The Generalized Fermat-Steiner Problem with Free Ends

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1 Introduction

Let $G = (e_1, e_2, \ldots, e_n)$ be a conneted graph such that the degree of its vertices are all 3 except for the end points. In other words, $G$ is a network with triple junctions. For a given region $\Omega \subset \mathbb{R}^2$, a set of line segment $\Gamma_G$ is called admissible for $G$ if $\Gamma_G$ is isomorphic to $G$ and all the end points of $\Gamma_G$ are on $\partial \Omega$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{An example of $\Gamma_G$.}
\end{figure}

We assign a positive number $\sigma_i$ to each edge $e_i$, which represents "surface energy." Denote by $\gamma_i$ ($i = 1, 2, \ldots, n$) component segments of $\Gamma_i$ which correspond to $e_i$. In this study we are concerned with the following problem:
Problem P. Find an admissible $\Gamma_G$ for $G$ that minimizes

$$E[\Gamma_G] = \sum_{i=1}^{n} \sigma_i |\gamma_i|,$$

where $|\gamma_i|$ denote the lengths of $\gamma_i$.

This problem arises in grain boundary motions of annealing pure metal. Critical points of $E[\Gamma_G]$ represent stationary states of a curvature-driven motion, which models the grain boundary motions. A curvature-driven motion with a triple junction has been introduced by Mullins [6]. Later, the motion was derived formally by Bronsard and Reitich [1] as the singular limit of a vector-valued Allen-Cahn equation. Bronsard and Reitich [1] also showed short-time existence of the motion. Let $\Gamma_i(t)$ ($i = 1, 2, 3$) represent curves at time $t > 0$ contained in a two-dimensional bounded region $\Omega$ with smooth boundary $\partial \Omega$. Suppose $\Gamma_i(t)$ ($i = 1, 2, 3$) meet at one point $m(t)$. The evolving interface that we consider is subject to the following laws:

(M1) The normal velocity of the interface is given by its curvature.

(M2) At the triple junction $m(t)$, the contact angle $\theta_k$ between $\Gamma_i(t)$ and $\Gamma_j(t)$ is given by Young's law, where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$. That is, for positive constants $\sigma_1, \sigma_2, \sigma_3$, we have:

$$\frac{\sin \theta_1}{\sigma_1} = \frac{\sin \theta_2}{\sigma_2} = \frac{\sin \theta_3}{\sigma_3},$$

where $0 < \theta_k < \pi$ and $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

(M3) At the other end of each curve, $\Gamma_i(t)$ touches $\partial \Omega$ at the right angle.

The interfaces have Energy $E(t)$, which decreases as time goes:

$$E(t) = \sigma_1|\Gamma_1(t)| + \sigma_2|\Gamma_2(t)| + \sigma_3|\Gamma_3(t)|,$$

where $|\Gamma_i(t)|$ ($i = 1, 2, 3$) mean the lengths of curves $\Gamma_i(t)$. Stationary interfaces of the motion can be viewed as critical points of the energy. In this connection Sternberg and Ziemer [7] have proved the existence of local minimizers of the energy in clover-like regions. Here we remark that stationary interfaces consist of straight line segments.

On the other hand, Ikota and Yanagida [4] have studied stabilities of stationary interfaces of the motion (M1)-(M3) by linearizing corresponding equations around the stationary interfaces. They linearized the equations formally and analyzed the resulting elliptic operator rigorously. Later they have extended their results to stationary interfaces of binary-tree type with more than one triple junctions[5]. The results are stated as follows.
Theorem 1.1. Let $\Gamma = \{ \gamma_i \}$ be a stationary interface that is homeomorphic to a binary tree. Denote by $L_i$ the length of $\gamma_i$. Define a characteristic index $D$ by

$$D = \sum_{\gamma_i \in \Gamma} \sigma_i L_i \times \prod_{\gamma_i \in B} h_i + \sum_{\gamma_i \in B} \left\{ \sigma_i \prod_{\gamma_j \in B \setminus \{\gamma_i\}} h_j \right\},$$

where $h_i$ denotes the curvature of $\partial \Omega$ at the point of contact with $\gamma_i \in B$. (Note that $h_i$ is taken to be nonpositive if $\Omega$ is convex.)

(i) The unstable dimension $N_\mathrm{U}$ is given by

$$N_\mathrm{U} = \begin{cases} 
  m - 1 & \text{for } (-1)^m D < 0, \\
  m & \text{for } (-1)^m D > 0,
\end{cases}$$

where $m = \# \{ h_i < 0 \}$.

(ii) The stationary interface is degenerate (i.e., there exists a zero eigenvalue) if and only if $D = 0$.

We remark that the index $D$ is independent of the topology of $\Gamma$.

Although Ikota and Yanagida have established a stability criterion assuming the existence of stationary interfaces, it has not been known whether given regions have stationary interfaces in general. The existence problem can be regarded as a variation of the Fermat-Steiner problem[2].

In [4] and [5], stabilities of stationary states have been studied on the assumption of the existence of stationary states.

In the present study we show that stationary states do exist for convex $\Omega$. Our problem can be regarded as a variant of the Fermat-Steiner problem, though the treatments are quite different.

