ T_x}{d\mu_x} = A(x) \exp(-\varphi).
\] (1)

Using the existence of Gibbs families we extend the result of semiconjugacies for maps of the interval to certain skew products where the maps \( T_x \ (x \in X) \) are maps of the interval.

2 Semiconjugacies for skew-products

In this section we prove our result about semiconjugacies. We begin with the case of piecewise monotone map \( T \) of a totally ordered Polish space \( X \). Recall that \( X \) is totally ordered if there exists an order relation \( \preceq \) such that for each \( x, y \in X \) either \( x \preceq y \) or \( y \preceq x \) and \( x \preceq y \preceq x \) implies that \( x = y \). This gives rise to a further relation \( \prec \), where \( x \prec y \) if \( x \preceq y \) and \( x \neq y \). With this setting, the notion of closed and open intervals can be easily extended to the space \( X \) and these intervals will be denoted by \([a, b]\) and \((a, b)\), respectively. The topology on \( X \) is assumed to be generated by the open intervals or in other words, the topology on \( X \) is the order topology.

The map \( T \) is referred to be piecewise continuous and monotone if there exists a finite partition \( \alpha \) of \( X \) into intervals such that for each \( a \in \alpha \) the restriction \( T|_a \) is continuous and monotone. Let \( m \) be a non-atomic and nonsingular probability measure on \( X \) and let \( \Pi : X \to [0, 1] \subset \mathbb{R} \) and \( S : [0, 1] \to [0, 1] \) be defined as follows.

\[
\Pi : X \to [0, 1], \quad x \mapsto m(\{z \in X \mid z \preceq x\}),
\]

\[
S : [0, 1] \to [0, 1], \quad y \mapsto \Pi(Tx) \text{ where } x \in \Pi^{-1}(\{y\}).
\]

Note that, since \( m \) has no atoms and is nonsingular, the map \( \Pi \) is onto, and \( S \) is well defined. Moreover, we have that \( S \circ \Pi = \Pi \circ T \) and \( S \) is piecewise continuous and monotone on \( \Pi(a) \) for each \( a \in \alpha \). Furthermore, we obtain the following immediate result.

**Proposition 2.1.** The map \( \Pi \) is continuous and semiconjugates \( T \) and \( S \). Furthermore, \( S \) is continuous and monotone on the interior \( (\Pi(a))^0 \) of \( \Pi(a) \) for each \( a \in \alpha \), and \( \lambda = m \circ \Pi^{-1} \) where \( \lambda \) refers to the Lebesgue measure. Moreover, if \( \Pi \) is a homeomorphism if and only if \( m((a, b]) \neq 0 \) for all \( a, b \in X, a \prec b \).
In case that \( m \) is a Gibbs measure for the potential \( \varphi \) the following proposition gives the relation between the derivative \( DS \) of \( S \) and \( \varphi \).

**Proposition 2.2.** Let \( m \) be a Gibbs measure for the potential \( \varphi \). Assume that \( y_0 \) belongs to the interior \( (\Pi(a))^o \) of \( \Pi(a) \) for some \( a \in \alpha \) and that \( \exp(\varphi) \) is constant on \( \Pi^{-1}(\{y_0\}) \) and continuous in \( \partial\Pi^{-1}(\{y_0\}) \). Then \( S \) is differentiable in \( y_0 \) and, for \( x \in \Pi^{-1}(\{y_0\}) \),

\[
DS(y_0) = \begin{cases} e^{\varphi(x)} & \text{if } T\mid_a \text{ is increasing} \\
-e^{\varphi(x)} & \text{if } T\mid_a \text{ is decreasing.}
\end{cases}
\]

**Proof.** Assume without loss of generality that \( S \) is monotone increasing on \( a \in \alpha \). For \( y, y_0 \in \Pi(a), y > y_0 \) and \( x, x_0 \in X \) such that \( \Pi(x) = y \) and \( \Pi(x_0) = y_0 \) we have that

\[
\frac{S(y) - S(y_0)}{y - y_0} = m((T(x_0), T(x)) \frac{m([T(x_0), T(x)])}{m([x_0, x])}.
\]

