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Power series solutions of the inhomogeneous heat equation

Werner Balser*
Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm, Germany
balser@mathematik.uni-ulm.de

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Abstract

We investigate formal solutions of the inhomogeneous heat equation, where the inhomogenuity is a k-summable formal power series in t with coefficients that are holomorphic in a disc.

1 Introduction

Recently a new interest has arisen in power series solutions of partial differential equations, and in particular the *non-Kowalewskian case* of solutions with radius of convergence equal to zero has been studied: Various authors have been establishing the *Gevrey order* for such power series solutions, while some of the most recent work concerns the question of their summability. The case of a Cauchy problem for the complex heat equation in one spatial variable has been more or less completely analyzed in articles of *Lutz, Miyake, and Schäfke* [14], resp. *W. Balser* [1]. In subsequent articles, other PDE with constant, and in some cases holomorphic, coefficients have been treated, but up to now the theory is far from fully developped. Without claim of completeness, we list the following articles containing results in this direction: *M. Hibino* [8–12], *M. Miyake* [16–18], *Miyake and Hashimoto* [20], *Miyake and Yoshino* [21–23], *S. Ōuchi* [24–27], *Pliś and Ziemian* [28], *Balser and Miyake* [6], *Miyake* [19], *K. Ichinobe* [13], *Balser and Kostov* [5], *W. Balser* [3], *S. Malek* [15], and *O. Costin and S. Tanveer* [7].

In this article we shall investigate formal solutions for the *inhomogeneous heat equation*, finding their Gevrey order as well as determining their summability properties. This case has been briefly looked at in [5] and shall be investigated here in more detail. It appears possible that this result might be of importance in treating other equations with holomorphic coefficients, using a perturbation technique. In detail we shall use the following notation:

- Throughout this paper, let $\mathcal{D} = \mathcal{D}_r$ denote the open disc of radius $r > 0$ about the origin, where $r = \infty$ may occur, and let $f_j(z)$, for $j \in \mathbb{N}_0$, denote functions that all are holomorphic in $\mathcal{D}$. In terms of these functions, we shall be concerned with two formal power series in t given by

$$\hat{f}(t,z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} f_j(z), \qquad \hat{u}(t,z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} u_j(z), \qquad u_j(z) = \sum_{\substack{\nu,\mu\ge 0\\ \nu+\mu=j}} f_\nu^{(2\mu)}(z), \quad (1.1)$$

where $f_\nu^{(2\mu)}(z)$ denotes the (2μ)-th derivative of $f_\nu(z)$.

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The series $\hat{u}(t,z)$ can be easily seen to be the unique power series solution of the Cauchy problem for an inhomogeneous heat equation, in one spatial dimension, of the form

$$u_t = u_{zz} + \partial_t \hat{f}(t,z), \qquad u(0,z) = f_0(z).$$

Note that every inhomogeneous heat equation with an inhomogenuity that is a holomorphic function in a polydisc about the origin of $\mathbb{C}^2$, or a formal power series in t and z, can be written in this form. In view of this fact, it appears natural to assume that the power series $\hat{f}(t,z)$ converges – *however, even then the solution $\hat{u}(t,z)$ will, in general, be a formal series in the sense that it fails to converge for every $t \neq 0$.* For this reason, it is more suitable here to allow that the series $\hat{f}(t,z)$ is formal as well. In this situation, the correspondence $\hat{f}(t,z) \longmapsto \hat{u}(t,z)$ is a bijective mapping of $\mathcal{O}_{\mathcal{D}}[[t]]$ (denoting the differential algebra of all formal power series in t with coefficients that are holomorphic in the disc $\mathcal{D}$) into itself. The main problem addressed in this article is to give necessary and/or sufficient conditions on $\hat{f}(t,z)$ so that the corresponding solution $\hat{u}(t,z)$ is of Gevrey order $s \geq 0$, or even k-summable in a direction d. To see that such cases indeed exist, note that the general theory of formal power series and multisummability, presented, e. g., in [2], ensures that, in case the formal solution $\hat{u}(t,z)$ is of Gevrey order s, or k-summable in a direction d, or multisummable, then the same holds for its partial derivatives and antiderivatives, and therefore for $\hat{f}(t,z)$ as well. So the main problem is whether, and if so, how we can recognize in terms of $\hat{f}(t,z)$, or equivalently in terms of the functions $f_j(z)$, when these situations occur.

