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GLOBAL L^2-BOUNDEDNESS THEOREMS FOR A CLASS OF FOURIER INTEGRAL OPERATORS AND THEIR APPLICATION

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This article is based on the joint work with Michael Ruzhansky (Imperial College) which will appear in [13], [14], [15] and so on.

Fourier integral operators

We consider the following Fourier integral operator:

$$Tu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y)\, dy d\xi \tag{1}$$

($x \in \mathbb{R}^n$), where $a(x,y,\xi)$ is an amplitude function and $\phi(x,y,\xi)$ is a real phase function of the form

$$\phi(x,y,\xi) = x \cdot \xi + \varphi(y,\xi).$$

Note that, by the equivalence of phase function theorem, Fourier integral operators with the local graph condition can always be written in this form locally.

Local L^2 mapping property of (1) has been established by Hörmander [9] and Eskin [7]. The aim of this article is to present *global* L^2-boundedness properties of operators (1).

When is T globally L^2-bounded?

• **(Asada and Fujiwara** [1]**)** Assume that all the derivatives of $a(x,y,\xi)$ and all the derivatives of each entry of the matrix

$$D(\phi) = \begin{pmatrix} \partial_x \partial_y \phi & \partial_x \partial_\xi \phi \\ \partial_\xi \partial_y \phi & \partial_\xi \partial_\xi \phi \end{pmatrix}$$

are bounded. Also assume that $|\det D(\phi)| \geq C > 0$. Then T is $L^2(\mathbb{R}^n)$-bounded.

This result was used to construct the fundamental solution of Schrödinger equation in the way of Feynman's path integral.

(The result of Kumano-go [12] was used to construct the fundamental solution of hyperbolic equations, and it requires that

$$J(y,\xi) = \phi(x,y,\xi) - (x-y) \cdot \xi$$

satisfies

$$\left| \partial_y^\alpha \partial_\xi^\beta J(y,\xi) \right| \leq C_{\alpha\beta} (1+|\xi|)^{1-|\beta|}$$

for all α and β.)

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However, there one had to make a quite restrictive and not always natural assumption on the boundedness of $\partial_\xi \partial_\xi \phi$, which fails in many important cases.

The case we have in mind is

$$\phi(x,y,\xi) = x \cdot \xi - y \cdot \psi(\xi), \tag{2}$$

where $\psi(\xi)$ is a smooth function of growth order 1. If we take $\psi(\xi) = \xi$, then we have $\phi(x,y,\xi) = x \cdot \xi - y \cdot \xi$, and the operator T defined by it is a pseudo-differential operator.

We cannot use Asada-Fujiwara's result with our example (2), because the boundedness of the entries of $\partial_\xi \partial_\xi \phi$ fails generally. (We cannot use Kumano-go's either by the same reason.)

Why is the phase function (2) important?

Because it is used to represent a canonical transformations. In fact, if we take $a(x,y,\xi) = 1$, we have

$$Tu(x) = F^{-1}[(Fu)(\psi(\xi))](x) \tag{3}$$

hence

$$T \cdot \sigma(D) = (\sigma \circ \psi)(D) \cdot T.$$

Especially, for a positive and homogeneous function $p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ of degree 1, we have the relation

$$T \cdot (-\Delta) \cdot T^{-1} = p(D)^2 \tag{4}$$

if we take

$$\psi(\xi) = p(\xi) \frac{\nabla p(\xi)}{|\nabla p(\xi)|} \tag{5}$$

and assume that the hypersurface

$$\Sigma = \{\xi; p(\xi) = 1\}$$

has non-vanishing Gaussian curvature.

The curvature condition on Σ means that the Gauss map

$$\frac{\nabla p}{|\nabla p|} : \Sigma \to S^{n-1}$$

is a global diffeomorphism and its Jacobian never vanishes. (See Kobayashi and Nomizu [11].) Hence, we can construct the inverse C^∞-map $\psi^{-1}(\xi)$ of $\psi(\xi)$ defined by (5). On account of (3), the inverse T^{-1} can be given by replacing ψ by ψ^{-1}.

The L^2-property of the Laplacian $-\Delta$ is well known in various situations. Our objective is to know the L^2-property of the operator T, so that we can extract the L^2-property of the operator $p(D)^2$ from that of the Laplacian.