If \( \exp(\varphi) \) is constant on \( \Pi^{-1}(\{y_0\}) \) and is continuous in \( \partial\Pi^{-1}(\{y_0\}) \) the limit as \( y \to y_0 \) is independent of the choice of the representatives of \( y_0 \) in \( X \). Hence,

\[
\lim_{y \to y_0} \frac{S(y) - S(y_0)}{y - y_0} = \frac{dm \circ T}{dm} (x_0) = e^{\varphi(x_0)}.
\]

\[\square\]

Note that the latter condition for the existence of \( DS \) can be reformulated as follows. If the assignment \( y \mapsto \exp(\varphi(\hat{x})) \), where \( y \in (\Pi(a))^o \) and \( \hat{x} \in \Pi^{-1}(\{y\}) \), is independent of the choice of \( \hat{x} \) and extends to a continuous function in \( y \) then \( DS(y) \) exists. Furthermore, there is a straightforward generalization of these results to skew products of the following class. Let \( X \) be a topological space, \( Y \) be a totally ordered space as above and \( T : X \times Y \to X \times Y \), \((x, y) \mapsto (\tau(x), T_x(y))\) where each fiber map is monotone and continuous on each atom of the partition \( \alpha_x \) of \( Y \). Moreover, assume that \( \{\mu_x \mid x \in X\} \) is a family of non–atomic, nonsingular Borel probability measures on \( Y \) such that \( x \mapsto \mu_x \) is weak* continuous. We then have, for

\[
\Pi_x : Y \to [0, 1], \quad y \mapsto (x, \mu_x(\{z \mid z \leq y\}))
\]

\[
S : X \times [0, 1] \to X \times [0, 1], \quad (x, y) \mapsto (\tau(x), \Pi_{\tau(x)}(T_x(y)) = y_0 \in \Pi^{-1}_x(\{x\}).
\]

**Proposition 2.3.** The map \( \Pi : X \times Y = X \times [0, 1], (x, y) \mapsto (x, \Pi_x(y)) \) semi-conjugates the skew products \( T \) and \( S \), and \( S_x \) is continuous and monotone on \( (\Pi_x(a))^o \) for each atom \( a \in \alpha_x \). The map \( \Pi \) is a homeomorphism if and only if \( \mu_x([a, b]) \neq 0 \) for all \( x \in X, a, b \in Y, a < b \).

Let \( \{\mu_x \mid x \in X\} \) be a weak* continuous Gibbs family for the continuous potential \( \varphi \) and continuous gauge function \( A : X \to \mathbb{R} \) having no atom on each
fiber. We then have for \( x \in X \) and \( y \in (\Pi_x(a))^a \) for \( a \in \alpha_x \), such that the assignment \( y \mapsto \exp(\varphi(y)) \) is independent of the choice of \( y \in \Pi^{-1}_x(\{y\}) \) and continuous in \( y \),

\[
\begin{align*}
\mathcal{D}S_x(y) = \begin{cases} 
  e^{\varphi(x,y)} : & T_x|_a \text{ is increasing} \\
  -e^{\varphi(x,y)} : & T_x|_a \text{ is decreasing}.
\end{cases}
\end{align*}
\]

**Proof.** Since the assertions concerning the fiber maps follow by Propositions 2.1 and 2.2 it is left to show that \( (x, y) \mapsto (x, \Pi_x(y)) \) is continuous. So assume that \( ((x_n, y_n)) \) is a sequence in \( X \times Y \) converging to \( (x, y) \). Since \( \mu_x \) has no atoms for each \( n \in \mathbb{N} \), \( \lim_{n \to \infty} \Pi_x y_m = \Pi_x(y) \). Furthermore, the weak* continuity of \( x \mapsto \mu_x \) gives that \( \lim_{n \to \infty} \mu_x \{ \{ z \mid z \leq y_m \} \} = \mu_x \{ \{ z \mid z \leq y_m \} \} \) for all \( m \in \mathbb{N} \). This essentially gives the assertion. \( \square \)

Note that sufficient conditions for the existence of weak* continuous Gibbs families can be deduced from [1]. A skew product \( T : X \times Y \to X \times Y \), where \( X \) and \( Y \) are compact metric spaces with metrics \( d_X \) and \( d_Y \), respectively, is called fiber expanding, if the fiber maps \( T_x : \{ x \} \times Y \to \{ \tau(x) \} \times Y \) are uniformly expanding in Ruelle's sense. This means that there exists \( a > 0 \) and \( \rho \in (0, 1) \) such that for \( x \in X \) and \( u, v' \in Y \) and \( d_Y(T_x(u), v') < 2a \), then there exists a unique \( v \in Y \) such that \( T_x(v) = v' \) and \( d_Y(u, v) < 2a \). Furthermore, we have that

\[
d_Y(u, v) \leq \rho d_Y(T_x(u), T_x(v)).
\]