The Fermat-Steiner problem is described as follows: for a given triangle $\triangle ABC$, find a point $P$ that minimizes the sum of lengths

$$|PA| + |PB| + |PC|.$$

This problem was proposed by Fermat to Torricelli. Afterwards Steiner considered the same problem and gave a systematic solution. In [2] Gueron and Tessler solved the weighted Fermat-Steiner problem. They also gave an interesting historical survey of the problem.

Now we are in a position to state our result.
Theorem 1.2. Suppose $\Omega$ is convex. Let $n$ be a positive integer and $G$ a binary tree with $n$ triple junctions. Then there exists at least one critical interface of $E$ which is admissible for $G$.

2 Outline of Proof

Before proceeding with Problem $P$, we consider the two phase separation problem with no triple junctions as an illustration. Let $\Omega$ be a convex domain in $\mathbb{R}^2$. Suppose two points $P_1$ and $P_2$ are on the boundary $\partial \Omega$. We seek a critical interface of $E(P_1 P_2) = |P_1 P_2|$, the length of a line segment $P_1 P_2$.

A simple calculation shows that $P_1 P_2$ is critical if and only if $P_1 P_2$ intersects with $\partial \Omega$ at the right angle. Thus all we have to do is to find $P_1 P_2$ such that $P_1 P_2$ are orthogonal to $\partial \Omega$ at both $P_1$ and $P_2$.

We parameterize $\partial \Omega$ by an arc length parameter $s$: $s \mapsto P(s) = (x(s), y(s)) \in \partial \Omega$. By $\tau(s)$ we denote the tangential vector to $\partial \Omega$ at $P(s)$, that is $\tau(s) = (\partial / \partial s)(x(s), y(s))$. For any point $P_1 = P_1(s_1) \in \partial \Omega$, we can choose $s = s_2$ so that $\tau(s_2)$ is parallel to $\tau(s_1)$.

![Figure 2: Lines $l_1$ and $l_2$ are rotated along $\partial \Omega$.](image)

Then we move $s_1$ and observe variations of the distance $d(l_1, l_2)$, where $l_i$ are tangential lines to $\partial \Omega$ at $P(s_i)$ ($i = 1, 2$). We can easily see that the distance $d(l_1, l_2)$ is critical if and only if $P_1 P_2$ intersects with $\partial \Omega$ orthogonally. Since $d(l_1, l_2)$ has a maximum (and a minimum), the energy $E(P_1 P_2)$ has a critical interface.

Now we turn our attention to Problem $P$. We consider the case where $G$ has a single triple junction; for more triple junctions we can show the existence of critical interfaces
by induction. We can easily verify that $\Gamma_G = (\gamma_1, \gamma_2, \gamma_3)$ is critical if and only if the following two conditions are satisfied:

1. $\angle(\gamma_i, \gamma_j) = \theta_k$ \hspace{1em} $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

2. $\gamma_i \perp \partial \Omega$ \hspace{1em} $(i = 1, 2, 3)$.

Let $\nu(s_i)$ be the unit normal to $\partial \Omega$ at $P(s_i)$ pointing inside $\Omega$. Likewise in the analysis of the two phase problem, we can choose $s_2$ and $s_3$ for $s_1$ such that

$$\angle(\nu(s_i), \nu(s_j)) = \theta_k, \hspace{1em} (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2).$$

![Figure 3: Lines $n_1, n_2, n_3$ are rotated.](image)

Let $T$ be the triangle composed of $l_1, l_2, l_3$, where $l_i$ are again the tangential lines at $P_i = P(s_i)$, and $T(s)$ the area of $T$. Denote by $n_i$ the normal line to $\partial \Omega$ at $P_i$. Then we can prove that $n_1, n_2, n_3$ meet at one point if and only if $\partial T/\partial s = 0$. This indicates that the $n_i$ $(i = 1, 2, 3)$ make a critical $\Gamma_G$.

3 Concluding Remarks

If $\Omega$ is not convex, the approach we took in the previous section does not work in general. We illustrate it in the two phase problem.

Let $a, b$ be positive constants. We introduce two graphs in $\mathbb{R}^2$:

$$y = g_1(x) = (x - a)^3,$$

$$y = g_2(x) = x^3 + b.$$
Suppose $\partial \Omega$ is represented by $g_1(x)$ and $g_2(x)$ locally. We parameterize the two parts as $(\xi, (\xi - a)^3)$ and $(-\xi, -\xi^3 + b)$ respectively. Here $\xi$ runs over some interval $(-\delta, \delta)$. Then the distance between $l_1$ and $l_2$ are given by
\[
d(l_1, l_2) = \frac{(4\xi^3 + 3a\xi^2 + b)}{\sqrt{9\xi^4 + 1}}.
\]
Straightforward calculation shows that
\[
\left. \frac{\partial}{\partial \xi} d(l_1, l_2) \right|_{\xi=0} = 0.
\]
However the two normal lines at $\xi = 0$ do not coincide.

\[\text{Figure 4: The critical lines of the distance } d(l_1, l_2) \text{ do not coincide.}\]

References