2 Definitions and technical results

In the definitions and results of this section, we shall consider an arbitrary formal power series in $\mathcal{O}_{\mathcal{D}}[[t]]$, written in the form

$$\hat{x}(t,z) = \sum_0^\infty \frac{t^j}{j!}\, x_j(z), \qquad f_j(z) \in \mathcal{O}_{\mathcal{D}}.$$

Due to the form chosen here, we set $s_+ = s+1$ and say that such a series is *of Gevrey order* $s \geq 0$ provided that we can find constants $\rho \in (0,r]$ and $C, K > 0$ such that

$$|x_j(z)| \leq C K^j \Gamma(1+s_+ j) \qquad \forall\, j \geq 0, \quad |z| < \rho. \tag{2.1}$$

Note that this definition, when the functions $x_j(z)$ all are constants, coincides with the standard definition of the Gevrey order of power series. Expanding $x_j(z) = \sum_0^\infty z^n\, x_{jn}/n!$ for $|z| < \rho$, we define

$$y(t,z) = \sum_{j=0}^\infty \frac{t^j}{\Gamma(1+s_+ j)}\, x_j(z), \qquad y_n(t) = \sum_{j=0}^\infty \frac{t^j}{\Gamma(1+s_+ j)}\, x_{jn} \qquad \forall\, n \geq 0. \tag{2.2}$$

In these terms, we can rephrase the definition of Gevrey order as follows:

Lemma 1 *For power series $\hat{x}(t,z)$, $y(t,z)$, and $y_n(t)$ as above, the following statements are equivalent:*

(a) *$\hat{x}(t,z)$ is of Gevrey order $s \geq 0$.*

(b) *There exist $\rho, C, K > 0$, with $\rho \leq r$, so that*

$$|x_{jn}| \leq C K^j \rho^{-n}\, n!\, \Gamma(1+s_+ j) \qquad \forall\, j,n \geq 0. \tag{2.3}$$

(c) *There exist $\rho, C, K > 0$, so that all $y_n(t)$ converge for $|t| < \rho$, and*

$$|y_n(t)| \leq C K^n n! \qquad \forall\, n \geq 0, \quad t \in \mathcal{D}_\rho. \tag{2.4}$$

(d) *There exist $\rho_1, \rho_2 > 0$, with $\rho_2 \leq r$, so that $y(t,z)$ converges for $|t| < \rho_1$ and $|z| < \rho_2$.*

Proof: Suppose (a). Using Cauchy's formula, we conclude from (2.1) that (2.3) holds, which shows (b). Thus we have $|y_n(t)| \leq C\rho^{-n}\, n! \sum_0^\infty (K|t|)^j$, from which (c) follows, for suitable $C, K, \rho > 0$ different from those in (2.1). Since $y(t,z) = \sum_n y_n(t)/n!$, we conclude from (b) that $|y(t,z)| \leq C \sum_n (K|z|)^n$, which implies (d). Finally, if (d) holds, then $y(t,z)$ is bounded for $|t| \le r_1 < \rho_1$ and $|z| \le r_2 < \rho_2$ and this, together with Cauchy's formula, implies (a). □

For $\hat{x}(t,z)$ as above, and for $k > 0$ and $d \in \mathbb{R}$, we say that this series *is k-summable in the direction d*, if the following two conditions are satisfied:

- There exist $\rho \in (0, r]$, and $R > 0$ that may depend upon ρ, such that for $s_+ = 1 + 1/k$ the power series $y(t,z)$, defined in (2.2), converges absolutely for $|z| < \rho$ and $|t| < R$. In other words, this says that $\hat{x}(t,z)$ is of Gevrey order $s = 1/k$.
- There exists a $\delta > 0$ so that for every $z \in \mathcal{D}_\rho$ the function $y(t,z)$ can be continued with respect to t into the sector $S_{d,\delta} = \{t : 2\,|d - \arg t| < \delta\}$. Moreover, for every $\delta_1 < \delta$ there exist constants $C, K > 0$ so that
$$\sup_{|z|<\rho} |y(t,z)| \;\le\; C \exp[K\,|t|^k] \qquad \forall\, t \in S_{d,\delta_1}\,. \tag{2.5}$$
It shall be convenient to say that this means that $y(t,z)$ is of exponential growth at most of order k in the sector $S = S_{d,\delta}$, by which we mean to say implicitly that the growth estimate (2.5) is uniform in z, for z on a sufficiently small disc.

