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Expansive automorphisms and expansive endomorphisms of the shift (a survey)

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Abstract: Some major problems on expansive automorphisms and expansive endomorphisms of onesided and twosided shifts of finite type are explained, and a quick survey of the results on them which have been obtained by various people is given from the author's viewpoint.

1. Introduction

A *commuting system* (X, τ, φ) means an ordered pair of continuous maps $\tau : X \to X$ and $\varphi : X \to X$ of a compact metric space with

$$\varphi\tau = \tau\varphi.$$

If (X, τ, φ) is a commuting system, then φ is called an *endomorphism* of the dynamical system (X, τ) (and hence τ is an endomorphism of the dynamical system (X, φ)). When φ is a homeomorphism in this definition, then φ is an *automorphism* of (X, τ).

Two commuting systems (X, τ, φ) and (X', τ', φ') are said to be *conjugate* and written by

$$(X, \tau, \varphi) \cong (X', \tau', \varphi'),$$

if there is a *conjugacy* $\psi : (X, \tau, \varphi) \to (X', \tau', \varphi')$, i.e., a homeomorphism $\psi : X \to X'$ which gives conjugacies $\psi : (X, \tau) \to (X, \tau')$ and $\psi : (X, \varphi) \to (X, \varphi')$ between dynamical systems at the same time.

Throughout this article, "conjugacy" and "conjugate" mean "topological conjugacy" and "topologically conjugate". Further, "transitive" means "topologically transitive" (i.e., having a dense forward orbit) and "mixing" means "topologically mixing".

Let A be an alphabet (i.e., a nonempty finite set of symbols). Let $A^{\mathbb{Z}} = \{(a_j)_{j\in\mathbb{Z}} \mid a_j \in A\}$ be endowed with a metric compatible with the product topology of the discrete topology on A. Let $\sigma_A : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be defined by $\sigma_A((a_j)_{j\in\mathbb{Z}}) = (a_{j+1})_{j\in\mathbb{Z}}$. The dynamical system $(A^{\mathbb{Z}}, \sigma_A)$ is called the *full shift* over A or the *full N-shift* if the number of symbols in A is N. Let X be a closed subset of $A^{\mathbb{Z}}$ with $\sigma_A(X) = X$. Let $\sigma = \sigma_A|X$. Then we have a dynamical system (X, σ), which is called a *subshift* over A. Let

$A^{\mathbb{N}} = \{(a_j)_{j\in\mathbb{N}} \mid a_j \in A\}$ be endowed with a metric compatible with the product topology of the discrete topology on A. Let $\tilde{\sigma}_A : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be defined by $\tilde{\sigma}_A((a_j)_{j\in\mathbb{N}}) = (a_{j+1})_{j\in\mathbb{N}}$. The dynamical system $(A^{\mathbb{N}}, \tilde{\sigma}_A)$ is called the *onesided full shift* over A or the *onesided full N-shift* if the number of symbols in A is N. For a subshift (X, σ) over A , let $\tilde{X} = \{(a_j)_{j\in\mathbb{N}} \mid \exists(a_j)_{j\in\mathbb{Z}} \in X\}$. Then with the onto continuous map $\tilde{\sigma} = \tilde{\sigma}_A|\tilde{X}$ we have a dynamical system $(\tilde{X}, \tilde{\sigma})$, which is called a *onesided subshift* over A and is said to be *induced* by (X, σ). A subshift (X, σ) over an alphabet A is called a *subshift of finite type* (SFT) if there is a finite set F of words (blocks) over A such that X is the set of all points $(a_j)_{j\in\mathbb{Z}}$ in $A^{\mathbb{Z}}$ such that $a_j \dots a_{j'} \notin F$ for all $j, j' \in \mathbb{Z}$ with $j \leq j'$. A *onesided subshift of finite type* (onesided SFT) is a onesided subshift induced by an SFT.

- (Hedlund [H], Reddy [R]) Let X be a compact, 0-dimensional metric space. If $\tau : X \to X$ is an expansive homeomorphism, then (X, τ) is conjugate to a subshift. If $\tau : X \to X$ is a positively expansive onto continuous map, then (X, τ) is conjugate to a onesided subshift.

Let (X, σ) be a subshift over an alphabet A. For $n \geq 1$, let

$$L_n(X) = \{a_1 \dots a_n \mid \exists(a_j)_{j\in\mathbb{Z}} \in X, a_j \in A\}.$$

Let (X, σ) and (X', σ') be subshifts. Let $m, n \geq 0$. A mapping $\phi : X \to X'$ is called a *block map of (m, n)-type* or a *block map* if there is a *local rule* $f : L_{m+n+1}(X) \to L_1(X')$ such that

$$\phi((a_j)_{j\in\mathbb{Z}}) = (b_j)_{j\in\mathbb{Z}} \quad \text{with} \quad b_j = f(a_{j-m} \dots a_{j+n}) \ \forall j \in \mathbb{Z}.$$

A block map of (0,0)-type is called a *1-block map.*

- (Curtis, Lindon and Hedlund [H]) Let (X, σ) and (X', σ') be subshifts. Then a mapping $\phi : X \to X'$ is continuous with $\phi\sigma = \sigma'\phi$ if and only if ϕ is a block map.

