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The role of local energy decay in L^p-estimates for the wave equation with time-dependent dissipation†

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1. Introduction

This paper is the résumé of papers [4, 5]. Some of proofs of theorems, propositions and lemmas are omitted.

Let Ω be an unbounded domain having the compact and smooth boundary $\partial\Omega$, and let $\mathbb{R}^n \setminus \Omega$ be star-shaped with respect to the origin such that $\mathbb{R}^n \setminus \Omega \subset B_{\rho_0}$ for some $\rho_0 > 0$, where we set $B_{\rho_0} = \{x \in \mathbb{R}^n; |x| < \rho_0\}$. We consider L^p-estimates and scattering rates for the following initial-boundary value problem in odd space dimension n with $n \geq 3$:

$$(\mathrm{P}) \begin{cases} u_{tt} - \Delta u + a(x,t)u_t = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

We make the following assumption on $a(x,t)$:

Assumption A. (i) *$a(x,t)$ is nonnegative on $\overline{\Omega} \times [0,\infty)$.*
(ii) *$a(x,t)$ belongs to $\mathfrak{B}^\infty\left(\overline{\Omega} \times [0,\infty)\right)$.*
(iii) *The support of $a(x,t)$ is contained in a time-dependent domain*

$$\Omega(t) \equiv \left\{ x \in \overline{\Omega};\ |x| < (R+t)^\alpha \right\}$$

for some $R > \rho_0$ and α with $0 < \alpha < \frac{1}{2}$. If $\alpha = 0$, we assume that the support of $a(x,t)$ is contained uniformly in $\Omega \cap B_R$, B_R being the ball centered at the origin with radius R.

The condition $0 \leq \alpha < \frac{1}{2}$ means that the support of $a(x,t)$ expands at a speed strictly less than the wave speed. The equation of this kind was first treated by Tamura (see [14]), and it was proved that if the data have compact supports, then the local energy decays exponentially. Since then, there is no work of asymptotic behaviour for the problem (P). The difficulty of analysis lies in the fact that

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the coefficient a in the dissipative term depends on space-time variables. For example, Wirth (see [16]) has treated delicately the equation $\Box u + \mu(1+t)^{-1}u_t = 0$ ($\mu > 0$) through the Fourier representation formulae and obtained L^p-L^q-estimates.

Recently, the present author obtained L^p-estimates and scattering rates for the problem (P) (see [4, 5]). In deriving L^p-estimates we used the time-dependent cut-off method, which gives an extention of Shibata (see [11]) and Shibata and Tsutsumi (see [12]). As is well-known, the local energy decay plays a crucial role in this cut-off method. We provide this estimate in Proposition 2.2 which the integral region is given by time-dependent domain $\Omega(t)$, and apply it to cut-off method.

In order to state results we introduce the notation of Sobolev norms : for $s \geq 1$, we set

$$I_s = \|u_0\|_{H^s(\Omega)} + \|u_1\|_{H^{s-1}(\Omega)},$$
$$I_s^{(e)} = \|e^{|\cdot|}u_0\|_{H^s(\Omega)} + \|e^{|\cdot|}u_1\|_{H^{s-1}(\Omega)}, \quad e = \text{the Napier number},$$
$$I_s^{(\gamma)} = \|\langle\cdot\rangle^{\gamma}u_0\|_{H^s(\Omega)} + \|\langle\cdot\rangle^{\gamma}u_1\|_{H^{s-1}(\Omega)},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $H^s(\Omega)$ is the fractional order Sobolev space, and

$$\overline{D}u = (\partial_t^j\nabla^{\mu}u\,;\, j+|\mu| \leq 1), \quad Du = (\partial_t^j\nabla^{\mu}u\,;\, j+|\mu| = 1),$$

$$\Omega(L) = \Omega \cap B_L \quad (L > 0).$$

Then we proved

Theorem 1 ([4]). *Assume that Assumption A is satisfied with $0 < \alpha < \frac{1}{2}$. Let m be a nonnegative integer and set $M = [\frac{n}{2}] + 1$, $[\frac{n}{2}]$ being the integer part of $\frac{n}{2}$. Let the data u_0, u_1 satisfy*

$$u_0 \in H^{3M+m}(\Omega), \quad u_1 \in H^{3M+m-1}(\Omega), \quad I_{3M+m}^{(e)} < \infty,$$

and the compatibility condition of order $3M+m-1$. Then the solution u of problem (P) satisfies the following estimates : Let p be a number with $2 \leq p \leq \infty$. Then there exists a constant C such that

$$\|\overline{D}u(t)\|_{W^{m,p}(\Omega)} \leq C I^{(e)}_{3M-\frac{2(M-1)}{p}+m}(1+t)^{-\frac{n-1}{2}(1-\frac{2}{p})}.$$

