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Removable singularities for solutions to k-curvature equations

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1. INTRODUCTION

We are concerned with the removability of singular sets of solutions to the so-called curvature equations of the form

$$(1.1)_k \qquad H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = \psi \quad \text{in } \Omega \setminus K,$$

where Ω is a bounded domain in $\mathbb{R}^n$ and K is a compact set contained in Ω. Here, for a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of the graph of the function u, namely, the eigenvalues of the matrix

$$(1.2) \quad \mathcal{C} = D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right) D^2u,$$

and $S_k, k = 1, \ldots, n$, denotes the k-th elementary symmetric function, that is,

$$(1.3) \qquad S_k(\kappa) = \sum \kappa_{i_1} \cdots \kappa_{i_k},$$

where the sum is taken over increasing k-tuples, $i_1, \ldots, i_k \subset \{1, \ldots, n\}$. The family of equations $(1.1)_k$, $k = 1, \ldots, n$ contains some well-known and important equations in geometry and physics.

The case $k = 1$ corresponds to the **mean curvature equation**;

The case $k = 2$ corresponds to the **scalar curvature equation**;

The case $k = n$ corresponds to **Gauss curvature equation**.

In this article, we call the equation $(1.1)_k$ "k-curvature equation."

The classical Dirichlet problem, in which the inhomogeneous term ψ in $(1.1)_k$ is a smooth function, has been studied in Caffarelli, Nirenberg and Spruck [4], and Ivochkina [9]. Trudinger [21] established the existence and uniqueness of Lipschitz solutions of the Dirichlet problem in the viscosity sense, under natural geometric restrictions and under relatively weak regularity hypotheses on ψ, for instance, $\psi^{\frac{1}{k}} \in C^{0,1}(\overline{\Omega})$.

Let us consider the following problem.

Problem: Is it always possible to extend a "solution" to $(1.1)_k$ as a "solution" to $H_k[u] = \psi$ in the *whole* domain Ω?

For the case of $k = 1$, such removability problems were extensively studied. Bers [1], Nitsche [15] and De Giorgi-Stampacchia [8] proved the removability of isolated singularities for solutions to the equation of minimal surface ($\psi \equiv 0$) or constant mean curvature (ψ is a constant function). Serrin [16, 17] studied the same problem for a more general class of quasilinear equations of mean curvature type. He proved that any weak solution u to the mean curvature type equation in $\Omega \setminus K$ can be extended to a weak solution in Ω if the singular set K is a compact set of vanishing $(n-1)$-dimensional Hausdorff measure. For various semilinear and quasilinear equations, there are a number of papers concerning removability results.

We remark here that $(1.1)_k$ is a quasilinear equation for $k = 1$ while it is a *fully nonlinear* equation for $k \geq 2$. It is much harder to study the fully nonlinear equations' case. For Monge-Ampère equations' case, there are some results about the removability of isolated singularities (see, for example, [2, 10]). However, until recently, no results are known for other types of fully nonlinear elliptic PDEs except for the recent work of Labutin [11, 12, 13] who have studied for the case of uniformly elliptic equations and Hessian equations.

We note that there exist solutions to $(1.1)_n$ with non-removable singularity at a single point. For example,

$$u(x) = \alpha|x|, \quad x \in \Omega = B_1(0) = \{|x| < 1\} \tag{1.4}$$

where $\alpha > 0$, satisfies the equation $(1.1)_n$ with $\psi \equiv 0$ and $K = \{0\}$, in the classical sense as well as in the viscosity and generalized sense (the notion of generalized solutions is stated below). However, u does not satisfy $H_n[u] = 0$ in $\Omega = B_1(0)$ (see Example 3.1 (1)). Accordingly, it is sufficient to discuss our **Problem** for $1 \leq k \leq n-1$.

We state our main results in this article.

(1) Removability of the isolated singularity of *viscosity solutions* to $(1.1)_k$. (Section 2)

First we consider the simplest case that K is a single point. Moreover, we consider our **Problem** in the framework of the theory of viscosity solutions. We shall prove that for $1 \leq k \leq n-1$, isolated singularities are always removable under the convexity assumption on the solution.