Therefore in particular, an endomorphism of a full shift is a cellular automaton (map). We can say that an endomorphism of a subshift is a cellular automaton over a subshift space.

By the theorem of Hedlund and Reddy, we know that every expansive automorphism of a subshift is conjugate to a subshift. That is, if (X, σ, φ) is a commuting system such that (X, σ) is a subshift and φ is an expansive homeomorphism, then (X, φ) is conjugate to a subshift. Our problem is, typically speaking, "What subshift is this?" This can be interpreted as the problem of determining the dynamics of a given expansive cellular automaton over a subshift space.

The following are major problems in this line which have been studied.
(1) To what subshift is an expansive automorphism of an SFT conjugate?
(2) To what onesided subshift is a positively expansive onto endomorphism

of an SFT conjugate?
(3) To what subshift is an expansive automorphism of a onesided SFT conjugate?
(4) To what onesided subshift is a positively expansive onto endomorphism of a onesided SFT conjugate?

It is convenient to use the notion of "commuting subshifts" as some people have used. A subshift or a onesided subshift is said to *commute with* another subshift or onesided subshift if there is a commuting system (X, τ, φ) such that the former is conjugate to (X, τ) and the latter is conjugate to (X, φ).

The following are the kernels of our problems.
(1) Is a subshift commuting with an SFT also an SFT?
(2) Is a onesided subshift commuting with an SFT a onesided SFT?
(3) Is a subshift commuting with a onesided SFT an SFT?
(4) Is a onesided subshift commuting with a onesided SFT also a onesided SFT?

We will quickly survey from our viewpoint the results on and around the problems above which have been obtained by various people so far. Of course we can not cover all the results.

Our viewpoint is mainly that of "textile systems" introduced in [N2]. All people except Nasu have obtained their results on the problems above without using textile systems, but many of their results can be explained by using textile systems with the benefit of hindsight.

The reader is referred to [Ki] or [LMar] for a comprehensive introduction to symbolic dynamics, and to [AH] for information on topological dynamics.

2. Textile systems

Let G be a graph. Here a *graph* means a directed graph which may have multiple arcs and multiple loops. Let A_G and V_G denote the arc-set and the vertex-set, respectively, of G. Let $i_G : A_G \to V_G$ and $t_G : A_G \to V_G$ be the mappings such that for arc $a \in A_G$, $i_G(a)$ and $t_G(a)$ are the initial and terminal vertices, respectively, of a. Hence the graph G is represented by $V_G \xleftarrow{i_G} A_G \xrightarrow{t_G} V_G$. Let X_G be the set of all points $(a_j)_{j\in\mathbb{Z}}$ in $A_G^{\mathbb{Z}}$ such that $t_G(a_j) = i_G(a_{j+1})$ for all $j \in \mathbb{Z}$. Then we have a subshift (X_G, σ_G) and a onesided subshift $(\tilde{X}_G, \tilde{\sigma}_G)$, which are called the *topological Markov shift* and the *onesided topological Markov shift*, respectively, defined by G. If M is a nonnegative integral square matrix and G is the graph such that $M_G = M$, then (X_M, σ_M) and $(\tilde{X}_M, \tilde{\sigma}_M)$ denote (X_G, σ_G) and $(\tilde{X}_G, \tilde{\sigma}_G)$, respectively, where $M_G = (M_G(u, v))_{u,v\in V_G}$ with $M_G(u, v)$ equal to the number of arcs a in G starting from u and ending in v.

Topological Markov shifts are SFTs, and SFTs are subshifts which are conjugate to topological Markov shifts. The relation between onesided topological Markov shifts and onesided SFTs is similar.

For graphs Γ and G, a *graph-homomorphism* h of Γ into G, written by $h : \Gamma \to G$, is a pair (h_A, h_V) of mappings $h_A : A_\Gamma \to A_G$ (*arc-map*) and $h_V : V_\Gamma \to V_G$ (*vertex-map*) such that the following diagram is commutative.

$$\begin{tikzcd} V_\Gamma \arrow[d, "h_V"'] & A_\Gamma \arrow[l, "i_\Gamma"'] \arrow[r, "t_\Gamma"] \arrow[d, "h_A"'] & V_\Gamma \arrow[d, "h_V"] \\ V_G & A_G \arrow[l, "i_G"'] \arrow[r, "t_G"] & V_G \end{tikzcd}$$

A graph-homomorphism $h : \Gamma \to G$ gives 1-block maps $\phi_h : X_\Gamma \to X_G$ and $\tilde{\phi}_h : \tilde{X}_\Gamma \to \tilde{X}_G$ by

$$\begin{aligned} \phi_h((\alpha_j)_{j\in\mathbb{Z}}) &= (h_A(\alpha_j)_{j\in\mathbb{Z}}), \quad (\alpha_j)_{j\in\mathbb{Z}} \in X_\Gamma,\ \alpha_j \in A_\Gamma, \\ \tilde{\phi}_h((\alpha_j)_{j\in\mathbb{N}}) &= (h_A(\alpha_j)_{j\in\mathbb{N}}), \quad (\alpha_j)_{j\in\mathbb{N}} \in \tilde{X}_\Gamma,\ \alpha_j \in A_\Gamma. \end{aligned}$$