Theorem 1 imposes the exponential weight on the initial data u_0, u_1. This condition is too restrictive. For the case of $\alpha = 0$, i.e., the support of $a(x,t)$ is contained uniformly in $\Omega \cap B_R$, we can relax it to polynomially weighted condition.

Our result reads as follows :

Theorem 2 ([5]). *Assume that Assumption A is satisfied with $\alpha = 0$. Let m be a nonnegative integer. Let the data u_0, u_1 satisfy*

$$u_0 \in H^{3M+m}(\Omega), \quad u_1 \in H^{3M+m-1}(\Omega), \quad I^{(\gamma)}_{3M+m} < \infty$$

for some γ with $\gamma > n-1$, and the compatibility condition of order $3M+m-1$. Then the solution u of problem (P) satisfies the following estimates : Let p be a number with $2 \le p \le \infty$. Then there exists a constant C such that

$$\|\overline{D}u(t)\|_{W^{m,p}(\Omega)} \le C I^{(\gamma)}_{3M-\frac{2(M-1)}{p}+m}(1+t)^{-\frac{n-1}{2}(1-\frac{2}{p})}.$$

Based on Theorems 1 and 2, we can argue the existence of scattering states and determine its asymptotic rates.

Theorem 3 ([4, 5]). *Let u be the solution in Theorems 1 and 2. Then there exists a free wave w^+ in Ω with finite energy such that*

$$\|u(t) - w^+(t)\|_E = \begin{cases} O\left(t^{-\frac{n-2}{2}+\frac{\alpha(n-\delta)}{2}}\right), & \text{if } 1 < \delta < n, \\ O\left(t^{-\frac{n-2}{2}} \log^{\frac{1}{2}}(2+t^{\alpha})\right), & \text{if } \delta = n, \\ O\left(t^{-\frac{n-2}{2}}\right), & \text{if } \delta > n, \end{cases}$$

as $t \to \infty$, where $\|\cdot\|_E$ is energy norm defined by

$$\|u(t)\|_E^2 = \frac{1}{2}\int_\Omega \left(|\nabla u(t)|^2 + u_t(t)^2\right) dx.$$

Here, we say that w^+ is free wave in Ω if w^+ satisfies the following initial-boundary value problem :

$$\begin{cases} w_{tt} - \Delta w = 0, & (x,t) \in \Omega \times (0,\infty), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \Omega, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

It is well-known in Mochizuki [7, 8] (cf. Mochizuki and Nakazawa [9]) that the energy does in general not decay. Thus the scattering problem is meaningful. The proof of the last theorem depends deeply on L^∞-estimate. For the proof, see [4, 5].

2. Local energy decay and L^2-bound

Let $\tau \geq 0$ be fixed, and let $v(x,t;\tau)$ be a finite energy solution of the following problem

$$\begin{cases} v_{tt} - \Delta v + a(x,t)v_t = 0, & (x,t) \in \Omega \times (\tau, \infty), \\ v(x,t;\tau) = 0, & (x,t) \in \partial\Omega \times (\tau,\infty) \end{cases} \tag{2.1}$$

with the initial data $f_1(x,\tau)$, $f_2(x,\tau)$ of compact supports in $\Omega(\tau)$. If we reconsider the proof of [14, Tamura] for $0 < \alpha < \frac{1}{2}$ and [8, Mochizuki] for $\alpha = 0$ carefully, the following proposition can be obtained.