(2) To introduce a concept of *generalized solutions* to k-curvature equations. (Section 3)

There is a notion of generalized solutions to Gauss curvature equation ($k = n$) when the inhomogeneous term ψ is a Borel measure, since it belongs to a class of Monge-Ampère type. We introduce a concept of generalized solutions to other k-curvature equations. We shall prove that if k-curvature equation has a convex solution, then ψ must be a Borel measure.

(3) Removability of singular sets of *generalized solutions* to $(1.1)_k$. (Section 4)

Thanks to the integral nature of generalized solutions, we can examine the removability of singular sets as well as a singular point. We shall prove that if for $1 \le k \le n-1$, any generalized solutions to $(1.1)_k$ can be extended as a generalized solutions to $H_k[u] = \psi$ in Ω, provided the removable set K has the vanishing $(n-k)$-dimensional Hausdorff measure. This is a Serrin type removability result for $(1.1)_k$.

Remark 1.1. It is well known that minimal surfaces are characterized as critical points of the area functional. Indeed, the variational derivative of the functional $I_1(u) = \int_\Omega \sqrt{1+|Du|^2}\,dx$ is

(1.5)
$$\frac{\delta}{\delta u} I_1(u) = -H_1[u]. \quad (H_1[u] = \text{mean curvature of the graph of } u)$$

The following proposition says that other k-curvature equations also have a variational nature. The proof is given in [9].

Proposition 1.1. *Let $u \in C^2(\overline{\Omega})$ be a solution to $H_k[u] = \psi(x,u)$ in Ω. Then u is a critical point of the functional*

$$I_k(u) = \int_\Omega \left(\sqrt{1+|Du|^2} H_{k-1}[u] + k\Psi(x,u)\right) dx, \tag{1.6}$$

where $\frac{\partial}{\partial u}\Psi(x,u) = \psi(x,u)$.

2. Removability result in the class of viscosity solutions

In the first part of this section, we define the notion of viscosity solutions to the equation

$$H_k[u] = \psi(x) \quad \text{in } \Omega, \tag{2.1$_k$}$$

where Ω is an arbitrary open set in $\mathbb{R}^n$ and $\psi \in C^0(\Omega)$ is a non-negative function. The theory of viscosity solutions to the first order equations and the second order ones was developed in the 1980's by Crandall, Evans, Ishii, Lions and others. See, for example, [5, 6, 7, 14].

We define the admissible set of k-th elementary symmetric function S_k by

$$\begin{aligned}(2.2)\qquad \Gamma_k &= \{\lambda \in \mathbb{R}^n \mid S_k(\lambda+\mu) \geq S_k(\lambda) \quad \text{for all } \mu_i \geq 0\} \\ &= \{\lambda \in \mathbb{R}^n \mid S_j(\lambda) \geq 0, \quad j = 1, \ldots, k\}.\end{aligned}$$

Let Ω be an open set in $\mathbb{R}^n$. We say that a function $u \in C^2(\Omega)$ is *k-admissible* if $\kappa = (\kappa_1, \ldots, \kappa_n)$ belongs to Γ_k for every point $x \in \Omega$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of the graph of u at x.

Proposition 2.1. *Let $1 \leq k \leq n$ and $u \in C^2(\Omega)$.*

(i) Γ_k is a cone in $\mathbb{R}^n$ with vertex at the origin, and

$$(2.3)\quad \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \Gamma_+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i \geq 0, \quad i = 1, \ldots, n\}.$$

(ii) u is n-admissible if and only if u is (locally) convex in Ω.

(iii) The operator H_k is degenerate elliptic for k-admissible functions.

Proof. (i) is obvious and (ii) can be readily proved from (i). For the proof of (iii), see [3, 4]. □

Now we define a viscosity solution to $(2.1)_k$. A function $u \in C^0(\Omega)$ is said to be a *viscosity subsolution* (resp. *viscosity supersolution*) to $(2.1)_k$ if for any k-admissible function $\varphi \in C^2(\Omega)$ and any point $x_0 \in \Omega$ which is a maximum (resp. minimum) point of $u - \varphi$, we have

$$(2.4)\qquad H_k[\varphi](x_0) \geq \psi(x_0) \quad (\text{resp. } \leq \psi(x_0)).$$

A function u is said to be a *viscosity solution* to $(2.1)_k$ if it is both a viscosity subsolution and supersolution. One can prove that a function $u \in C^2(\Omega)$ is a viscosity solution to $(2.1)_k$ if and only if it is a k-admissible classical solution. Therefore, the notion of viscosity solutions is weaker than that of classical solutions.