A *textile system T over* a graph G is defined to be an ordered pair of graph-homomorphisms $p : \Gamma \to G$ and $q : \Gamma \to G$ such that each $\alpha \in A_\Gamma$ is uniquely determined by the quadruple $(i_\Gamma(\alpha), t_\Gamma(\alpha), p_A(\alpha), q_A(\alpha))$. We write

$$T = (p, q : \Gamma \to G).$$

We have the following commutative diagram.

$$\begin{tikzcd} V_G & A_G \arrow[l, "i_G"'] \arrow[r, "t_G"] & V_G \\ V_\Gamma \arrow[u, "p_V"'] \arrow[d, "q_V"] & A_\Gamma \arrow[l, "i_\Gamma"'] \arrow[r, "t_\Gamma"] \arrow[u, "p_A"] \arrow[d, "q_A"'] & V_\Gamma \arrow[u, "p_V"] \arrow[d, "q_V"'] \\ V_G & A_G \arrow[l, "i_G"'] \arrow[r, "t_G"] & V_G \end{tikzcd}$$

If we observe this diagram vertically, then we have the ordered pair of graph-homomorphisms

$$\begin{tikzcd} V_G & A_G \arrow[l, "i_G"'] \\ V_\Gamma \arrow[u, "p_V"'] \arrow[d, "q_V"] & A_\Gamma \arrow[l, "i_\Gamma"'] \arrow[u, "p_A"'] \arrow[d, "q_A"'] \\ V_G & A_G \arrow[l, "i_G"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_G \arrow[r, "t_G"] & V_G \\ A_\Gamma \arrow[r, "t_\Gamma"] \arrow[u, "p_A"] \arrow[d, "q_A"'] & V_\Gamma . \arrow[u, "p_V"] \arrow[d, "q_V"'] \\ A_G \arrow[r, "t_G"] & V_G \end{tikzcd}$$

This defines another textile system

$$T^* = (p^*, q^* : \Gamma^* \to G^*)$$

called the *dual* of T, where $i_{\Gamma^*} = p_A, t_{\Gamma^*} = q_A$, $i_{G^*} = p_V$ and $t_{G^*} = q_V$.

Let $T = (p, q : \Gamma \to G)$ be a textile system. Let $\xi = \phi_p$ and let $\eta = \phi_q$. A two-dimensional configuration $(\alpha_{ij})_{i,j\in\mathbb{Z}}, \alpha_{ij} \in A_\Gamma$, is called a *textile* woven by T if $(\alpha_{ij})_{j\in\mathbb{Z}} \in X_\Gamma$ and $\eta((\alpha_{i-1,j})_{j\in\mathbb{Z}}) = \xi((\alpha_{ij})_{j\in\mathbb{Z}})$ for all $i \in \mathbb{Z}$. Let U_T denote the set of all textiles woven by T. Define

$$Z_T = \{(\alpha_{0j})_{j\in\mathbb{Z}} \mid \exists(\alpha_{ij})_{i,j\in\mathbb{Z}} \in U_T\}, \ X_T = \{\xi((\alpha_{0j})_{j\in\mathbb{Z}}) \mid \exists(\alpha_{ij})_{i,j\in\mathbb{Z}} \in U_T\}.$$

Then we have subshifts (Z_T, ς_T) and (X_T, σ_T). We call (X_T, σ_T) the *woof shift* of T and (X_{T^*}, σ_{T^*}) the *warp shift* of T. We also have the onesided subshifts $(\tilde{Z}_T, \tilde{\varsigma}_T)$ and $(\tilde{X}_T, \tilde{\sigma}_T)$ induced by (Z_T, ς_T) and (X_T, σ_T), respectively. We say that T is *nondegenerate* if $(X_T, \sigma_T) = (X_G, \sigma_G)$. We define onto maps $\xi_T : Z_T \to X_T$ and $\eta_T : Z_T \to X_T$ to be the restrictions of ξ and η, respectively. If T is *onesided 1-1*, i.e., ξ_T is 1-1, then an onto endomorphism φ_T of (X_T, σ_T) is defined by

$$\varphi_T = \eta_T \xi_T^{-1}.$$

If T is *1-1*, i.e., both ξ_T and η_T are 1-1, then φ_T is an automorphism of (X_T, σ_T). We also define onto maps $\tilde{\xi}_T : \tilde{Z}_T \to \tilde{X}_T$ and $\tilde{\eta}_T : \tilde{Z}_T \to \tilde{X}_T$ to be the restrictions of $\tilde{\phi}_p$ and $\tilde{\phi}_q$, respectively, and if $\tilde{\xi}_T$ is 1-1, we have an onto endomorphism $\tilde{\varphi_T}$ of $(\tilde{X}_T, \tilde{\sigma_T})$ by $\tilde{\varphi_T} = \tilde{\eta}_T \tilde{\xi_T}^{-1}$. If T is 1-1, then an onto continuous map $\chi_T : X_T \to X_{T^*}$ is defined by

$$\chi_T(\xi_T((\alpha_{1j})_{j\in\mathbb{Z}})) = \xi_{T^*}((\alpha_{i1})_{i\in\mathbb{Z}}), \quad (\alpha_{ij})_{i,j\in\mathbb{Z}} \in U_T, \ \alpha_{ij} \in A_\Gamma.$$

The following gives a typical fundamental relation between a textile system T and its dual T^*.