Main result

The following is our main result, which is expected to have many applications. For $m \in \mathbb{R}$, we set

$$\langle x\rangle^m = \left(1+|x|^2\right)^{m/2}.$$

Let $L^2_m(\mathbb{R}^n)$ be the set of functions f such that the norm

$$\|f\|_{L^2_m(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\langle x\rangle^m f(x)|^2\, dx\right)^{1/2}$$

is finite.

Theorem 1. *Let $\phi(x,y,\xi) = x\cdot\xi + \varphi(y,\xi)$. Assume that*

$$|\det \partial_y\partial_\xi\varphi(y,\xi)| \geq C > 0,$$

and all the derivatives of entries of $\partial_y\partial_\xi\varphi$ are bounded. Also assume that

$$\left|\partial_\xi^\alpha\varphi(y,\xi)\right| \leq C_\alpha\langle y\rangle \quad \text{for all } |\alpha|\geq 1,$$
$$\left|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)\right| \leq C_{\alpha\beta\gamma}\langle x\rangle^{-|\alpha|}$$

for all α, β, and γ. Then T is bounded on $L^2_m(\mathbb{R}^n)$ for any $m\in\mathbb{R}$.

Theorem 1 says that, if amplitude functions $a(x,y,\xi)$ have some decaying properties with respect to x, we do not need the boundedness of $\partial_\xi\partial_\xi\phi$ for the L^2-boundedness, as required in Asada-Fujiwara [1], and we can have weighted estimates, as well.

The same is true when both phase and amplitude functions have some decaying properties with respect to y.

Theorem 2. *Let $\phi(x,y,\xi) = x\cdot\xi + \varphi(y,\xi)$. Assume that*

$$|\det \partial_y\partial_\xi\varphi(y,\xi)| \geq C > 0.$$

Also assume that

$$\left|\partial_y^\alpha\partial_\xi^\beta\varphi(y,\xi)\right| \leq C_\alpha\langle y\rangle^{1-|\alpha|} \quad \text{for all } \alpha,\ |\beta|\geq 1,$$
$$\left|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)\right| \leq C_{\alpha\beta\gamma}\langle y\rangle^{-|\beta|}$$

for all α, β, and γ. Then T is bounded on $L^2_m(\mathbb{R}^n)$ for any $m\in\mathbb{R}$.

If the amplitude $a(x,y,\xi)$ is independent of the variable x or y, the decaying property can be automatically satisfied. Furthermore, we can reduce the regularity assumptions for amplitude and phase functions in this case.

Theorem 3. *Let $\phi(x,y,\xi) = x\cdot\xi + \varphi(y,\xi)$ and $a(x,y,\xi) = a(x,\xi)$. Assume that*

$$|\det \partial_y\partial_\xi\varphi(y,\xi)| \geq C > 0$$

and each entry $h(y,\xi)$ of $\partial_y\partial_\xi\varphi(y,\xi)$ satisfies

$$\left|\partial_y^\alpha h(y,\xi)\right| \leq C_\alpha, \qquad \left|\partial_\xi^\beta h(y,\xi)\right| \leq C_\beta$$

for $|\alpha|, |\beta| \leq 2n+1$. Also assume

$$\partial_x^\alpha\partial_\xi^\beta a(x,\xi) \in L^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$$

for one of the followings:

(i) $\alpha, \beta \in \{0,1\}^n$, (ii) $|\alpha|, |\beta| \leq [n/2]+1$,

(iii) $|\alpha| \leq [n/2]+1$, $\beta \in \{0,1\}^n$, (iv) $\alpha \in \{0,1\}^n$, $|\beta| \leq [n/2]+1$.

Then T is $L^2(\mathbb{R}^n)$-bounded.

Theorem 3 with $\varphi(y,\xi) = -y \cdot \xi$ is a refined version of known results on the L^2-boundedness of pseudo-differential operators with non-regular symbols: (i) with $\alpha, \beta \in \{0,1,2,3\}^n$ is due to Calderón and Vaillancourt [3], (ii) is due to Cordes [6], and conditions (iii) with $|\alpha| \leq [n/2]+1$, $\beta \in \{0,1,2\}^n$, is due to Coifman and Meyer [5].