Observe that the series representing $y(t,z)$ is *not* the formal Borel transform of $\hat{x}(t,z)$. Therefore, the definition given above is that of a certain type of *moment summability* which, however, was proven in [2] to be *equivalent to the standard definition of k-summability* and is more suitable to series of the form that is investigated here. The sum $x(t,z)$ of the series $\hat{x}(t,z)$ is not given by the Laplace transform of order k of $y(t,z)$ – instead, one has to use another integral transformation that has been introduced by *J. Ecalle* under the name of *acceleration operator* and whose definition can also be found in [2, Section11.1]. Nonetheless, it can be shown that this sum is holomorphic in $G_d \times D_\rho$, with a sectorial region G_d of opening larger than π/k and bisecting direction $\arg t = d$. For the case when all the functions $x_j(z)$ are constants, the above definition of k-summability is equivalent to *J.-P. Ramis'* [29] original one.

The functions $y(t,z)$ and $y_n(t)$ defined in (2.2) shall here be referred to as *associated to the formal series $\hat{x}(t,z)$*. Moreover, it shall also be convenient to introduce the formal power series

$$\hat{x}_n(t) \;=\; \partial_z^n\,\hat{x}(t,z)\big|_{z=0} \;=\; \sum_{j=0}^{\infty} x_{jn}\,\frac{t^j}{j!} \qquad \forall\, n \ge 0\,. \tag{2.6}$$

As an alternative interpretation of summability of series in two variables in terms of series in one variable, we now state a result that has been proven in [4] and is quite analogous to the lemma shown above:

Lemma 2 *For power series $\hat{x}(t,z)$, $\hat{x}_n(t)$, $y(t,z)$, and $y_n(t)$ as above, the following statements are equivalent:*

(a) *The formal series $\hat{x}(t,z)$ is k-summable in the direction d.*

(b) *The formal series $\hat{x}_n(t)$ all are k-summable in the direction d. Moreover, there exists a sectorial region G that is independent of n and has opening larger than π/k and bisecting direction d, in which all sums $x_n(t)$ of the series $\hat{x}_n(t)$ are holomorphic, for $n \ge 0$. Finally, for every closed subsector $\overline{S}$ in G there exist constants $C, K > 0$, independent of n, so that*
$$|x_n^{(\ell)}(t)| \;\le C\,K^{n+\ell}\,n!\,\Gamma(1 + s_+\,\ell) \qquad \forall\; n, \ell \ge 0\,,\; t \in \overline{S}\,. \tag{2.7}$$

(c) *The series $y_n(t)$ all converge for $|t| < r_1$, with some $r_1 > 0$ that is independent of n. Moreover, there exists a $\delta > 0$ so that all functions $y_n(t)$ can be holomorphically continued into the sector $S_{d,\delta}$. Finally, for every $\delta_1 < \delta$ there exist constants $C, K > 0$, independent of n, so that*
$$|y_n(t)| \;\le\; C^n\,n!\;\exp[K\,|t|^k] \qquad \forall\, t \in S_{d,\delta_1}\,,\quad \forall\, n \ge 0\,.$$

Remark 1: Roughly speaking, this lemma says that the summability of a series with coefficients that are holomorphic functions of a variable z is equivalent to uniform summability of countably many series with constant coefficients. While it is, in general, simpler to deal with series, or functions, in one instead of two variables, this advantage is counterbalanced by the fact that instead of one series we are left with infinitely many to verify their summability. However, as shall become clear in Section 4, at least for the case of the heat equation, for formal solutions of PDE these series are strongly interrelated, so that indeed it suffices to summability of finitely many of them. □

3 Gevrey order

Given a series $\hat{x}(t,z)$ as above, we have considered functions $x_j(z)$, resp. constants x_{jn} that, up to factorials, are the coefficients of this series, and defined other series $\hat{x}_n(t)$, resp. functions $y(t,z)$ and $y_n(t)$. For the series $\hat{u}(t,z)$ and $\hat{f}(t,z)$ in the introduction we shall define $u_j(z), f_j(z)$, u_{jn}, f_{jn}, and $\hat{u}_n(t), \hat{f}_n(t)$ accordingly, and shall use letters v and g, instead of y, to denote the corresponding functions. In these terms we shall now characterize the cases when $\hat{u}(t,z)$ is of Gevrey order s.