- ([N2]) Let T be a onesided 1-1 textile system.
  (1) φ_T is expansive if and only if T^* is 1-1.
  (2) If both T and T^* are 1-1, then

$$(X_T, \sigma_T, \varphi_T) \cong (X_{T^*}, \varphi_{T^*}, \sigma_{T^*})$$

  through the conjugacy χ_T.

This implies that if T is a 1-1 textile system with T^* 1-1, then the woof shift (X_T, σ_T) and the warp shift (X_{T^*}, σ_{T^*}) are commuting subshifts.

For a given expansive automorphism φ of a subshift (X, σ), we can easily construct a 1-1 textile system T such that

$$(X, \sigma, \varphi) \cong (X_T, \sigma_T, \varphi_T).$$

The dynamics of φ is given by (X_{T^*}, σ_{T^*}). However it is not easy in general to identify this subshift. If (X_{T^*}, σ_{T^*}) is an SFT, then we can identify this

SFT ([N2]). In particular, if T^* is nondegenerate, i.e., $X_{T^*} = X_{G^*}$, our work is done. Some "resolving" textile systems have this property.

A graph G is said to be *nondegenerate* if both i_G and t_G are onto. A graph-homomorphism $h : \Gamma \to G$ is said to be *onto* if both h_A and h_V are onto. An onto graph-homomorphism $h : \Gamma \to G$ between nondegenerate graphs is said to be *right resolving* if for each $u \in V_\Gamma$, the restriction of h_A on $i_\Gamma^{-1}(\{u\})$ is a bijection onto $i_G^{-1}(\{h_V(u)\})$. It is said to be *left resolving* if for each $u \in V_\Gamma$, the restriction of h_A on $t_\Gamma^{-1}(\{u\})$ is a bijection onto $t_G^{-1}(\{h_V(u)\})$.

A textile system $T = (p, q : \Gamma \to G)$ is said to be *LR* if p is left resolving and q is right resolving, and *LL* if both p and q are left resolving.

If a textile system T is LR, then T is nondegenerate and T^* is LR. If T is LL, then T is nondegenerate and $T^* = (p^*, q^* : \Gamma \to G^*)$ is *q-biresolving*, i.e., q^* is both left resolving and right resolving, but T^* is not generally nondegenerate. Every LR textile system is easily obtained.

- ([N2]) If M and N are nonnegative integral matrices such that (X_M, σ_M) and (X_N, σ_N) are the woof and warp shifts of an LR textile system, then M and N commute. If M and N are commuting nonnegative integral matrices, then an LR textile system whose woof and warp shifts are (X_M, σ_M) and (X_N, σ_N), respectively, is given by a *specified equivalence* between $\tilde{M}\tilde{N}$ and $\tilde{N}\tilde{M}$ (i.e., one-to-one correspondence between the terms of each entry of $\tilde{M}\tilde{N}$ and the terms of the corresponding entry of $\tilde{N}\tilde{M}$) , and vice versa, where $\tilde{M}$ denotes the symbolic representation matrix of M.

3. Automorphisms of SFTs

Our study started from the following result:

- (Boyle and Krieger [BoKr1]) If φ is an automorphism of a topological Markov shift (X, σ), then $(X, \varphi\sigma^k)$ is conjugate to a topological Markov shift for all k greater than $l \geq 0$ such that φ and φ^{-1} are block maps of (l, l)-type.

A *symbolic conjugacy* $\kappa : (X, \sigma) \to (X', \sigma')$ between subshifts is a conjugacy given by just renaming of symbols, i.e., there exists a bijection $k : L_1(X) \to L_1(X')$ such that κ maps $(a_j)_{j\in\mathbb{Z}}$ to $(k(a_j))_{j\in\mathbb{Z}}$.