The following theorems are comparison principles for viscosity solutions to $(2.1)_k$. Both of them are important materials for the proof of our removability result in this section.

Theorem 2.2. [21] *Let Ω be a bounded domain. Let ψ be a non-negative continuous function in $\overline{\Omega}$ and u, v be $C^0(\overline{\Omega})$ functions satisfying $H_k[u] \geq \psi + \delta$, $H_k[v] \leq \psi$ in Ω in the viscosity sense, for some positive constant δ. Then*

$$(2.5)\qquad \sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+.$$

Proposition 2.3. [20] *Let Ω be a bounded domain. Let ψ be a non-negative continuous function in $\overline{\Omega}$, $u \in C^0(\overline{\Omega})$ be a viscosity subsolution to $H_k[u] = \psi$, and $v \in C^2(\overline{\Omega})$ satisfying*

$$(2.6)\qquad \kappa[v(x)] \notin \{\lambda \in \Gamma_k \mid S_k(\lambda) \geq \psi(x)\}$$

for all $x \in \Omega$, where $\kappa[v(x)]$ denotes the principal curvatures of v at x. Then (2.5) holds.

We state a removability result for viscosity solutions to $(1.1)_k$.

Theorem 2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin, $K = \{0\}$ and $\psi \in C^0(\Omega)$ be a non-negative function in Ω. Let $1 \le k \le n-1$ and $u \in C^0(\Omega \setminus \{0\})$ be a viscosity solution to $(1.1)_k$. We assume that u can be extended to the continuous function $\tilde{u} \in C^0(\Omega)$. Then $\tilde{u}$ is a viscosity solution to $H_k[\tilde{u}] = \psi$ in Ω. Consequently, $\tilde{u} \in C^{0,1}(\Omega)$.*

The last part of Theorem 2.4 is a consequence of [21]. Note that one cannot expect much better regularity for a viscosity solution in general. In fact, let $k \ge 2$ and A be a positive constant. $u(x) = A\sqrt{x_1^2 + \cdots + x_{k-1}^2}$, where $x = (x_1, \ldots, x_n)$, satisfies $H_k[u] = 0$ in the viscosity sense, but is only Lipschitz continuous. Moreover, Urbas [22] proved that for any positive continuous function ψ, there exist an $\varepsilon > 0$ and a viscosity solution to $H_k[u] = \psi$ in $B_\varepsilon(0) = \{|x| < \varepsilon\}$ which does not belong to $C^{1,\alpha}(B_\varepsilon(0))$ for any $\alpha > 1 - \dfrac{2}{k}$.

Sketch of the proof. We denote u as the extended function $\tilde{u}$ in Ω. We divide the proof into two steps.

Step 1. (To control the behavior of the solution in the neighborhood of the origin)

We prove the following lemma.

Lemma 2.5. *Let $l(x) = u(0) + \sum_{i=1}^{n} \beta_i x_i$, where $\beta_1, \ldots, \beta_n \in \mathbb{R}$. Then there exist sequences $\{z_j\}, \{\tilde{z}_j\} \subset \Omega \setminus \{0\}$ such that $z_j, \tilde{z}_j \to 0$ as $j \to \infty$ and*

$$\liminf_{j\to\infty} \frac{u(z_j) - l(z_j)}{|z_j|} \le 0, \tag{2.7}$$

$$\limsup_{j\to\infty} \frac{u(\tilde{z}_j) - l(\tilde{z}_j)}{|\tilde{z}_j|} \ge 0. \tag{2.8}$$