Proposition 2.1. *Suppose that Assumption A is satisfied. Let $v(x,t;\tau)$ be the finite energy solution of problem (2.1), and let L, $L > \rho_0$, be fixed. Then there exist constants $C > 0$ and $\lambda > 0$, independent of R, L and τ, such that for $t \geq \tau$,*

$$\|Dv(t;\tau)\|^2_{L^2(\Omega(L))} \leq Ce^{\lambda(L^\beta + (R+\tau)^{\alpha\beta})} e^{-\lambda(t-\tau)^\beta} \|f(\tau)\|_E, \tag{2.2}$$

where $f(\tau) = \{f_1(\cdot,\tau), f_2(\cdot,\tau)\}$. If $\alpha = 0$, the right-hand side of (2.2) *should be replaced by*

$$Ce^{\lambda(L+R)}e^{-\lambda(t-\tau)}\|f(\tau)\|_E.$$

Remark. The constant β can be taken so that $\beta = (p+1)^{-1}$ with $p \geq \alpha(2+\gamma)(1-\alpha(2+\gamma))$, where $\alpha(2+\gamma) < 1$ and $0 < \gamma \leq 1$. For details, see [14]. Therefore we must assume that $0 \leq \alpha < \frac{1}{2}$.

Based on Proposition 2.1, we consider the following problem with forcing term :

$$(\mathrm{P})_f \begin{cases} u_{tt} - \Delta u + a(x,t)u_t = f(x,t), & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

Proposition 2.2. *Suppose that Assumption A is satisfied. Let m be a nonnegative integer, and let u be the solution of problem $(\mathrm{P})_f$ with data $\{u_0, u_1, f(\cdot,t)\} \in H^{m+1}(\Omega) \times H^m(\Omega) \times H^m(\Omega)$ such that u_0, u_1, $f(\cdot,0)$ satisfy the compatibility condition of order m and the supports of u_0, u_1 and $f(\cdot,t)$ are contained in $\Omega(R)$ and $\widetilde{\Omega}(t)$, respectively, where $\widetilde{\Omega}(t) = \{x \in \overline{\Omega};\ |x| < (R+t)^\alpha + 1\}$. If $\alpha = 0$, $\widetilde{\Omega}(t)$ should be replaced by $\Omega(R+1)$. Assume further that*

$$\sum_{j=0}^{m} \|\partial_t^j f(t)\|_{H^{m-j}(\Omega)} \leq \begin{cases} \Lambda e^{-\lambda t^\beta} & \text{if } 0 < \alpha < \frac{1}{2}, \\ \Lambda(1+t)^{-\gamma} & \text{if } \alpha = 0 \end{cases}$$

for some constants $\Lambda > 0$ and $\gamma > 0$. Then there exist constants C, λ, independent of the diameters of supports of u_0, u_1, $f(\cdot, 0)$, such that for $t \geq 0$,

$$\begin{cases} \|Du(t)\|_{H^m(\tilde{\Omega}(t))} \leq Ce^{2\lambda R^{\alpha\beta}} I_{m+1} e^{-\lambda' t^\beta} + C\Lambda e^{2\lambda R^{\alpha\beta}} e^{-\lambda' t^\beta} & \text{if } 0 < \alpha < \frac{1}{2}, \\ \|Du(t)\|_{H^m(\Omega(R+1))} \leq e^{2\lambda R} I_{m+1} e^{-\lambda t} + C\Lambda e^{2\lambda R}(1+t)^{-\gamma} & \text{if } \alpha = 0. \end{cases}$$

For the proof see [4, 5].

The following local energy decay estimate of free waves in odd space dimensions plays an important role in later discussion.

Proposition 2.3. *Let v be the smooth solution of the following Cauchy problem in odd space dimension $n = 2p + 1$ $(p = 1, 2, \ldots)$:*

$$\begin{cases} v_{tt} - \Delta v = 0, & (x,t) \in \mathbb{R}^n \times (0, \infty), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \mathbb{R}^n. \end{cases}$$