The system \((X \times Y, T)\) is called topologically exact along fibers if, for every \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \) such that, for any \( (x, y) \in X \times Y \) and \( n \geq N \), we have that

\[
T^n_x(B(y, \varepsilon)) = Y,
\]

where \( B(y, \varepsilon) \subseteq Y \) denotes the ball of radius \( \varepsilon \) centered at the point \( y \) and where

\[
T^n_x = T_{\tau^{-1}(x)} \circ T_x^{-1} \quad \text{for } n \geq 1.
\]

Under these conditions Gibbs families do exists (see [1]).

The (weak*) continuity of the Gibbs family depends on properties of the map

\[
i : X \times Y \to \{(x, (z, y)) \in X^2 \times Y : z = \tau(x)\}
\]

defined by \( i((x, y)) = (x, \tau((x, y))) \). In order that a Gibbs family is weak* continuous it is sufficient that \( i \) is a local homeomorphism.

## 3 Applications

Let \( S : [0, 1] \to [0, 1] \) be a piecewise monotone and continuous map. By this we mean that there are finitely many points \( 0 = p_0 < p_1 < \ldots < p_s = 1 \) partitioning the unit interval, so that for each \( k \in \{0, 1 \ldots s-1\} \), \( S|_{(p_k, p_{k+1})} \) can be extended to a monotone and continuous map on \( J_k = [p_k, p_{k+1}] \). We first recall the Hofbauer-Keller construction in [2]. Dividing each point \( p_k \) and all its forward and backward
iterates $p$ into two points $p^+ = \lim_{x \uparrow p} x$ and $p^- = \lim_{x \downarrow p} x$, one constructs a compact extension $(X, \widetilde{S})$ of $([0, 1], S)$, such that $\widetilde{S}$ is an open map and the natural projection $\pi : X \to [0, 1]$ is one-to-one except in countably many points. Hence for every continuous potential $\varphi : [0, 1] \to \mathbb{R}$ there is a Gibbs measure $\widetilde{m}$ on $X$ so that
\[
\int_X \widetilde{V} f(\pi(z)) \tilde{m}(dz) = \lambda \int_X f(\pi(z)) \tilde{m}(dz).
\]
If $\tilde{m}$ has no atoms, then $m = \tilde{m} \circ \pi$ defines a Gibbs measure on $[0, 1]$ for the potential $\varphi$.

**Proposition 3.1.** Let $S : [0, 1] \to [0, 1]$ be a continuous and piecewise monotone map with positive topological entropy $h(S)$. Then there exists a non-atomic Gibbs measure for the potential $\varphi = 0$ and $\lambda = e^{h(S)}$.

**Proof.** Let $(X, \tilde{S})$ denote the extension of $([0, 1], S)$ as above. Let $\tilde{m}$ denote the Gibbs measure for $\varphi \circ \pi$ on $X$. It is well known that for piecewise continuous maps of the interval topological entropy equals the asymptotic growth rate of the number of inverse branches of $S^n$. By inspecting the construction in [2] one can easily show that $\lambda$ is also equal to this asymptotic growth rate with respect to $\widetilde{S}^n$, which implies that $\lambda > 1$ by assumption. Let $x \in X$. Then $\tilde{m}(\{\widetilde{S}^n(x)\}) = \lambda^n \tilde{m}(\{x\})$. In case $x$ is non-periodic we have $\tilde{m}(\{\widetilde{S}^n(x)\}) \to \infty$ unless $\tilde{m}(\{x\}) = 0$, and in case $\widetilde{S}^n(x) = x$ for some $n \geq 1$ we get $\lambda = 1$ unless $\tilde{m}(\{x\}) = 0$. It follows that $\tilde{m}$ has no atoms, whence $\pi$ is a measure theoretic isomorphism and $m = \tilde{m} \circ \pi$ is a non-atomic Gibbs measure with $\lambda = \exp[h(S)]$.  

Applying Propositions 2.1 and 2.2 in this situation immediately gives the following result which is the advertised generalization of the result in [3].