Theorem 4. *Let $\phi(x,y,\xi) = x\cdot\xi + \varphi(y,\xi)$ and $a(x,y,\xi) = a(y,\xi)$. Assume that*

$$\left|\partial_y^\alpha \partial_\xi^\beta a(y,\xi)\right| \leq C_{\alpha\beta},$$

for $|\alpha|, |\beta| \leq 2n+1$. Also assume that

$$|\det \partial_y \partial_\xi \varphi(y,\xi)| \geq C > 0$$

and each entry $h(y,\xi)$ of $\partial_y\partial_\xi\varphi(y,\xi)$ satisfies

$$|\partial_y^\alpha h(y,\xi)| \leq C_\alpha, \qquad \left|\partial_\xi^\beta h(y,\xi)\right| \leq C_\beta$$

for $|\alpha|, |\beta| \leq 2n+1$. Then the operator T is $L^2(\mathbb{R}^n)$-bounded.

An example of how to use our results

Kato and Yajima [10] showed that the classical Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_x)u(t,x) = 0, \\ u(0,x) = g(x) \end{cases}$$

has the global smoothing estimate

(6) $$\left\|\langle x\rangle^{-1}\langle D\rangle^{1/2} u\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}_x^n)} \leq C\|g\|_{L^2(\mathbb{R}_x^n)},$$

where $n \geq 3$.

From this fact, we can extract a similar estimate for generalized Schrödinger equations

(7) $$\begin{cases} (i\partial_t - p(D)^2)u(t,x) = 0, \\ u(0,x) = g(x). \end{cases}$$

• **Assumption.** $p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ is homogeneous of order 1, $p(\xi) > 0$, and the hypersurface $\Sigma = \{\xi; p(\xi) = 1\}$ has non-vanishing Gaussian curvature.

Remember that we have the relation

$$T^{-1} \cdot p(D)^2 = (-\Delta) \cdot T^{-1}$$

by (4). Operating T^{-1} from the left hand side of equation (7), we have, by this relation,

$$\begin{cases} (i\partial_t - \triangle)T^{-1}u(t,x) = 0, \\ T^{-1}u(0,x) = T^{-1}g(x). \end{cases}$$

Hence, from (6) and Theorem 1, we obtain the following conclusion:

Theorem 5. *Suppose $n \geq 3$. Under the assumption above, the solution $u(t,x)$ to generalized Schrödinger equation* (7) has the same global smoothing estimate (6) as the classical one has.

Remark 1. Walther [16] consider the case of radially symmetric $p(\xi)^2$. Theorem 5 says that we can treat more general case.

Smoothing effect with a structure

By using the idea above, we can have more refined global smoothing estimates. In order to state them, we introduce some notations:

- Classical orbit determined by $p(D)^2$:

$$\begin{cases} \dot{x}(t) = \nabla_\xi p^2(\xi(t)), \quad \dot{\xi}(t) = 0 \\ x(0) = 0, \quad \xi(0) = k. \end{cases} \tag{8}$$

- The set of the path of all classical orbits:

$$\begin{aligned} \Gamma_p =& \{(x(t), \xi(t));\ \text{sol. of (8)},\ t \geq 0, k \in \mathbb{R}^n \setminus 0\} \\ =& \{(t\nabla p(\xi), \xi); \xi \in \mathbb{R}^n \setminus 0, t > 0\}. \end{aligned}$$

- Notation:

$$\sigma(x,\xi) \sim \langle x \rangle^a |\xi|^b$$

$$\Longleftrightarrow$$

$$\begin{cases} \sigma(x,\xi) \in C^\infty(\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus 0)), \\ \sigma(\lambda x, \xi) = \lambda^a \sigma(x,\xi)\,;\ (\lambda > 1,\ |x| \gg 1), \\ \sigma(x, \lambda\xi) = \lambda^b \sigma(x,\xi)\,;\ (\lambda > 0). \end{cases}$$

Theorem 6. *Suppose $n \geq 2$. Assume*

$$\sigma(x,\xi) = 0 \text{ on } \Gamma_p, \quad \sigma(x,\xi) \sim \langle x \rangle^{-1/2} |\xi|^{1/2}.$$

Then the solution u to equation (7) satisfies

$$\|\sigma(X,D)u\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|g\|_{L^2(\mathbb{R}^n_x)}.$$

Remark 2. Without the structure condition $\sigma(x,\xi)=0$ on Γ_p, we have the esitmate in Theorem 6 for the followings:

- $\sigma(x,\xi)=\langle x\rangle^{-s}|\xi|^{1/2}$ $(s>1/2)$ (Ben-Artzi and Klainerman [2])
- $\sigma(x,\xi)=|x|^{\alpha-1}|\xi|^{\alpha}$ $(0<\alpha<1/2)$ (Kato and Yajima [10])