To prove this, we construct appropriate subsolutions and supersolutions, and use comparison principles (Theorem 2.2 and Proposition 2.3). We only sketch the proof of the existence of $\{z_j\}$ satisfying (2.7). To the contrary, we suppose that there exists an affine function $l(x) = u(0) + \sum_{i=1}^{n} \beta_i x_i$ such that

$$u(x) > l(x) + m|x| \quad \text{for } x \in B_\rho(0) \setminus \{0\}, \tag{2.9}$$

for some $m, \rho > 0$. Rotating the coordinate system in $\mathbb{R}^{n+1}$ if necessary, we may assume that $Dl(x) = 0$, that is, $l(x) \equiv u(0)$. In the case $k \leq n/2$, We consider the auxiliary function w_ε in $\mathbb{R}^n \setminus B_\varepsilon(0)$ for fixed $\varepsilon > 0$ as follows:

$$w_\varepsilon(x) = u(0) + C_1 + C_2|x|^2 + C_3(\varepsilon) f_\varepsilon(x), \tag{2.10}$$

where $C_1, C_2, C_3(\varepsilon)$ are appropriate positive constants and

$$f_\varepsilon(x) = \int_{r_0}^{|x|} \frac{ds}{\sqrt{\left(\frac{M}{\binom{n}{k}} s^k + \left(\frac{s}{\varepsilon}\right)^{k-n}\right)^{-\frac{2}{k}} - 1}}, \tag{2.11}$$

is a radially symmetric solution to $H_k[u] = M = \sup_{B_\rho(0)} \psi$ and $r_0 \in (0, \rho)$ is also an appropriate constant. (In the case $k > n/2$, we have to modify the auxiliary function w_ε. See [20] for detail.) By direct calculations, one can see that

- w_ε is k-admissible and $H_k[w_\varepsilon] \geq \psi + \delta$ in $B_\rho(0) \setminus B_{2\varepsilon}(0)$ for some posive constant δ.
- $w_\varepsilon < u$ on $\partial B_{2\varepsilon}(0) \cup \partial B_{r_0}(0)$.

From the comparison principle, we obtain $w_\varepsilon \leq u$ in $\overline{B_{r_0}(0)} \setminus B_{2\varepsilon}(0)$. Now we fix $x \in B_{r_0}(0) \setminus \{0\}$, it follows that

$$u(x) \geq w_\varepsilon(x) \geq u(0) + C_1 + C_3(\varepsilon) f_\varepsilon(x). \tag{2.12}$$

We can also show that $\liminf_{\varepsilon \to 0} C_3(\varepsilon) f_\varepsilon(x) = 0$, also by direct calculations. As ε tends to 0 in (2.12), we obtain

$$u \geq u(0) + C_1 \quad \text{in } B_{r_0}(0) \setminus \{0\} \tag{2.13}$$

which contradicts the continuity of u at 0.

Step 2. (To prove that u is a viscosity solution to $H_k[u] = \psi$ in Ω)

To show that u is a viscosity subsolution to $H_k[u] = \psi$ in Ω, it is sufficient to prove that $H_k[P] \geq \psi(0)$ for any k-admissible quadratic polynomial P which touches u at the origin from above (supersolution case is similar). First we fix $\delta > 0$ and set $P_\delta(x) = P(x) + \delta|x|^2/2$. Then $P_\delta(x)$ satisfies the following properties:

$$P_\delta(0) = u(0), \quad P_\delta > u \quad \text{in } B_{r_0}(0) \setminus \{0\} \quad \text{for some } r_0 > 0. \tag{2.14}$$

Next there exists $\varepsilon = \varepsilon(\delta) > 0$ and $\tilde{\rho} = \tilde{\rho}(\delta) > 0$ such that $P_{\delta,\varepsilon}(x) = P_\delta(x) - \varepsilon(x_1 + \cdots + x_n)$ satisfies

$$P_{\delta,\varepsilon}(0) = u(0), \quad u < P_{\delta,\varepsilon} \quad \text{in } B_{r_0}(0) \setminus B_{\tilde{\rho}}(0). \tag{2.15}$$

where $\varepsilon(\delta) \to 0$ and $\tilde{\rho}(\delta) \to 0$ as $\delta \to 0$. Now we apply Lemma 2.5 for $l(x) = \langle DP_\delta(0), x\rangle + P_\delta(0)$. Passing to a subsequence if necessary, there exists a sequence $\{z_j\}$, $z_j \to 0$ as $j \to \infty$ such that all coordinates of every z_j are non-negative, and

$$u(z_j) - P_{\delta,\varepsilon}(z_j) > 0 \tag{2.16}$$

for any sufficiently large j. Thus there exists a point $x^\delta \in B_{r_0}(0) \setminus \{0\}$ such that

$$u(x^\delta) - P_{\delta,\varepsilon}(x^\varepsilon) = \max_{B_{r_0}(0)}(u - P_{\delta,\varepsilon}) > 0. \tag{2.17}$$