**Theorem 1.** Let $S : [0, 1] \to [0, 1]$ be a piecewise monotone and continuous transformation of the unit interval. Assume that

\[
\limsup_{n \to \infty} \frac{1}{n} \log c_n = h(S) = M > 0,
\]

where $c_n$ denotes the number of monotone branches of $S^n$. Then there exists a Gibbs measure $m$ for the constant potential with no atoms, and

\[
h(x) = m([0, x]) \quad 0 \leq x \leq 1
\]

defines a semiconjugacy between $S$ and a piecewise linear and continuous map $T$ of the interval with slope $e^M$.

**Remark 3.2.** The map $T : [0, 1] \to [0, 1]$ in Theorem 1 is defined as follows:
Let $p_0 = 0 < p_1 < \ldots < p_r = 1$ denote the coarsest partition so that $S$ is monotonie on each of the intervals $J_k = [p_k, p_{k+1}]$. Let $a_k = h(p_k)$. In case that $S$ is non-decreasing on $[p_0, p_1]$, for $a_k \leq y \leq a_{k+1}$

$$T(y) = h(S(p_0)) + e^M \left( 2 \sum_{j=1}^{k} (-1)^{j+1} a_j + (-1)^k y \right). \quad (2)$$

Similarly, if $S$ is non-increasing on $[p_0, p_1]$, for $a_k \leq y \leq a_{k+1}$

$$T(y) = h(S(p_0)) - e^M \left( 2 \sum_{j=1}^{k} (-1)^{j+1} a_j - (-1)^k y \right). \quad (3)$$

If $S$ is unimodal with turning point $p_1 = c$ and $T(0) = T(1) = 0$, then

$$T(y) = \begin{cases} 
  e^M y & \text{if } y \leq 1/2 \\
  e^M (1 - y) & \text{if } y \geq 1/2.
\end{cases}$$

It is also immediately clear that $h$ is a conjugacy if the Gibbs measure $m$ is positive on non-empty open intervals. This occurs for example, if the map $T$ is piecewise expanding.

We give a short proof of (2) and (3). For $x \in [p_k, p_{k+1})$ and $S(x) \geq S(p_k)$ one has

$$h(S(x)) = m([0, S(x)]) = m([0, S(p_k)]) + m(S(p_k, x])
= h(S(p_k)) + e^M m((p_k, x)] = h(S(p_k)) + e^M (h(x) - h(p_k)).$$

Similarly, for $x \in [p_k, p_{k+1})$ and $S(x) \leq S(p_k)$ one has

$$h(S(x)) = m([0, S(x)]) = m([0, S(p_k)]) - m(S(p_k, x])
= h(S(p_k)) - e^M m((p_k, x)] = h(S(p_k)) - e^M (h(x) - h(p_k)).$$

By induction one shows in case that $S$ is non-decreasing on the first interval

$$h(S(p_k)) = h(S(p_0)) + 2e^M \sum_{j=1}^{k-1} (-1)^{j+1} a_j + e^M (-1)^{k+1} a_k,$$

and similarly if $S$ is non-increasing on the first interval. If $T$ is defined as in Remark 3.2, we get $h \circ S = T \circ h$.

Suppose $T$ is semiconjugate to the piecewise linear map $S$ with slope $\lambda$ and with semiconjugacy $h$. Clearly, $h$ defines a probability measure $m$ on $[0,1]$ and satisfies

$$h(T(x)) = m([0, T(x)]) = h(T(p_k)) \pm \lambda m([p_k, x]).$$
for $x \in [p_k, p_{k+1}]$. This implies that $m$ is a Gibbs measure. If this Gibbs measure is unique, there is only one semiconjugacy to a piecewise linear map $S$ with constant slope.

In case of skew products, the existence of a Gibbs family is equivalent to the existence of an eigenspace for some relative version of the transfer operator. Namely, for a skew product $(X \times Y, T)$ and a Borel measurable function $\varphi : X \times Y \to \mathbb{R}$ the family $\{\mu_x \mid x \in X\}$ is a Gibbs family (cf. section 1) for $\varphi$ if and only if there exists a Borel measurable function $A_\varphi : X \to \mathbb{R}$ such that for $x \in X$ and $f \in L_1(\mu_x)$ we have that

$$\int V_x f(y) \mu_y(x)(dy) = A_\varphi(x) \int f(y) \mu_y(dy),$$

where $V_x f(y) := \sum_{T_x(y')}=y f(y') e^{\varphi(y')}$ denotes the relative transfer operator.