Let $L > 0$ be fixed and m a nonnegative integer. Then we have the following assrtions : (i) *There exist constants λ, λ' and C, independent of L, such that for $t \geq 0$,*

$$\|v(t)\|_{H^m(|x|<L)} \leq Ce^{\lambda' L} J^{(e)}_{2M+m-1} e^{-\lambda t},$$
$$\|Dv(t)\|_{H^m(|x|<L)} \leq Ce^{\lambda' L} J^{(e)}_{2M+m} e^{-\lambda t},$$

where

$$J^{(e)}_{2M+m-1} = \sum_{i=0}^{1} \sum_{0 \leq |\mu| \leq m} \sum_{k=0}^{p-i} \|e^{|\cdot|} \partial_r^k \nabla^\mu v_i(\cdot)\|_{H^M(\mathbb{R}^n)}, \quad \partial_r = \frac{x}{|x|} \cdot \nabla,$$

$$J^{(e)}_{2M+m} = \sum_{i=0}^{1} \sum_{0 \leq |\mu| \leq m} \Big(\sum_{k=0}^{p-i} \|e^{|\cdot|} \partial_r^k \nabla^\mu \nabla v_i(\cdot)\|_{H^M(\mathbb{R}^n)} + \sum_{k=0}^{p+1-i} \|e^{|\cdot|} \partial_r^k \nabla^\mu v_i(\cdot)\|_{H^M(\mathbb{R}^n)} \Big).$$

(ii) *Let γ be a constant with $\gamma > n - 1$. Then there exist a constant $C(L)$ depending on L such that for $t \geq 0$,*

$$\|v(t)\|_{H^m(|x|<L)} \leq C(L) J^{(\gamma)}_{2M+m-1} (1+t)^{-\gamma + \frac{n-1}{2}},$$
$$\|Dv(t)\|_{H^m(|x|<L)} \leq C(L) J^{(\gamma)}_{2M+m} (1+t)^{-\gamma + \frac{n-1}{2}},$$

where

$$J^{(\gamma)}_{2M+m-1} = \sum_{i=0}^{1} \sum_{0\le|\mu|\le m} \sum_{k=0}^{p-i} \left\| \langle\cdot\rangle^{\gamma}\partial_r^k \nabla^{\mu} v_i(\cdot)\right\|_{H^M(\mathbb{R}^n)},$$

$$J^{(\gamma)}_{2M+m} = \sum_{i=0}^{1} \sum_{0\le|\mu|\le m} \Big(\sum_{k=0}^{p-i} \left\| \langle\cdot\rangle^{\gamma}\partial_r^k \nabla^{\mu} \nabla v_i(\cdot)\right\|_{H^M(\mathbb{R}^n)}$$
$$+ \sum_{k=0}^{p+1-i} \left\| \langle\cdot\rangle^{\gamma}\partial_r^k \nabla^{\mu} v_i(\cdot)\right\|_{H^M(\mathbb{R}^n)} \Big).$$

For the proof see [4, 5].

The final proposition is concerned with an L^2-estimate.

Proposition 2.4. *Let v be the smooth solution of the following Cauchy problem :*

$$\begin{cases} v_{tt} - \Delta v = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \mathbb{R}^n. \end{cases}$$

Let m be a nonnegative integer. Then there exists a constant $C > 0$ such that for $t \ge 0$,

$$\|\overline{D}^m v(t)\|_{L^2(\mathbb{R}^n)} \le C \left(\|(\overline{D}^m v)(0)\|_{L^2(\mathbb{R}^n)} + \|v_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \right). \tag{2.3}$$

For the proof see Ikehata [1] (cf. Ikehata and Matsuyama [2]).

3. L^p-ESTIMATES

In this section we give an outline of proof of Theorems 1 and 2. The existence theorem is well known in [6, Mizohata].

We use the cut-off method as in Shibata and Tsutsumi [12] and Nakao [10]. Let L be any fixed number with $L > R + 1$. Let us take a smooth function $\mu(x)$ so that $\mu(x) = 0$ if $|x| \le L + 1$ and $= 1$ if $|x| \ge L + 2$. Then the solution u of problem (P) can be expressed by $u = \widetilde{u} + \widehat{u}$, where $\widetilde{u}$ and $\widehat{u}$ are solutions of the following problems $\widetilde{(\mathrm{P})}$ and $\widehat{(\mathrm{P})}$, respectively :

$$\widetilde{(\mathrm{P})} \begin{cases} \widetilde{u}_{tt} - \Delta\widetilde{u} + a(x,t)\widetilde{u}_t = 0, \quad (x,t) \in \Omega \times (0,\infty), \\ \widetilde{u}(x,0) = \mu(x)u_0(x), \quad \widetilde{u}_t(x,0) = \mu(x)u_1(x), \quad x \in \Omega, \\ \widetilde{u}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \end{cases}$$