Let C and D be disjoint alphabets. Let (Z, σ) be a *bipartite subshift with respect to* (C, D), that is, $L_1(Z) = C \cup D$ and for all $(a_j)_{j\in\mathbb{Z}} \in Z$ and for all $j \in \mathbb{Z}$, $a_j a_{j+1} \in CD \cup DC$, where $CD = \{cd \mid c \in C, d \in D\}$. Let (Z_{CD}, σ_{CD}) and (Z_{DC}, σ_{DC}) be the subshifts over the alphabets CD and DC, respectively, defined by

$$Z_{CD} = \{(a_{2j}a_{2j+1})_{j\in\mathbb{Z}} \mid (a_j)_{j\in\mathbb{Z}} \in Z, a_0 \in C\},$$
$$Z_{DC} = \{(a_{2j}a_{2j+1})_{j\in\mathbb{Z}} \mid (a_j)_{j\in\mathbb{Z}} \in Z, a_0 \in D\}.$$

The conjugacies $\zeta^+_{C,D}$ and $\zeta^-_{C,D}$ of (Z_{CD}, σ_{CD}) onto (Z_{DC}, σ_{DC}) defined by

$$\zeta^+_{C,D} : (c_j d_j)_{j\in\mathbb{Z}} \mapsto (d_j c_{j+1})_{j\in\mathbb{Z}}, \quad \zeta^-_{C,D} : (c_j d_j)_{j\in\mathbb{Z}} \mapsto (d_{j-1} c_j)_{j\in\mathbb{Z}}$$

are called the *forward bipartite conjugacy* and *the backward bipartite conjugacy*, respectively, induced by (Z, σ).

Any topological conjugacy ϕ between subshifts has a κ-ζ *factorization*:

$$\phi = \kappa_n \zeta_n \kappa_{n-1} \dots \kappa_1 \zeta_1 \kappa_0,$$

where κ_i is a symbolic conjugacy and ζ_i is a forward or backward bipartite conjugacy([N1]).

An automorphism φ of a subshift (X, σ) is said to be *forward* if it has a κ-ζ factorization with all ζ_i forward; it is said to be *backward* if it has a κ-ζ factorization with all ζ_i backward.

Hence, for any automorphism φ of a subshift (X, σ), $\varphi\sigma^n$ is forward for all sufficiently large n and $\varphi\sigma^{-n}$ is backward for all sufficiently large n.

- ([N2]) If φ is an automorphism of a topological Markov shift (X, σ), then φ is forward if and only if φ is *LR*, i.e., there is a 1-1 LR textile system T with $(X_T, \sigma_T, \varphi_T) = (X, \sigma, \varphi)$.

This implies that every automorphism of a topological Markov shift (X, σ) is obtained by an LR automorphism of (X, σ) composed by σ^{-n} for some integer n.

It also implies that if φ is a forward automorphism of a topological Markov shift (X_M, σ_M), where M is a nonnegative integral matrix, then there exists a nonnegative integral matrix P commuting with M such that $(X, \varphi^k \sigma^l)$ is conjugate to $(X_{P^k M^l}, \sigma_{P^k M^l})$ for all $k \geq 0$ and $l \geq 1$, and if φ is expansive, then (X, φ) is conjugate to (X_P, σ_P). The matrix P is unique for φ and explicitly given as follows ([N2]):

Suppose $\varphi = \kappa_n \zeta_n \kappa_{n-1} \dots \kappa_1 \zeta_1 \kappa_0$ is a forward κ-ζ factorization. If $n \geq 1$ and the κ-ζ factorization induces self "strong shift equivalence" (cf. [W])

$$M = P_1 Q_1, Q_1 P_1 = P_2 Q_2, \dots, Q_{n-1} P_{n-1} = P_n Q_n, Q_n P_n = M,$$

then $P = P_1 \dots P_n$, and if $n = 0$ and κ_0 induces a graph-automorphism $k = (k_A, k_V)$ of the graph G with $M_G = M$, then P is the permutation matrix corresponding to k_V.

Let X be a compact, 0-dimensional metric space. Let $H(X)$ denote the group of homeomorphisms of X onto itself. Let $E(X)$ denote the set of all expansive homeomorphisms in $H(X)$.

Let $\varphi \in H(X)$ and let $\tau \in E(X)$ with $\varphi\tau = \tau\varphi$. Then φ is called an *essentially forward automorphism* of (X, τ) if (X, τ, φ) is conjugate to some commuting system $(X_0, \sigma_0, \varphi_0)$ such that (X_0, σ_0) is a subshift and φ_0 is a forward automorphism of (X_0, σ_0). It is known that φ is an essentially forward automorphism of (X, τ) if and only if (X, τ, φ) is conjugate to some commuting system $(X_0, \sigma_0, \varphi_0)$ such that (X_0, σ_0) is a subshift and φ_0 is an invertible block map of $(0, k)$-type and φ_0^{-1} is a block map of $(l, 0)$-type for some $k, l \geq 0$ ([N3], Section 6).

Furthermore φ is called an *directionally essentially forward automorphism* of (X, τ) if there are positive integers m and n such that φ^m is an essentially forward automorphism of (X, τ^n).

For $\varphi, \tau \in H(X)$, we write $\varphi \rightsquigarrow \tau$ to mean that $\tau \in E(X)$ and φ is a directionally essentially forward automorphism of (X, τ), and write $\varphi \leftrightsquigarrow \tau$ to mean that $\varphi \rightsquigarrow \tau$ and $\tau \rightsquigarrow \varphi$.

The following was stated without proof in [N3].

