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無限個の不連続点をもつ目的関数に対する最適化問題の可解性

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Abstract
In this paper we extend invex properties to a method of generalized directional differentials and also we get the existence and the uniqueness theorems concerning non-continuous optimization problems with almost smoothness.

Keywords: invex analysis ; generalized directional differentiation; convex set; non-continuous optimization problem; variational inequality problem ; upper semi-continuity.

1 Convex analysis and invex analysis

Denote by $\mathbf{R}^n$ the $n-$dimensional real vector space with a positive integer n. Let a subset C be compact and convex in $\mathbf{R}^n$ and let f be a convex function from C to $\mathbf{R}$. Consider a compact convex problem:

$$minimize\ f(x)\ subject\ to\ x \in C \tag{P}$$

By analyzing the convexity of feasible set and objective function, we discuss the existence of optimal solutions of (P) as follows.

Since the definition of the convexity of f, it follows that $f(\lambda y+(1-\lambda)x) \leq \lambda f(y)+(1-\lambda)f(x)$ for $x, y \in C$and $0 \leq \lambda \leq 1$, which means that, by putting $\eta(y,x) = y - x$,

$$\frac{f(x+\lambda\eta(y,x)) - f(x)}{\lambda} \leq f(y) - f(x)$$

for $\lambda > 0$. Here $x + \lambda\eta(y,x) \in C$, because C is convex. Moreover assume that f is C^1-class, then we have

$$\eta(y,x)^T \bigtriangledown f(x) \leq f(y) - f(x)$$

for $x, y \in C$. Denote by x^T the transpose of $x \in \mathbf{R}^n$. Since C is compact, by Fan's method (see Section 4), there exists a solution $x_0 \in C$ such that the following variational inequality problem :

$$(y - x_0)^T \bigtriangledown f(x_0) \geq 0. \qquad (VIP)$$

Therefore $f(x_0) \leq f(y)$ for any $y \in C, i.e.$, x_0 is also an optimal solution of (P).

In what follows we discuss the existence of solutions for optimization problems and variational inequality problems by applying a generalized directional differential. The generalized directional differential, considering a parameter function η from $C \times C$ to $\mathbf{R}^n$, means a Dini's derivative of f to η for $y, x \in C$ as

$$f'(x; \eta(y,x)) = \limsup_{\lambda \to 0+} \frac{f(x + \lambda\eta(y,x)) - f(x)}{\lambda}.$$

Ruiz-Garzón et al. [5] discussed the existence of (VIP) and (P) by invex analysis which is the essential the idea of generalized directional differential. They defined invex sets and invex functions, which are including the properties of convex of sets and functions, respectively.

Definition 1 *Let η be a function from $C \times C$ to $\mathbf{R}^n$. A set C is called* **invex set**(IX) *to η, if $x + \lambda\eta(y,x) \in C$ for $x, y \in C$ and $0 \leq \lambda \leq 1$. A differentiable $f : C \to \mathbf{R}$ is called* **invex function**(IX) *to η $\eta(y,x)^T \bigtriangledown f(x) \leq f(y) - f(x)$ for $x, y \in C$.*

When C is (IX) to $\eta = y - x$, then $x + \lambda(y - x) \in C$, which means that C is convex. When $C = ([1,\infty) \times \mathbf{R}) \cup (\mathbf{R} \times [1,\infty)) \subset \mathbf{R}^2$ is (IX) to $\eta(y,x) = y$. When f is differentiable and convex, then it follows that $(y - x)^T \bigtriangledown f(x) \leq f(y) - f(x)$ for $x, y \in C$.

2 Existence theorem for (VLIP) and (P) by invex analysis

In [5] the existence criteria for (VLIP) and (P) are given under conditions that f is differentiable and $y^T \bigtriangledown f(x + ty)$ is continuous in t.

Theorem R

The following conditions (i)-(iv) *hold:*

(i) *C is non-empty, compact and convex in $\mathbf{R}^n$;*

(ii) *$\eta(y,x)$ is linear in y and $\eta(y,x) + \eta(x,y) = 0$ for $y, x \in C$;*

(iii) *f is differentiable and f' is pseudo invex monotone*(PIM) *to η;*

(iv) *$y^T \bigtriangledown f(x + ty)$ is continuous in $t \in [0,1]$ for $y, x \in C$.*

Then f is pseudo invex(PIX) to η. Moreover there exists a solution $x_0 \in C$ of the following variational-like inequality problem

$$\eta(y,x_0)^T \bigtriangledown f(x_0) \geq 0 \qquad (VLIP)$$

for $y \in C$ and also x_0 is an optimal solution of (P).