Theorem 1 *For $\hat{f}(t,z)$ and $\hat{u}(t,z)$ as above, the following two cases occur:*

(a) *For $s \geq 1$, the series $\hat{u}(t,z)$ is of Gevrey order s if, and only if, the series $\hat{f}(t,z)$ has Gevrey order s as well.*

(b) *For $0 \leq s < 1$, the series $\hat{u}(t,z)$ is of Gevrey order s if, and only if, the series $\hat{f}(t,z)$ has Gevrey order s and, in addition, the series $v_0(t)$ and $v_1(t)$ have positive radius of convergence.*

Proof: If $\hat{u}(t,z)$ is of Gevrey order $s \geq 0$, then the same holds for partial derivatives and antiderivatives, and therefore for $\hat{f}(t,z)$ as well. Moreover, convergence of all $v_n(t)$ follows from Lemma 1, so one direction of both (a) and (b) holds true. To show the converse, assume that $\hat{f}(t,z)$ is of Gevrey order s, hence (2.3) holds with f_{jn} in place of x_{jn}. Setting $u_{-1,n} = 0$ for every $n \geq 0$, we conclude from (1.1) that

$$u_{jn} = \sum_{\substack{\nu,\mu\geq 0\\ \nu+\mu=j}} f_{\nu,n+2\mu} = f_{jn} + u_{j-1,n+2} \qquad \forall\, j,n \geq 0\,. \tag{3.1}$$

Estimating as usual, we then obtain for every $j,n \geq 0$

$$|u_{jn}| \leq C \sum_{\substack{\nu,\mu\geq 0\\ \nu+\mu=j}} K^{\nu}\rho^{-n-2\mu}(n+2\mu)!\,\Gamma(1+s_+\nu) \leq C K^j \rho^{-n} \sum_{\substack{\nu,\mu\geq 0\\ \nu+\mu=j}} K^{-\mu}\rho^{-2\mu}\,\Gamma(1+n+2\mu+s_+\nu)\,.$$

In case (a), i. e. $s_+ \geq 2$, we have $\Gamma(1+n+2\mu+s_+\nu) \leq \Gamma(1+n+s_+j)$, which essentially is of the same "magnitude" as $n!\,\Gamma(1+s_+j)$, as far as the question of Gevrey order of $\hat{x}(t,z)$ is concerned. Hence the converse conclusion of (a) is correct. For $s_+ < 2$, however, we have to proceed differently: We conclude from (3.1) that

$$u_{j,2n} = u_{j+n,0} - \sum_{\mu=0}^{n-1} f_{j+n-\mu,2\mu}\,, \qquad u_{j,2n+1} = u_{j+n,1} - \sum_{\mu=0}^{n-1} f_{j+n-\mu,2\mu+1} \qquad \forall\, j,n \geq 0\,. \tag{3.2}$$

By assumption we have in case (b) that $v_0(t)$ has positive radius of convergence, so that for sufficiently large $C, K > 0$

$$|u_{j,0}| \leq C K^j\,\Gamma(1+s_+j) \qquad \forall\, j \geq 0\,.$$

Using this and the same estimate for f_{jn} as above, we obtain from (3.2) that

$$|u_{j,2n}| \leq C K^{j+n}\left[\Gamma\big(1+s_+(j+n)\big) + \sum_{\mu=0}^{n-1} K^{-\mu}\rho^{-2\mu}(2\mu)!\,\Gamma\big(1+s_+(j+n-\mu)\big)\right] \qquad \forall\, j,n \geq 0\,.$$

Since $s_+ < 2$, we have $(2\mu)!\,\Gamma(1+s_+(j+n-\mu)) < \Gamma(1+2n+s_+j)$, and from the same arguments as above we then obtain (2.3), with u_{jn} in place of x_{jn}, for all even n. To prove the same for odd n, we can proceed analogously, using that by assumption $v_1(t)$ has positive radius of convergence. □

Remark 2: Theorem 1 shows that, if the Gevrey order of $\hat{f}(t,z)$ is larger than 1, then both series $\hat{u}(t,z)$ and $\hat{f}(t,z)$ always are of the same Gevrey order. On the other hand, if $\hat{f}(t,z)$ has small Gevrey order or even converges – and this situation naturally occurs in applications – *then the Gevrey order of $\hat{x}(t,z)$ is at most equal to* 1 *but will, in general, be larger than that of $\hat{f}(t,z)$.* To see that such cases can occur, let $f_j(z) \equiv 0$ for $j \geq 1$, i. e., consider a homogeneous Cauchy problem. In this case, $\hat{f}(t,z)$ is independent of t and therefore has Gevrey order $s = 0$. One may check that $u_j(z) = f_0^{(2j)}(z)$, and $u_{jn} = f_{0,n+2j}$, for all $n, j \geq 0$. Therefore, the Gevrey order of $\hat{u}(t,z)$ is at most equal to 1 and will, in fact, be equal to 1 except for the following cases: For $0 \leq s < 1$, the Gevrey order of $\hat{u}(t,z)$, according to Theorem 1, is equal to s if, and only if, $\hat{u}_0(t)$ and $\hat{u}_1(t)$ both have Gevrey order s, which in turn is equivalent to existence of $C, K > 0$ for which $|f_{0,2n}|, |f_{0,2n+1}| \leq C K^n \Gamma(1 + s_+ n)$, for every $n \geq 0$. Such an estimate holds exactly when $f_0(z) = \sum_n f_{0n}\, z^n/n!$ is entire and of exponential growth (in every sector) at most of order $2/(1-s)$. In particular, we rediscover the classical result that *the formal solution of the homogeneous Cauchy problem for the heat equation converges if, and only if, the initial condition is entire and of exponential growth at most* 2. □