- Let $\tau, \tau_1, \tau_2, \varphi, \varphi_1, \varphi_2 \in H(X)$.
  (1) If $\tau \in E(X)$, then $\tau \leftrightsquigarrow \tau$.
  (2) If $\tau_1 \in E(X)$ and $\tau_1 \rightsquigarrow \tau_2$, then $\tau_1 \leftrightsquigarrow \tau_2$.
  (3) If $\tau\tau_2 = \tau_2\tau$, then $\tau \rightsquigarrow \tau_1$ and $\tau_1 \rightsquigarrow \tau_2$ imply $\tau \rightsquigarrow \tau_2$.
  (4) If $\tau \in E(X)$ with $\varphi\tau = \tau\varphi$, then there is $m \geq 1$ such that $\varphi\tau^n \rightsquigarrow \tau$ for all $n \geq m$.
  (5) If $\varphi_1 \rightsquigarrow \tau$, $\varphi_2 \rightsquigarrow \tau$ and $\varphi_1\varphi_2 = \varphi_2\varphi_1$ and if there are $m, n \geq 0$ with $\varphi_1^m\varphi_2^n \in E(X)$, then $\varphi_1\varphi_2 \leftrightsquigarrow \tau$.

For $\tau \in E(X)$ and for a commutative subgroup K of $H(X)$ with $K \ni \tau$, we define

$$C_K(\tau) = \{\varphi \in K \mid \varphi \rightsquigarrow \tau\}$$

and call $C_K(\tau)$ the *directionally essentially forward cone containing τ in K*.

A directionally essentially forward cone can contain non-expansive homeomorphisms on the "boundary" of the cone.

The notion of a directionally essentially forward cone is closely related with that of an "expansive component of 1-frames" for a $\mathbb{Z}^d$ action in the theory "expansive subdynamics" of Boyle and Lind [BoL], which extensively studied the dynamics of commuting homeomorphisms of compact metric spaces in a different, more comprehensive framework considering not only "rational directions" but also "irrational directions".

- ([N2],[N3]) If (X, τ) with $\tau \in H(X)$ is conjugate to an SFT and φ is a directionally essentially forward automorphism of (X, τ) (i.e., $\varphi \rightsquigarrow \tau$), then φ is an essentially forward automorphism of (X, τ).

Therefore, a directionally essentially forward cone containing $\tau \in H(X)$ with (X, τ) conjugate to an SFT in a commutative subgroup K of $H(X)$ is called an *essentially forward cone* or *essentially LR cone* containing τ in K. Every expansive homeomorphism in an essentially LR cone is cojugate to an SFT (and every non-expansive homeomorphism has pseudo orbit tracing property). Hence an essentially LR cone is closely related with a "Markov component of 1-frames" of Boyle and Lind [BoL], on which we will mention a few from [BoL] in Section 7.

We will continue treating the topic above in Section 5.

Now we state known results on the problem: "Is a subshift commuting with an SFT is also an SFT?"

• (D. Fiebig [F]) There is an expansive automorphism φ of a non-transitive SFT (X, σ) such that (X, φ) is not conjugate to an SFT.

Hence now our problem is:

Open problem: Is a subshift commuting with a transitive SFT also an SFT?

A *sofic shift* is defined to be a subshift which is the image of an SFT under a block map. A block map between subshifts is said to be *left closing* if it never collapses distinct forwardly asymptotic points ; it is *right closing* if it never collapses distinct backwardly asymptotic points; it is *biclosing* if it is left closing and right closing. A sofic shift is said to be *almost Markov* if it is the image of an SFT under a biclosing block map [BoKr2].

Recently Mike Boyle has given the following result together with important fundamental results in [Bo2] (this paper includes the examples of D. Fiebig showing the above as Appendix).

• (Boyle [Bo2]) A strictly sofic almost Markov shift cannot commute with a nonwandering SFT.

4. Endomorphisms of onesided SFTs

• ([N2]) Suppose $\tilde{\varphi}$ is an endomorphism of a onesided SFT $(\tilde{X}, \tilde{\sigma})$ with $(\tilde{X}, \tilde{\varphi})$ conjugate to a onesided SFT. Then there exists a onesided 1-1, LR textile system T with T^* onesided 1-1 such that $(\tilde{X}_T, \tilde{\sigma}_T, \tilde{\varphi}_T) \cong (\tilde{X}, \tilde{\sigma}, \tilde{\varphi})$, and hence there are commuting nonnegative integral matrices M and N such that for all $k, l \geq 0, (k, l) \neq (0, 0)$, $(\tilde{X}, \tilde{\varphi}^k \tilde{\sigma}^l)$ is conjugate to the onesided topological Markov shift $(\tilde{X}_{N^k M^l}, \tilde{\sigma}_{N^k M^l})$.

• (Kůrka [Ku]) If $\tilde{\varphi}$ is a positively expansive endomorphism of a transitive onesided SFT $(\tilde{X}, \tilde{\sigma})$, then $(\tilde{X}, \tilde{\varphi})$ is conjugate to a onesided SFT.

(This was independently proved by Nasu under the additional condition that $\tilde{\varphi}$ is onto, by using textile systems; see [N3]. See [BoFF] and Section 6 of [BoKi] for Kůrka's proof.)