Definition 2 *Let η be a parameter function from $C \times C$ to $\mathbf{R}^n$ and let f from C to $\mathbf{R}$ be differentiable.*
$\bigtriangledown f$ is called **invex monotone**(IM) *to η, if $\eta(y,x)^T[\bigtriangledown f(y) - \bigtriangledown f(x)] \geq 0$ for $y, x, \in C$.*
$\bigtriangledown f$ is called **pseudo invex monotone**(PIM) *to η, if*
$\eta(y,x)^T \bigtriangledown f(y) \geq 0$ as long as $\eta(y,x)^T \bigtriangledown (x) \geq 0$ for $y, x \in C$.
f is called **pseudo invex**(PIX) *to η, if $f(y) - f(x) \geq 0$ as long as $\eta(y,x)^T \bigtriangledown f(x) \geq 0$ for $y, x \in C$.*

For the same parameter η, if f' is (IM), then f' is (PIM). For the same parameter η, if f is (IX), then f is (PIX).

Example 1 (1) *Denote $f(x) = x^2$ on $C = \{x \geq 0\}$. Then f is (IM) to $\eta(y,x) = e^y - e^x$, because*

$$\begin{aligned}&\eta(y,x)[f'(y) - f'(x)]\\ &= (y-x)(1 + \frac{y+x}{2!} + \frac{y^2+yx+x^2}{3!}\\ &\quad + \cdots)2(y+x)(y-x)\\ &\geq 0.\end{aligned}$$

(2) *Denote $f(x) = -x$ $(x < 0)$; $f(x) = 0$ $(x \geq 0)$. Then f' is (IM) and (PIM) to the same $\eta(y,x) = e^y - e^x$.*
(3) *Denote $f(x) = 2x + \sin x$ on $C = \mathbf{R}$. Then $f'(x) = 2 + \cos x > 0$ for $x \in C$. Let $\eta(y,x) = \dfrac{f(y) - f(x)}{f'(x)}$. Then it follows that f is* (PIX) *and f' is (PIM) to the same $\eta(y,x)$. If $\eta(y,x)f'(x) \geq 0$, then $f(y) - f(x) \geq 0$, which means that f is (PIX). If $\eta(y,x)f'(x) \geq 0$, then $\eta(y,x)f'(y) = (f(y) - f(x))\frac{f'(y)}{f'(x)} \geq 0$, which means that f' is (PIM).*

3 Generalized directional differential

Consider the optimization problem (P) and the following variational-like inequality problem

$$f'(x_0; \eta(y,x_0)) \geq 0 \qquad (VLIP)$$

for $y \in C$. Here C is compact convex in $\mathbf{R}^n$ and f isn't necessarily continuous but satisfies Hypothesis (H), where there exists a covering of $C = \cup_i C_i$ such that f is locally Lipschizian on C_i for any i. Under (H) we consider the Dini's derivative to η for y, which is a generalized directional differential. Assume that the following properties of f concerning some kind of smoothness.

Hypothesis (H) *Assume that C is non-empty and compact in $\mathbf{R}^n$. Let η be a function from $C \times C$ to $\mathbf{R}$. The objective function f satisfies the following condition* (i) *and* (ii).

(i) *There exists a convex covering $\{C_i \subset C : convex,$ $i = 1, 2, \cdots,\}$ such that $\cup_i C_i = C$ and that f is locally Lipschizian on C_i for each i. Denote by $O_i, i = 1, 2, \cdots$, the maximal open sets in C_i. f is C^1-class on O_i for each i. If $x \notin O_i$ for each i and $y \in C$, then $f'(x; \eta(y, x)) \geq 0$.*

(ii) *Let $x, y \in C$ and $\eta : C \times C \to \mathbf{R}^n$. If $f(y) < f(y - \lambda\eta(y, x))$ for $0 < \lambda \leq 1$, then there exits $\mu = \mu(\lambda) \in [0, 1]$ such that $\mu > \lambda$ and $f(y) \geq f(y - \mu\eta(y, x))$.*

By the above (i), there exists the **Dini's derivative** to η for $y, x \in C_i$ by

$$f'(x; \eta(y, x)) = \limsup_{\lambda \to 0+} \frac{f(x + \lambda\eta(y, x)) - f(x)}{\lambda}.$$

Then $f'(\cdot; \eta(\cdot, y)) : C_i \to \mathbf{R} \cup \{+\infty\}$ for η and $y \in C_i$ with $x + \lambda\eta(y, x) \in C_i$. By the above (ii), it follows that

$$f'(x; \eta(y, x)) = \eta(y, x)^T \bigtriangledown f(x)$$

at $x \in O_i$ for $y \in C$. The following example illustrates two cases: a finite and an infinite number of covering of C.