We notice that $x^\delta \in B_{\tilde{\rho}}(0)$ from (2.15) which implies that $x^\delta \to 0$ as $\delta \to 0$. We introduce the polynomial

$$Q_{\delta,\varepsilon}(x) = P_{\delta,\varepsilon}(x) + u(x^\delta) - P_{\delta,\varepsilon}(x^\delta). \tag{2.18}$$

From (2.15), (2.17), we see that $Q_{\delta,\varepsilon}$ touches u at $x^\delta \neq 0$ from above. Since u is a subsolution to $(1.1)_k$ in $\Omega \setminus \{0\}$, we deduce that

$$\psi(x^\delta) \leq H_k[Q_{\delta,\varepsilon}] = H_k\left[P + \frac{\delta}{2}|x|^2 - \varepsilon(x_1 + \cdots + x_n)\right]. \tag{2.19}$$

Finally, as $\delta \to 0$, we conclude that $H_k[P] \geq \psi(0)$ holds. □

3. The notion of generalized solutions

In this section we give the definition of generalized solutions to k-curvature equations, which is introduced by the author [18].

We state some notations which we shall use. Let Ω be an open, convex and bounded subset of $\mathbb{R}^n$ and we look for solutions in the class of convex and (uniformly) Lipschitz functions defined in Ω. For a point $x \in \Omega$, let $\mathrm{Nor}(u; x)$ be the set of downward normal unit vectors to u at $(x, u(x))$. For a non-negative number ρ and a Borel subset η of Ω, we set

$$Q_\rho(u; \eta) = \{z \in \mathbb{R}^n \mid z = x + \rho v,\ x \in \eta,\ v \in \gamma_u(x)\}, \tag{3.1}$$

where $\gamma_u(x)$ is a subset of $\mathbb{R}^n$ defined by

$$\gamma_u(x) = \{(a_1, \ldots, a_n) \mid (a_1, \ldots, a_n, a_{n+1}) \in \mathrm{Nor}(u; x)\}. \tag{3.2}$$

The following theorem, which is an analogue of the so-called Steiner type formula, plays an important part in the definition of generalized solutions.

Theorem 3.1. ([18, Theorem 1.1]) *Let Ω be an open convex bounded set in $\mathbb{R}^n$, and let u be a convex and Lipschitz function defined in Ω. Then the following hold.*

(i) For every Borel subset η of Ω and for every $\rho \geq 0$, the set $Q_\rho(u; \eta)$ is Lebesgue measurable.

(ii) There exist $n+1$ non-negative, finite Borel measures $\sigma_0(u;\cdot),\ldots,\sigma_n(u;\cdot)$ such that

$$\mathcal{L}^n(Q_\rho(u;\eta)) = \sum_{m=0}^{n}\binom{n}{m}\sigma_m(u;\eta)\rho^m \tag{3.3}$$

for every $\rho \geq 0$ and for every Borel subset η of Ω, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

Remark 3.1. The measures $\sigma_k(u;\cdot)$ determined by u are characterized by the following two properties.

(i) If $u \in C^2(\Omega)$, then for every Borel subset η of Ω,

$$\binom{n}{k}\sigma_k(u;\eta) = \int_\eta H_k[u](x)\,dx. \tag{3.4}$$

(The proof is given in [18, Proposition 2.1].)

(ii) If u_i converges uniformly to u on every compact subset of Ω, then

$$\sigma_k(u_i;\cdot) \rightharpoonup \sigma_k(u;\cdot) \quad \text{(weakly)} \tag{3.5}$$

Therefore we can say that for $k = 1,\ldots,n$, the measure $\binom{n}{k}\sigma_k(u;\cdot)$ generalizes the integral of the function $H_k[u]$.