Boyle, Fiebig and Fiebig [BoFF] gave an example of a positively expansive onto endomorphism $\tilde{\varphi}$ of a non-transitive onesided SFT $(\tilde{X}, \tilde{\sigma})$ such that $(\tilde{X}, \tilde{\varphi})$ is not conjugate to a onesided SFT.

Two dynamical systems (X_1, τ_1) and (X_2, τ_2) are said to be *eventually conjugate* if (X_1, τ_1^n) and (X_2, τ_2^n) are conjugate for all sufficiently large n.

• (Blanchard and Maass [BlMaa]) If $\tilde{\varphi}$ is a positively expansive endomorphism of the onesided full N-shift with $N \geq 1$, then there exists $J \geq 1$ such that

(1) $(\tilde{X}, \tilde{\varphi})$ is eventually conjugate to the onesided full J-shift, and

(2) J and N are divisible by the same primes.

The result (1) above is best in the sense that a positively expansive endomorphism of a onesided full shift need not be conjugate to a onesided full shift (Boyle, Fiebig and Fiebig [BoFF]).

The following generalizes the result (1) above.

• (Boyle, Fiebig and Fiebig [BoFF]) Let M and N be nonnegative, integral $m \times m$ and $n \times n$ matrices, respectively. If the onesided topological Markov shifts $(\tilde{X}_M, \tilde{\sigma}_M)$ and $(\tilde{X}_N, \tilde{\sigma}_N)$ commute, then M^m and N^n have the same number of distinct columns.

The following is a generalization of (2) of Blanchard-Maass's result.

• Let M and N be irreducible and aperiodic, nonnegative, integral matrices and let λ_M and λ_N be their spectral radii (Perron eigenvalues), respectively. If $(\tilde{X}_M, \tilde{\sigma}_M)$ and $(\tilde{X}_N, \tilde{\sigma}_N)$ commute, then λ_M and λ_N are algebraic integers which generate the same algebraic number field, and in the ring of algebraic integers in this field, the principal ideals (λ_M) and (λ_N) are divisible by the same prime ideals. In particular, if either one of λ_M and λ_N is a rational integer, then both are rational integers which are divisible by the same rational primes.

• (Boyle, Fiebig and Fiebig [BoFF]) If $\tilde{\varphi}$ is a positively expansive endomorphism of a mixing onesided SFT $(\tilde{X}, \tilde{\sigma})$, then $\tilde{\varphi}$ and $\tilde{\sigma}$ have the same measure of maximal entropy.

5. Automorphisms of SFTs, again

Since the properties of positively expansive endomorphism of onesided SFTs stated above can come from the properties of LR textile systems, similar results are also obtained for essentially LR or essentially forward, expansive automorphisms of SFTs.

By the result ([N2], Proposition 8.8) that all sufficiently large powers of an essentially LR automorphism of a topological Markov shift are LR automorphisms of the shift, or by Theorem 8.6 of [BoL], we have:

• An expansive essentially forward automorphism of a full shift is eventually conjugate to a full shift.

It is not known whether in this result "eventually conjugate" can be replaced by "conjugate" or not. Is an SFT which is eventually conjugate to a full shift conjugate to the full shift? This is a long-standing open problem in symbolic dynamics.

• Let φ be an expansive essentially forward automorphism of a mixing SFT (X, σ). Then $e^{h(\sigma)}$ and $e^{h(\varphi)}$ are algebraic integers which generate the same algebraic number field, and in the ring of algebraic integers in this field, the principal ideals $(e^{h(\sigma)})$ and $(e^{h(\varphi)})$ are divisible by the same prime ideals, where $h(\cdot)$ denotes topological entropy. In particular, if either one of $e^{h(\sigma)}$ and $e^{h(\varphi)}$ is a rational integer, then both of them are rational integers which are divisible by the same rational primes.

• ([N2], Section 10) Let M and N be matrices given by

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then the topological Markov shifts (X_M, σ_M) and (X_N, σ_N) commute, but the spectral radii λ_M and λ_N of M and N generate different algebraic number fields. (The characteristic polynomials of M and N are $x^2 - 3x + 1$ and $(x+1)^2(x^3 - 2x^2 + x - 1)$, respectively.)

As far as the author knows, no example of an expansive automorphism of a full shift which is not conjugate to a full shift has been given.

Open Problem: Let φ be an automorphism of a mixing topological Markov shift (X, σ). Let K be the subgroup of $H(X)$ generated by σ and φ. Determine all the directionally essentially forward cones in K.

This is closely related with several open problems presented by Boyle and Lind in their more general framework in Section 9 of [BoL].

6. Endomorphisms of SFTs and automorphisms of onesided SFTs

By using LL textile systems, the following was obtained:

• ([N2]) If (X, τ, φ) is a commuting system such that (X, τ) is cojugate to a subshift with τ^n transitive for all $n \geq 1$ and (X, φ) is conjugate to a onesided SFT, then (X, φ) is conjugate to a onesided full shift.

• (Kůrka [Ku]) If (X, τ, φ) is a commuting system with (X, τ) conjugate to a transitive SFT and φ is positively expansive, then (X, φ) is conjugate to a onesided SFT.