and

$$
(\widehat{\mathrm{P}}) \begin{cases} \widehat{u}_{tt} - \Delta \widehat{u} + a(x,t)\widehat{u}_t = 0, \quad (x,t) \in \Omega \times (0,\infty), \\ \widehat{u}(x,0) = (1-\mu(x))\, u_0(x), \quad \widehat{u}_t(x,0) = (1-\mu(x))\, u_1(x), \quad x \in \Omega, \\ \widehat{u}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty). \end{cases}
$$

We need the next estimates.

Proposition 3.1. *For the solution $\widetilde{u}$ of problem $(\widetilde{\mathrm{P}})$, there exists a constant $C > 0$ such that for $t \geq 0$,*

$$
\left.\begin{aligned} \|\overline{D}\widetilde{u}(t)\|_{W^{m,\infty}(\Omega)} &\leq C I^{(e)}_{3M+m}(1+t)^{-\frac{n-1}{2}} \\ \|\overline{D}\widetilde{u}(t)\|_{H^m(\Omega)} &\leq C I^{(e)}_{2M+m+1} \end{aligned}\right\} \quad if\ 0 < \alpha < \frac{1}{2},
$$

$$
\left.\begin{aligned} \|\overline{D}\widetilde{u}(t)\|_{W^{m,\infty}(\Omega)} &\leq C I^{(\gamma)}_{3M+m}(1+t)^{-\frac{n-1}{2}} \\ \|\overline{D}\widetilde{u}(t)\|_{H^m(\Omega)} &\leq C I^{(\gamma)}_{2M+m+1} \end{aligned}\right\} \quad if\ \alpha = 0.
$$

For the proof of Proposition 3.1, we consider the following Cauchy problem :

$$
(\mathrm{CP}) \begin{cases} v_{tt} - \Delta v = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) = \mu(x) u_0(x), \quad v_t(x,0) = \mu(x) u_1(x), \quad x \in \mathbb{R}^n. \end{cases}
$$

Then it is known (cf. Klainerman [3] and W. von Wahl [15]) that if $n \geq 2$, then the solution v to the problem (CP) satisfies

(3.1)

$$
\|v(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n-1}{2}} \left(\|v(0)\|_{W^{M,1}(\mathbb{R}^n)} + \|v_t(0)\|_{W^{M-1,1}(\mathbb{R}^n)} \right).
$$

Now let us take a smooth function so that $\psi(x) = 1$ if $|x| \geq R+1$ and $= 0$ if $|x| \leq R$. Then ψv satisfies the following Cauchy problem :

$$
\begin{cases} (\psi v)_{tt} - \Delta(\psi v) = g(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ (\psi v)(x,0) = \mu(x) u_0(x), \quad (\psi v)_t(x,0) = \mu(x) u_1(x), \quad x \in \mathbb{R}^n, \end{cases}
$$

where we set $g(x,t) = -2\nabla\psi \cdot \nabla v - (\Delta\psi)v + 2\psi_t v_t + \psi_{tt} v$. Then, setting $w = \widetilde{u} - \psi v$, we see that w satisfies the following initial-boundary value problem :

$$
(\mathrm{P})_w \begin{cases} w_{tt} - \Delta w + a(x,t) w_t = -g(x,t), & (x,t) \in \Omega \times (0,\infty), \\ w(x,0) = w_t(x,0) = 0, & x \in \Omega, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}
$$

For the local energy of w in the domains $\widetilde{\Omega}(t)$ and $\Omega(R+1)$, combining Propositions 2.2, 2.3, 2.4 with Poincare's inequality, we have the following estimates.