We conclude describing two setups when Proposition 2.3 can be applied.

**Example 1.** Let $(X \times [0,1], T)$ be a skew product where $\tau : X \to X$ is bounded-to-one and each fiber map $T_x$ is a piecewise continuous and monotone map of the interval $Y = [0,1]$. Like in the case of an interval map as above we split each point in the partition $p_0(x) < p_1(x) < \ldots < p_s(x)(x)$ for the fiber map $T_x$ over $x \in X$ into two points, as well as their grand orbits. This procedure does not give a continuous extension in general, but we assume here it does. The extended system is then a fibered system (no longer a skew product in general), denoted by $(\tilde{Y}, \tilde{T})$. Taking the order topology we may assume w.l.o.g. that for each $x \in X$ the map $T_x$ is open. If this Hofbauer-Keller extension is fiberwise expanding and exact along fibers we can proceed by taking $\varphi : X \times [0,1] \to \mathbb{R}$ to be constant, hence its lift $\tilde{\varphi} : \tilde{Y} \to \mathbb{R}$ is Hölder continuous in the order space topology. Hence by [1], if $i : \tilde{Y} \to X \times \tilde{Y}$, $i(\tilde{y}) = (\tau(\tilde{y}), \tilde{T}(\tilde{y}))$ is a local homeomorphism, where $\pi : \tilde{Y} \to X$ denotes the canonical projection, the semiconjugacy of $T$ exists according to Proposition 2.3.

**Example 2.** If $T : X \times Y \to X \times Y$ is an open map and bounded-to-one, the operator $V_x : C(\{x\} \times Y) \to C(\{\tau(x)\} \times Y)$ acts on continuous functions for each $x \in X$. Moreover, we consider the map

$$V^* : C(X, C^*(Y)) \to C(X, C^*(Y))$$

defined by

$$\int f dV^* d\mu_x = \int V_x f(\tau(x), \cdot) d\mu_\tau(x),$$

where $\mu \in C(X, C^*(Y))$ and $f \in C(Y)$.

For $\mu \in C(X, C^*(Y))$ define

$$(L\mu)_x = V^* \mu_x / V^* \mu_x(Y),$$
and note that it is continuous since
\[ \|V^*\mu\|_\infty = \sup_{x \in X} \sup_{f \in C(Y)\|f\|_\infty = 1} \| \int fV_x^*d\mu_x \| \leq \|\mu\|_\infty \sup_{x \in X} \|V_x\|_\infty. \]

Define \( M \) to be the set of all \( \mu = (\mu_x)_{x \in X} \in C(X, C^*(Y)) \) such that for all \( f \in C(Y) \) with \( \|f\|_\infty \leq 1 \) the map \( x \mapsto \int f d\mu_x \) is Hölder continuous with Hölder exponent \( s \) and Hölder constant bounded by some \( M \) (independently of \( f \)).

**Proposition 3.3.** Let \((X \times Y, T)\) be a skew product with open map \( T \) and assume that \( L \) leaves \( M \) invariant. For every continuous potential \( \varphi : X \times Y \rightarrow \mathbb{R} \) there exists a Gibbs family \( \{\mu_x : x \in X\} \). Moreover, for this family the map \( x \mapsto \mu_x \) is continuous in the weak* topology.

**Proof.** As it easily can be seen the set \( M \) is convex. Assume that \( (\mu^n)_{n \in \mathbb{N}} \) is a sequence in \( M \) converging pointwise to \( \mu \). By the triangle inequality, for any \( f \in C(Y) \) with \( \|f\|_\infty \leq 1 \) and \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) so that for \( n \geq n_0 \)
\[ |\int f d\mu_x - \int f d\mu_y| \leq |\int f d\mu_x - \int f d\mu^n_x| + |\int f d\mu^n_x - \int f d\mu^n_y| + |\int f d\mu^n_y - \int f d\mu_y| \leq Md(x, y)^s + 2\epsilon. \]

Clearly \( \mu \in M \), whence the set \( M \) is compact. The proposition follows from the Schauder-Tychonoff fixed point theorem.

**References**