Definition 3 *Assume that Hypothesis (H) holds. f' is called* **pseudo invex monotone**(*PIM*) *to η for $y, x \in C$, if $f'(y; \eta(y, x)) \geq 0$ as long as $f'(x; \eta(y, x)) \geq 0$ for $x \in C$.*

f' is called **strictly pseudo invex monotone**(*SPIM*) *to η for $y, x \in C$ with $y \neq x$, if $f'(y; \eta(y, x)) > 0$ as long as $f'(x; \eta(y, x)) \geq 0$ for $y, x \in C$.*

f is called **pseudo invex**(*PIX*) *at $x \in C$ to η for $y \in C$, if $f(y) - f(x) \geq 0$ as long as $f'(x; \eta(y, x)) \geq 0$ for $y, x \in C$.*

f is called **strictly pseudo invex**(*SPIX*) *at $x \in C$ to η for $y \in C$ with $y \neq x$, if $f(y) - f(x) > 0$ as long as $f'(x; \eta(y, x)) \geq 0$ for $y, x \in C$.*

In case where f' is (SPIM) to η, it follows that f' is (PIM). In Theorem 1 we show that (PIM) gives the existence of solution for (VLIP). In Theorem 2 Property (PIM) guarantees the existence of optimal solutions of (P) and (PIM) means (PIX) at the optimal solution to the same η. In the theorem we need not to prove the Property (PIX) of f at each $x \in C$ but to show the property concerning the solution $x_0 \in C$ of (VLIP). Finally (SPIM) is significant for the uniqueness of solutions in Theorem 3.

Example 2 (1) *Denote* $C = [0,2]$, *where* $c, d \in \mathbf{R}$. *Denote* $J_1 = \{0 < x < 1\}$ *and* $J_2 = \{1 < x < 2\}$. *Denote*

$$f_{c,d}(x) = \begin{cases} p(x) & \text{for } x \in J_1 \\ g(x) & \text{for } x \in J_2 \end{cases}$$

and $f_{c,d}(0) = c; f_{c,d}(1) = 1; f_{c,d}(2) = d$. *Denote* $\eta(y,x) = y - x$. *Here* p, q *are* C^1*-class on* J_1, J_2, *respectively with* $\lim_{x \to 1-0} p(x) \le 1 \le \lim_{x \to 1+0} q(x)$ *and* $\lim_{x \to 2-0} q(x) \le d$. *A covering* $\{\{0\}, J_1, \{1\}, J_2, \{2\}\}$ *of* C *is finite. If* $c \le p(1-0)$, *then* (H) *holds. Then* f' *is* (PIM) *and* f *is* (PIX) *at* $x = 0$ *to the* η. *If* $c > p(1-0)$, *then* (H) *isn't satisfied because* $f'(0; \eta(y,0)) = -\infty$.

(2) *Let* $f : C = [0,\ c] \to \mathbf{R}$ *with* $0 < c \le 1$. *Denote* $a_n = \frac{1}{2n}$, $b_n = \frac{1}{2n-1}$ *for* $n = 1, 2 \cdots$. *Let* $\eta(y,x) = y - x$ *for* $y, x \in C$.

$$f(x) = \begin{cases} 0 & (x = 0) \\ \frac{(a_n^{5/2} - b_n^3)(x - b_n)}{b_n - a_n} + b_n^3 & (a_n < x \le b_n, \quad n = 1, 2 \cdots) \end{cases}$$

Then we have the following observation as follows.

Consider a covering $\{\{0\}, (a_n, b_n] : n = 1, 2, \cdots\}$ *of* C. *Then* f *is* C^1*-class on* $O_n = (a_n, b_n)$ *for* $n = 1, 2 \cdots$ *and*

$$\begin{aligned} f'(x, \eta(y,x)) &= \overline{\lim_{\lambda \to 0+}} \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \\ &= (y - x) f'(x) \\ &= (y - x) \frac{(a_n^{5/2} - b_n^3)}{b_n - a_n} \end{aligned}$$

for $x \in (a_n, b_n], y \in \mathbf{R}$. *Then* $f'(x) \to 0+$ *as* $n \to +\infty$. *It can be seen that* $x = 0 \notin O_n$ *for any* n *and* $f'(0, \eta(y,0)) = 0$. *If* $f'(x, \eta(y,x)) \ge 0$, *then* $f'(y, \eta(y,x)) > 0$ *for* $y \ne x$. f' *is* $(SPIM)$ *and* f *is* $(SPIX)$ *at* $x = 0$ *to* η *for* $y, x \in C$. $f'(x, \eta(y,x))$ *is upper semi-continuous in* x *to* η *and* $y \in C$. *There exists a unique minimal point* $x = 0$ *and* $\min f(x) = 0$.

In the following theorem we get a main result for the existence theorem of solutions for (VLIP).