Now we state the definition of a generalized solution to k-curvature equation.

Definition 3.2. Let Ω be an open convex bounded set in $\mathbb{R}^n$ and ν be a non-negative finite Borel measure on Ω. A convex and Lipschitz function $u \in C^{0,1}(\Omega)$ is said to be a *generalized solution* to

$$H_k[u] = \nu \quad \text{in } \Omega, \tag{3.6}$$

if it holds that

$$\binom{n}{k}\sigma_k(u;\eta) = \nu(\eta) \tag{3.7}$$

for every Borel subset η of Ω.

We note that one can also define the notion of generalized solutions stated above when Ω is merely an open set, not necessarily convex and u is a locally convex function in Ω. Indeed, we shall say that u is a generalized solution to (3.6) if for any point $x \in \Omega$ and for any ball $B = B_R(x) \subset \Omega$, (3.7) holds for every Borel subset η of $B_R(x)$.

Here are some examples of generalized solutions.

Example 3.1. Let $B_1(0)$ be a unit ball in $\mathbb{R}^n$ and α be a positive constant.

(1) Let $u_1(x) = \alpha|x|$. One can easily see that u_1 is a classical solution to $H_n[u_1] = 0$ in $B_1(0)\setminus\{0\}$, but that u_1 is not a solution to $H_n[u_1] = 0$

in $B_1(0)$ in the classical sense nor viscosity sense. However, u_1 is a generalized solution to

$$H_n[u_1] = \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^n \omega_n \delta_0 \quad \text{in } B_1(0), \tag{3.8}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and δ_0 is the Dirac measure at 0.

(2) Let $u_2(x) = \alpha\sqrt{x_1^2 + \cdots + x_k^2}$, where $x = (x_1, \ldots, x_n)$. One can see that u_2 cannot be a viscosity solution to $H_k[u_2] = \psi$ in $B_1(0)$ for any $\psi \in C^0(B_1(0))$. However, u_2 is a generalized solution to

$$H_k[u_2] = \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^k \omega_k \mathcal{L}^{n-k} \lfloor T \quad \text{in } B_1(0), \tag{3.9}$$

where ω_k denotes the k-dimensional measure of the unit ball in $\mathbb{R}^k$ and $T = \{(x_1, \ldots, x_n) \in B_1(0) \,|\, x_1 = \cdots = x_k = 0\}$.

We state some properties of generalized solutions to (3.6) defined above. Here we note that for $k = n$ which corresponds to Gauss curvature equation, there is a notion of generalized solutions, since they are in a class of Monge-Ampère type.

Proposition 3.3. *Let Ω be an open convex bounded set in $\mathbb{R}^n$, ν be a non-negative finite Borel measure on Ω and u be a locally convex function in Ω.*

(i) If $u \in C^2(\Omega)$ is a generalized solution to (3.6), then u is a classical solution to $H_k[u] = \psi$ for some $\psi \in C^0(\Omega)$ and $\nu = \psi(x)\,dx$.

(ii) For $k = n$, the definition of generalized solutions for Monge-Ampère type equations coincides with the one introduced in Definition 3.2.

(iii) Let $1 \le k \le n$ and ψ be a positive function with $\psi^{1/k} \in C^{0,1}(\overline{\Omega})$. If u is a viscosity solution to $H_k[u] = \psi$ in Ω, then u is a generalized solution to $H_k[u] = \nu$ in Ω, where $\nu = \psi(x)\,dx$. Therefore, we can say that the notion of generalized solutions is weaker than that of viscosity solutions under convexity assumptions.

Proof. (i) can be proved by the standard argument. The proof of (ii) is given in [18, Theorem 3.3]. (iii) is proved in [19]. □

4. Removability result in the class of generalized solutions

We establish results concerning the removability of a singular set of a generalized solution to k-curvature equation. We present our result — Serrin type removability result.