(This was independently proved by Nasu under the additional condition that φ is onto, by using textile systems; see [N3].)

By the two results above we know:

• ([N3]) A onesided subshift commuting with a mixing SFT is conjugate to a onesided full shift.

• (D. Fiebig [F]) Kůrka's result above can be extended to nonwandering SFTs and any SFT having a wandering point does not have a positively expansive endomorphism.

Using dimension groups arising in symbolic dynamics, Boyle and Maass proved:

• (Boyle and Maass [BoMaa]) Let $N \geq 1$. An SFT commuting with the onesided full N-shift is eventually conjugate to some full J-shift such that J and N are divisible by the same primes.

Together with this, Boyle and Maass [BoMaa] gave three conjectures, two of which have been proved by using q-biresolving textile systems.

• ([N5]) The first conjecture of Boyle and Maass is true, that is, a subshift commuting with a onesided full shift is an SFT.

• ([N6]) The third conjecture of Boyle and Maass is true, that is, if a positively expansive endomorphism φ of a mixing SFT is N-to-one, then the "left and right multipliers" l_φ and r_φ (developed by Boyle [Bo1]) are rational integers such that all N, l_φ and r_φ are divisible by the same primes, and in particular, if a prime p divides N, then p^2 divides N.

Boyle and Maass [BoMaa] also conjectured the following together with proving the sufficiency of the condition in it:

• ([BoMaa] and [N6]) For $J, N \geq 1$, the following are a necessary and sufficient condition for the existence of the full J-shift and the onesided full N-shift which commute:
(1) J and N are divisible by the same primes, and
(2) if a prime p divides N, then p^2 divides N.

The second conjecture of Boyle and Maass has not been settled:

Conjecture (Boyle and Maass [BoMaa]): An SFT commuting with a onesided full shift is conjugate to a full shift.

As was stated above, the result of Boyle and Maass above was proved by using dimension groups. In particular, Boyle and Maass proved the part that an SFT commuting with a onesided full shift is eventually conjugate to a full shift, using "bilateral dimension groups" introduced by Krieger [Kr]. No proof other than their proof has not been known for this part, though the other part can be recovered by a proof which does not use dimension groups ([N6]).

• (Boyle and Maass [BoMaa]) If (X, τ, φ) is a commuting system with (X, τ) conjugate to an SFT and (X, φ) conjugate to a onesided full shift, then τ and φ have the same measure of maximal entropy.

7. Commuting systems on compact metric spaces

Similar problems to those treated in the preceding sections can be considered for commuting continuous maps of compact metric spaces which are not necessarily 0-dimensional, as were done in [BoL] and [N4]. In this section, we mention two results from Section 8 of [BoL] without any explanation of the notion and terminology, and mention how much the problems corresponding to the kernels of our problems stated in Introduction have been solved.

- (Boyle and Lind [BoL]) If a vector in an expansive component of 1-frames for a $\mathbb{Z}^d$ action is Markov, so are all the vectors in the component.

- (Boyle and Lind [BoL]) For any two integral vectors in a mixing Markov component of 1-frames, the homeomorphisms corresponding to the vectors have the same unique measure of maximal entropy.

As was stated in Section 3, the following problem is open even for the 0-dimensional case.

Open problem (cf. Problem 9.6 of [BoL]): Suppose (X, τ, φ) is a commuting system such that τ and φ are expansive homeomorphisms. If τ is transitive and has pseudo orbit tracing property (POTP), then does φ have POTP?

- ([N4]) Suppose (X, τ, φ) is a commuting system such that τ is an expansive homeomorphism and φ is positively expansive and onto. If τ is transitive and has POTP, then φ has POTP.

As is seen by the first and sixth results stated in Section 6, the following problem was solved affirmatively for the 0-dimensional case under the condition that φ^n is transitive for all $n \geq 1$.

Open problem: Suppose (X, τ, φ) is a commuting system such that τ is positively expansive and onto and φ is an expansive homeomorphism. If τ has POTP, then does φ have POTP?

- ([N4]) Suppose (X, τ, φ) is a commuting system with τ and φ positively expansive and onto. If τ is transitive and has POTP, then φ has POTP.

An onto continuous map $\varphi : X \to X$ of a compact metric space is *expansive* if there is $\delta > 0$ such that if $(x_i)_{i \in \mathbb{Z}}$ and $(x'_i)_{i \in \mathbb{Z}}$ are orbits of φ and the distance between x_i and x'_i is less than δ for all $i \in \mathbb{Z}$, then $(x_i)_{i \in \mathbb{Z}} = (x'_i)_{i \in \mathbb{Z}}$.

Finally we mention a result which unifies and generalizes some known basic results for the 0-dimensional case.

- ([N4]) Let (X, τ, φ) be a commuting systems with τ and φ onto. If τ is expansive and has POTP and φ is mixing, then τ is mixing.

Hence in particular, if one of commuting two "subshifts" each of which is an SFT or a onesided SFT is mixing, then so is the other, as three of the four special cases of this were proved in [BoKr1], [N2] and [BoFF], separately.

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