Theorem 1 *The following conditions* (i)-(iv) *hold:*

(i) C *is non-empty, compact and convex in* $\mathbf{R}^n$;

(ii) $\eta(y,x)$ *is linear in* y *and* $\eta(x,x) = 0$ *for* $x \in C$;

(iii) (H) *holds and* f' *is* (PIM) *to* $\eta(y,x)$ *for* $y \in C$;

(iv) $f'(x;\eta(y,x))$ *is upper semi-continuous in* $x \in C$ *for* η *and* $y \in C$.

Then there exists a solution $x_0 \in C$ *of* $(VLIP)$.

Invex properties of f guarantees that solutions of non-continuous (VLIP) to $\eta(y,x) = y - x$ will become optimal solutions of non-continuous (P).

Theorem 2 *Assume that the same conditions* (i)-(iv) *in Theorem* 1. *Moreover Conditions* (ii)-(iv) *are satisfied for* $\eta(y,x) = y - x$.

Then there exists a solution $x_0 \in C$ *of* $(VLIP)$ *to* η *and* x_0 *is an optimal solution of* (P).

In order to guarantee the existence and uniqueness of solutions of non-continuous (VLIP) and (P) we introduce new definitions of (SPIM) and (SPIX) at some point under Hypothesis (H).

Theorem 3 *In replacing Property* (PIM) *of Condition* (iii) *in Theorem* 2 *with* $(SPIM)$ *of* f', *the following conclusions* (I)-(III) *hold.*

(I) *There exists a unique solution* $x_0 \in C$ *of* $(VLIP)$ *to* η;

(II) f *is* $(SPIX)$ *at* x_0 *to* η *for* $y, x \in C$;

(III) x_0 *is a unique optimal solution of* (P).

4 KKM-functions

At first we show the existence of solutions of (VLIP) to η provided with the compactness of C in the similar way of [4]. See [5].

Definition 4 *Let a function* V *be from a subset* $C \subset \mathbf{R}^n$ *to the power set* $2^{\mathbf{R}^n}$. *If, for every finite subset*

$$A = \{x_1, x_2, \cdots, x_m\} \subset C$$

and every m *is positive integer, it follows that the convex hull*

$$conv(A) \subset \cup\{V(x_i) : x_i \in A\},$$

then V *is called a* **KKM-function**.

See [4, 5].

Lemma F *Let C be non-empty and $V : C \to 2^{\mathbf{R}^n}$ a KKM-function. If $V(x)$ is compact for $x \in C$, then* $\cap\{V(x) : x \in C\} \neq \emptyset$.

The following lemma is an extension of Lemma 2 of [5].

Lemma 1 *Under the same Conditions* (i)-(iv) *of Theorem* 1 *without assuming the compactness of C the following statements* (I) *and* (II) *are mutually equivalent for $y \in C$.*

(I) *$x_0 \in C$ is a solution of $(VLIP)$ to η.*

(II) *$x_0 \in C$ satisfies $f'(y; \eta(y, x_0)) \geq 0$.*

In the following lemma Hypothesis (H) shows the set of solutions for non-continuous (VLIP) to η will become a KKM-function.

Lemma 2 *Assume that the same Conditions* (i)-(iv) *of Theorem* 1 *hold without assuming the compactness of C. Denote*

$$S_c(y) = \{x \in C : f'(y; \eta(y, x)) \geq 0\}$$

for $y \in C$.

Then $S_c(y)$ is a KKM-functions for $y \in C$.

Compact KKM-functions play an important role in discussing the existence of non-continuous (VLIP) to η. By the above lemmas we can prove Theorem 1.

When f is (PIM) to $\eta(y, x) = y - x$, it can be seen that f is (PIX) at the solution x_0 of (VLIP) to the η as in the following lemma.

Lemma 3 *Assume that Conditions* (i)-(iv) *of Theorem* 1 *and the following condition* (v) *hold.*

(v) *Conditions* (ii)-(iv) *are satisfied for $\eta(y, x) = y - x$.*

Then there exists at least one solution $x_0 \in C$ of $(VLIP)$ and f is (PIX) at x_0 to $\eta(y, x_0)$ for $y \in C$.

From the above lemma we get Theorem 2 immediately.

When f' is (SPIM) to η, it can be proved that f is (SPIX) at the (VLIP)-solution x_0 to the $\eta(y, x_0)$ in the similar way as Lemma 3.

Lemma 4 *Assume that Conditions* (i)-(v) *of Theorem* 3 *hold.*

Then there exists a unique solution $x_0 \in C$ of $(VLIP)$ and f is $(SPIX)$ at x_0 to $\eta(y, x_0)$ for $y \in C$.

From the above lemma, we get the proof of Theorem3.

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