Theorem 4.1. *Let Ω be a convex domain in $\mathbb{R}^n$ and $K \Subset \Omega$ be a compact set whose $(n-k)$-dimensional Hausdorff measure is zero. Let $1 \le k \le n-1$, $\psi \in L^1(\Omega)$ be a non-negative function, and u be a continuous function in $\Omega \setminus K$. We assume that u is a locally convex function in Ω and a generalized solution to $H_k[u] = \psi\, dx$ in $\Omega \setminus K$. Then u can be defined in the whole domain Ω as a generalized solution to $H_k[u] = \psi\, dx$ in Ω.*

Before giving a proof of Theorem 4.1, we introduce some notations. We write $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$. $B_r^{n-1}(x') \subset \mathbb{R}^{n-1}$ denotes the $(n-1)$-dimensional open ball of radius r centered at x'.

Proof. The proof is split into two steps.

Step 1. (Extension of u to a convex function in Ω)

Here we prove that u can be extended to a convex function in the whole domain Ω. The idea of the proof is adapted from that of Yan [23].

Let y, z be any two distinct points in $\Omega \setminus K$. Without loss of generality we may assume that y is the origin and $z = (0, \dots, 0, 1)$. First we prove the following lemma.

Lemma 4.2. *There exist sequences $\{y_j\}_{j=1}^{\infty}, \{z_j\}_{j=1}^{\infty} \subset \Omega \setminus K$ such that $y_j \to y, z_j \to z$ as $j \to \infty$ and*

$$[y_j, z_j] = \{ty_j + (1-t)z_j \mid 0 \le t \le 1\} \subset \Omega \setminus K. \tag{4.1}$$

Proof. To the contrary, we suppose that there exist $\delta > 0$ such that for every $\tilde{y} \in B_\delta(y)$ and for every $\tilde{z} \in B_\delta(z)$, there exists $\tilde{t} \in (0,1)$ such that $\tilde{t}\tilde{y} + (1-\tilde{t})\tilde{z} \in K$. Here we note that $\tilde{t}\tilde{y} + (1-\tilde{t})\tilde{z}$ must be in Ω since Ω is assumed to be convex. In particular, if we set $\tilde{y} = (a_1, \dots, a_{n-1}, 0), \tilde{z} = (a_1, \dots, a_{n-1}, 1)$ with $a' = (a_1, \dots, a_{n-1}) \in B_\delta^{n-1}(0)$, one sees that there exists $t_{a'} \in (0,1)$ such that $(a', t_{a'}) \in K$. We define the set V by

$$V = \{(a', t_{a'}) \mid a' \in B_\delta^{n-1}(0)\}. \tag{4.2}$$

Clearly $V \subset K$.

The assumption on K implies that the $(n-1)$-dimensional Hausdorff measure of K is zero. Hence there exist countable balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ such that

$$K \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i) \quad \text{and} \quad \sum_{i=1}^{\infty} r_i^{n-1} < \delta^{n-1}. \tag{4.3}$$

It follows that V is also covered by $\{B_{r_i}(x_i)\}_{i=1}^\infty$. By projecting both V and $\{B_{r_i}(x_i)\}_{i=1}^\infty$ onto $\mathbb{R}^{n-1}\times\{0\}$, we have that

$$B_\delta^{n-1}(0)\subset\bigcup_{i=1}^\infty B_{r_i}^{n-1}(x_i'). \tag{4.4}$$

Taking $(n-1)$-dimensional measure of each side of (4.4), we obtain that

$$\omega_{n-1}\delta^{n-1}\le\sum_{i=1}^\infty\omega_{n-1}r_i^{n-1}<\omega_{n-1}\delta^{n-1}, \tag{4.5}$$

which is a contradiction. Lemma 4.2 is thus proved. □

Let $\lambda\in[0,1]$ and set $x=\lambda y+(1-\lambda)z\in\Omega\setminus K$. From the above lemma and the local convexity of u, it follows that

$$u(x)\le\lambda u(y_j)+(1-\lambda)u(z_j) \tag{4.6}$$

for all $j\in\mathbb{N}$, where $\{y_j\}_{j=1}^\infty$ and $\{z_j\}_{j=1}^\infty$ are sequences which we obtained in Lemma 4.2. Since u is locally convex in $\Omega\setminus K$, u is continuous in $\Omega\setminus K$. Taking $j\to\infty$,