Proposition 3.2. *Let w be a solution of problem $(\mathrm{P})_w$. Then we have the following assertions:* (i) *If $0<\alpha<\frac{1}{2}$, there exists a constant $C>0$ such that for $t\geq 0$,*

$$\|\overline{D}w(t)\|_{W^{m,\infty}(\widetilde{\Omega}(t))}\leq CI^{(e)}_{3M+m}e^{-\lambda' t^{\beta}},$$
$$\|\overline{D}w(t)\|_{H^{m}(\widetilde{\Omega}(t))}\leq CI^{(e)}_{2M+m}e^{-\lambda' t^{\beta}},$$

where $\widetilde{\Omega}(t)=\{x\in\overline{\Omega};\,|x|<(R+t)^{\alpha}+1\}$. (ii) *Let γ be a number with $\gamma>n-1$. If $\alpha=0$, then there exists a constant $C(R)$ depending on R such that for $t\geq 0$,*

$$\|\overline{D}w(t)\|_{W^{m,\infty}(\Omega(R+1))}\leq C(R)I^{(\gamma)}_{3M+m}(1+t)^{-\gamma+\frac{n-1}{2}},$$
$$\|\overline{D}w(t)\|_{H^{m}(\Omega(R+1))}\leq C(R)I^{(\gamma)}_{2M+m}(1+t)^{-\gamma+\frac{n-1}{2}}.$$

We must estimate $w(t)$ outside of the domains $\widetilde{\Omega}(t)$ and $\Omega(R+1)$. For this purpose, we set $\widetilde{w}=\psi w$. Then $\widetilde{w}$ satisfies the following Cauchy problem :

$$\begin{cases}\widetilde{w}_{tt}-\Delta\widetilde{w}=g(x,t)+\widetilde{g}(x,t), & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ \widetilde{w}(x,0)=\widetilde{w}_t(x,0)=0, & x\in\mathbb{R}^n,\end{cases}$$

where $\widetilde{g}(x,t)=2\psi_t w_t+\psi_{tt}w-2\nabla\psi\cdot\nabla w-(\Delta\psi)w$. It follows from Duhamel's principle and the decay estimate (3.1) that L^∞ and L^2-norms of $\overline{D}\widetilde{w}(t)$ are estimated as follows :

Lemma 3.3. *Let m be a nonnegative integer. Then we have the following assertions :* (i) *If $0<\alpha<\frac{1}{2}$, there exists a constant C such that*

$$\|\overline{D}\widetilde{w}(t)\|_{W^{m,\infty}(\mathbb{R}^n)}\leq CI^{(e)}_{3M+m}(1+t)^{-\frac{n-1}{2}},$$
$$\|\overline{D}\widetilde{w}(t)\|_{H^{m}(\mathbb{R}^n)}\leq CI^{(e)}_{2M+m+1}.$$

(ii) *If $\alpha=0$, there exists a constant $C(R)$ depending on R such that for $t\geq 0$,*

$$\|\overline{D}\widetilde{w}(t)\|_{W^{m,\infty}(\mathbb{R}^n)}\leq CI^{(\gamma)}_{3M+m}(1+t)^{-\frac{n-1}{2}},$$
$$\|\overline{D}\widetilde{w}(t)\|_{H^{m}(\mathbb{R}^n)}\leq CI^{(\gamma)}_{2M+m+1}.$$

The following proposition is an immediate consequence of Lemma 3.3, and we can obtain estimates of $w(t)$ outside of the domains $\widetilde{\Omega}(t)$ and $\Omega(R+1)$ as follows:

Proposition 3.4. *Let m be a nonnegative integer. Then we have the following assertions :* (i) *If $0<\alpha<\frac{1}{2}$, there exists a constant C such* *that*

$$\|\overline{D}\widetilde{w}(t)\|_{W^{m,\infty}(\widetilde{\Omega}(t)^c)} \le CI^{(e)}_{3M+m}(1+t)^{-\frac{n-1}{2}},$$
$$\|\overline{D}\widetilde{w}(t)\|_{H^m(\widetilde{\Omega}(t)^c)} \le CI^{(e)}_{2M+m+1}.$$

(ii) *If* $\alpha = 0$, *there exists a constant* $C(R)$ *depending on* R *such that for* $t \ge 0$,

$$\|\overline{D}\widetilde{w}(t)\|_{W^{m,\infty}(\Omega(R+1)^c)} \le CI^{(\gamma)}_{3M+m}(1+t)^{-\frac{n-1}{2}},$$
$$\|\overline{D}w(t)\|_{H^m(\Omega(R+1)^c)} \le CI^{(\gamma)}_{2M+m+1}.$$