We have a similar result for inhomogeneous equations

$$\begin{cases} \left(i\partial_t - p(D)^2\right) u(t,x) = f(t,x) \\ u(0,x)=0. \end{cases} \tag{9}$$

Theorem 7. *Suppose $n\geq 2$. Assume*

$$\sigma(x,\xi)\geq 0, \quad \sigma(x,\xi)=0 \text{ on } \Gamma_p$$
$$\sigma(x,\xi)\sim \langle x\rangle^{1/2}|\xi|.$$

Then the solution u to (9) satisfies the estimate

$$\|\sigma(X,D_x)u\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\left\|\langle x\rangle^{3/2} f\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}.$$

Combining Theorems 6 and 7, we have an estimate for the equation

$$\begin{cases} \left(i\partial_t - p(D)^2\right) u(t,x) = f(t,x) \\ u(0,x)=g(x). \end{cases} \tag{10}$$

Corollary 8. *Suppose $n\geq 2$ and $s,\tilde{s}\geq 0$. Assume*

$$\sigma(x,\xi)\geq 0, \quad \sigma(x,\xi)=0 \text{ on } \Gamma_p$$
$$\sigma(x,\xi)\sim |\xi|.$$

Then the solution u to (10) satisfies the estimate

$$\left\|\langle x\rangle^{1/2}\sigma(X,D_x)u\right\|_{H^s_t(H^{\tilde{s}}_x)} \leq C\left\|\langle x\rangle\langle D_x\rangle^{2s+\tilde{s}+1/2} g\right\|_{L^2(\mathbb{R}^n_x)} + C\left\|\langle x\rangle^{3/2} f\right\|_{H^s_t(H^{\tilde{s}}_x)}.$$

Derivative Nonlinear Schrödinger Equation

Finally, we refer to further applications. We consider the following nonlinear Schrödinger equation:

$$(11)\qquad \begin{cases} (i\partial_t + \Delta_x)\, u(t,x) = |\nabla u(t,x)|^N \\ \quad u(0,x) = g(x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n. \end{cases}$$

What is the condition of the initial data $g(x)$ for equation (11) to have time global solution? There are some answers:

- $N \geq 3$ (Chihara [4]). Smooth, rapidly decay, and sufficiently small.
- $N \geq 2$ (Hayashi, Miao and Naumkin [8]). $g \in H^{[n/2]+5}$, rapidly decay, and sufficiently small.

Question: Can we weaken the smoothness assumption for $g(x)$?

Answer: Yes if the non-linear term has a "*structure*"!

Instead of (11), we consider

$$(12)\qquad \begin{cases} (i\partial_t - p(D)^2)\, u(t,x) = |\sigma(X, D_x)u|^N \\ \quad u(0,x) = g(x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \end{cases}$$

where

$$(13)\qquad \begin{cases} \sigma(x,\xi) \geq 0, \quad \sigma(x,\xi) = 0 \quad \text{on} \quad \Gamma_p, \\ \sigma(x,\xi) \sim |\xi|. \end{cases}$$

- Examples of nonlinear terms which satisfy (13) in the case $p(D)^2 = -\Delta_x$:

$$\sigma(x,\xi) = \left| \frac{x}{|x|} \wedge \xi \right|^2 |\xi|^{-1} \quad \text{for large } |x|$$

Theorem 9. *Suppose $n \geq 2$, $s > (n+3)/2$, and $N \geq 3$. Assume that $\langle x\rangle\langle D_x\rangle^s g \in L^2$ and its L^2-norm is sufficiently small. Then equation* (12) has a time global solution. (In the case $N = 2$, we need more structure.)

Key point to the proof of Theorem 9. Use Corollary 8 with $f = |\sigma(X, D_x)u|^N$. The space $H^s_t(H^{\tilde{s}}_x)$ is an algebra if $s > 1/2$ and $\tilde{s} > n/2$. Then we have

$$\begin{aligned} \left\| \langle x\rangle^{3/2} |\sigma(X,D_x)u|^N \right\|_{H^s_t(H^{\tilde{s}}_x)} &\leq \left\| \langle x\rangle^{1/(2N)+1/N} \sigma(X,D_x)u \right\|^N_{H^s_t(H^{\tilde{s}}_x)} \\ &\leq \left\| \langle x\rangle^{1/2} \sigma(X,D_x)u \right\|^N_{H^s_t(H^{\tilde{s}}_x)} \end{aligned}$$

if $N \geq 3$.

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