$$u(x)\le\lambda u(y)+(1-\lambda)u(z). \tag{4.7}$$

Next let U be the supergraph of u, that is,

$$U=\{(x,w)\mid x\in\Omega\setminus K, w\ge u(x)\}\subset\mathbb{R}^{n+1}, \tag{4.8}$$

and for every set $X\subset\mathbb{R}^{n+1}$, $\operatorname{co}X$ denotes the convex hull of X. Now we define the function $\tilde{u}$ by

$$\tilde{u}(x)=\inf\{w\in\mathbb{R}\mid(x,w)\in\operatorname{co}U\}. \tag{4.9}$$

One can easily show that the convex hull of $\Omega\setminus K$ (in $\mathbb{R}^n$) is Ω, so that $\tilde{u}$ is defined in the whole Ω. Moreover, $\tilde{u}$ is a convex function due to the convexity of $\operatorname{co}U$. Finally, we show that $\tilde{u}$ is an extension of u defined in $\Omega\setminus K$. To see this, fix a point $x\in\Omega\setminus K$. The definition of $\tilde{u}$ follows that $\tilde{u}(x)\le u(x)$. Taking the infimum of the right-hand side of (4.7) over all $y,z\in\Omega\setminus K$, we have that $u(x)\le\tilde{u}(x)$. Consequently, it holds that $u\equiv\tilde{u}$ in $\Omega\setminus K$. $\tilde{u}$ is the desired function.

Step 2. (Removability of the singular set K)

We denote the extended function constructed in Step 1 by the same symbol u. Theorem 3.1 implies that there exists a non-negative Borel measure ν whose support is contained in K such that

$$H_k[u]=\psi\,dx+\nu\quad\text{in }\Omega \tag{4.10}$$

in the generalized sense. We fix arbitrary $\varepsilon > 0$. By the assumption we can cover K by countable open balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} r_i^{n-k} < \varepsilon. \tag{4.11}$$

For any $\rho \geq 0$,

$$\begin{aligned} \omega_n(r_i+\rho)^n &\geq \mathcal{L}^n(Q_\rho(u; B_{r_i}(x_i))) \\ &= \sum_{m=0}^{n} \binom{n}{m} \sigma_m(u; B_{r_i}(x_i))\rho^m \\ &\geq \binom{n}{k} \sigma_k(u; B_{r_i}(x_i))\rho^k \\ &= \left(\int_{B_{r_i}(x_i)} \psi \, dx + \nu(B_{r_i}(x_i)) \right) \rho^k \geq \nu(B_{r_i}(x_i))\rho^k. \end{aligned} \tag{4.12}$$

The first inequality in (4.12) is due to the fact that $Q_\rho\,(u; B_{r_i}(x_i)) \subset B_{r_i+\rho}(x_i)$, since taking any $z \in Q_\rho\,(u; B_{r_i}(x_i))$ we obtain

$$|z - x_i| = |y + \rho v - x_i| \leq |y - x_i| + \rho|v| < r_i + \rho, \tag{4.13}$$

for some $y \in B_{r_i}(x_i)$, $v \in \gamma_u(y)$. Inserting $\rho = r_i$ in (4.12), we obtain that

$$\omega_n 2^n r_i^n \geq \nu\left(B_{r_i}(x_i)\right) r_i^k. \tag{4.14}$$

Consequently, it holds that

$$\nu\left(B_{r_i}(x_i)\right) \leq \omega_n 2^n r_i^{n-k}. \tag{4.15}$$

Now taking the summation for $i \geq 1$, we have that

$$\begin{aligned} \nu(K) &\leq \nu\left(\bigcup_{i=1}^{\infty} B_{r_i}(x_i)\right) \\ &\leq \sum_{i=1}^{\infty} \nu(B_{r_i}(x_i)) \\ &\leq \sum_{i=1}^{\infty} \omega_n 2^n r_i^{n-k} \\ &< \omega_n 2^n \varepsilon. \end{aligned} \tag{4.16}$$

Since we can take $\varepsilon > 0$ arbitrarily, we see that $\nu(K) = 0$. Therefore, $\nu \equiv 0$. We conclude that K is a removable set. $\square$

We see from Example 3.1 (2) that the number $(n-k)$ in Theorem 4.1 is optimal, since the Hausdorff dimension of T is $n-k$.

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