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. It suffices for our purpose to prove the case $0 < \alpha < \frac{1}{2}$, since the case $\alpha = 0$ can be handled in a similar way. It follows from Propositions 3.2 and 3.3 that

$$\begin{aligned}\|\overline{D}w(t)\|_{W^{m,\infty}(\Omega)} &\le \|\overline{D}w(t)\|_{W^{m,\infty}(\widetilde{\Omega}(t))} + \|\overline{D}w(t)\|_{W^{m,\infty}(\widetilde{\Omega}(t)^c)} \\ &\le CI^{(e)}_{3M+m}(1+t)^{-\frac{n-1}{2}},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\|\overline{D}w(t)\|_{H^m(\Omega)} &\le \|\overline{D}w(t)\|_{H^m(\widetilde{\Omega}(t))} + \|\overline{D}w(t)\|_{H^m(\widetilde{\Omega}(t)^c)} \\ &\le CI^{(e)}_{2M+m+1}.\end{aligned} \tag{3.3}$$

Notice that

$$\|\mu u_0\|_{W^{M+m+1,1}(\mathbb{R}^n)} + \|\mu u_1\|_{W^{M+m,1}(\mathbb{R}^n)} \le CI^{(e)}_{M+m+1},$$

$$\|\mu u_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \le C\left(\int_\Omega e^{2|x|}u_1^2\,dx\right)^{\frac{1}{2}} = C\|e^{|\cdot|}u_1\|_{L^2(\Omega)}.$$

Then we see from the decay estimate (3.1) and Proposition 2.4 that

$$\begin{aligned}&\|\overline{D}(\psi v)(t)\|_{W^{m,\infty}(\mathbb{R}^n)} \\ &\le C\left(\|\mu u_0\|_{W^{M+m+1,1}(\mathbb{R}^n)} + \|\mu u_1\|_{W^{M+m,1}(\mathbb{R}^n)}\right)(1+t)^{-\frac{n-1}{2}} \\ &\le CI^{(e)}_{M+m+1}(1+t)^{-\frac{n-1}{2}},\end{aligned} \tag{3.4}$$

$$\|\overline{D}^m(\psi v)(t)\|_{L^2(\mathbb{R}^n)} \le C\left(I_{m+1} + \|e^{|\cdot|}u_1\|_{L^2(\Omega)}\right). \tag{3.5}$$

Since $\widetilde{u} = w + \psi v$, we combine (3.2) and (3.3) with (3.4) and (3.5), respectively, to complete the proof of Proposition 3.1. □

Next, we introduce the decay for the solution $\widehat{u}$ of problem $\widehat{(\mathrm{P})}$.

Proposition 3.5. *For a nonnegative integer m, there exists a constant $C > 0$ such that for $t \geq 0$,*

$$\|\overline{D}\widehat{u}(t)\|_{W^{m,\infty}(\Omega)} \leq CI_{M+m+1}(1+t)^{-\frac{n-1}{2}},$$

$$\|\overline{D}\widehat{u}(t)\|_{H^m(\Omega)} \leq CI_{M+m+1}.$$

The proof of this proposition can be given much easier than the previous arguments, if we note that the supports of data $\widehat{u}(0)$, $\widehat{u}_t(0)$ are compact. Thus we may omit the details.

Proof of Theorems 1 and 2 completed. It follows from Propositions 3.1 and 3.5 that if $0 < \alpha < \frac{1}{2}$, then

$$\|\overline{D}u(t)\|_{W^{m,\infty}(\Omega)} \leq I^{(e)}_{3M+m}(1+t)^{-\frac{n-1}{2}}, \tag{3.6}$$

$$\|\overline{D}u(t)\|_{H^m(\Omega)} \leq I^{(e)}_{2M+m}. \tag{3.7}$$

Interpolating between L^∞-estimate (3.6) and L^2-estimate (3.7), we obtain L^p-estimate

$$\|\overline{D}u(t)\|_{W^{m,p}(\Omega)} \leq CI^{(e)}_{3M-\frac{2(M-1)}{p}+m}(1+t)^{-\frac{n-1}{2}(1-\frac{2}{p})}.$$

For more details see [4]. As to the case $\alpha = 0$, we can prove the same argument as above (see [5]). The proofs of Theorems 1 and 2 are complete.

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