4 Summability properties

In this section we shall investigate the summability properties in the case of a series $\hat{u}(t,z)$ of Gevrey order $s \leq 1$. Motivated by Lemma 2, we shall define series $\hat{u}_n(t)$ analogously to (2.6), with x_{jn} replaced by u_{jn}. We then can reformulate part (a) of Theorem 1 as saying that the Gevrey order of the formal solution $\hat{u}(t,z)$ equals s if, and only if, the series $\hat{u}_0(t)$, $\hat{u}_1(t)$, and $\hat{f}(t,z)$ all have the same Gevrey order s. Since $s = 0$ is nothing but saying that these series converge, we shall now restrict to $s > 0$. For this case we can prove a result on $(1/s)$-summability of $\hat{u}(t,z)$ that is completely analogous to Theorem 1:

Theorem 2 *Let $0 < s \leq 1$, and set $k = 1/s$. Then the power series $\hat{u}(t,z)$ is k-summable in a direction d if, and only if, the series $\hat{u}_0(t)$, $\hat{u}_1(t)$, and $\hat{f}(t,z)$ all are k-summable in the direction d.*

Proof: If $\hat{u}(t,z)$ is k-summable in a direction d, general results on k-summability imply the same for partial derivatives and antiderivatives, and hence we can conclude from (1.1) that $\hat{f}(t,z)$ is k-summable in the direction d, too. Moreover, the same holds for $\hat{u}_0(t)$ and $\hat{u}_1(t)$, owing to Lemma 2. To prove the converse, observe that (3.2) implies

$$\hat{u}_{2n}(t) \;=\; \hat{u}_0^{(n)}(t) \;-\; \sum_{\mu=0}^{n-1} \hat{f}_{2\mu}^{(n-\mu)}(t)\,, \qquad \hat{u}_{2n+1}(t) \;=\; \hat{u}_1^{(n)}(t) \;-\; \sum_{\mu=0}^{n-1} \hat{f}_{2\mu+1}^{(n-\mu)}(t) \qquad \forall\, n \geq 0\,. \tag{4.1}$$

From Lemma 2 we obtain that k-summability in the direction d of $\hat{f}(t,z)$ implies the same for all $\hat{f}_n(t)$, and since derivatives are also summable in the same sense, we see that (4.1) ensures k-summability in the direction d for every $\hat{u}_n(t)$. Moreover, the sums $f_n(t)$ of $\hat{f}_n(t)$ all are holomorphic in a sectorial region G with opening larger than π/k and bisecting direction d, and Lemma 2 says that this G does not depend upon n. From the general theory of k-summability we then conclude that the sums $x_n(t)$ of $\hat{x}_n(t)$ also are holomorphic on G, and that (4.1) holds, if we replace all formal series by their sums, for every $t \in G$. In view of Lemma 2, this leaves to prove an estimate of the form (2.7), for $u_n(t)$ in place of $x_n(t)$. This, however, can be done as follows: k-summability in the direction d of $\hat{f}(t,z)$ implies that (2.7) holds for $f_n(t)$, and k-summability of $\hat{u}_0(t)$, $\hat{u}_1(t)$ implies for their sums

$$|u_0^{(\ell)}(t)|, |u_1^{(\ell)}(t)| \;\leq\; C\,K^\ell\,\Gamma(1 + s_+\ell) \qquad \forall\, t \in G,\; \ell \geq 0\,,$$

provided that we take C, K sufficiently large. Hence (4.1) implies

$$|u_{2n}^{(\ell)}(t)| \;\leq\; C\,K^{n+\ell}\left[\Gamma\big(1 + s_+(n+\ell)\big) \;+\; \sum_{\mu=0}^{n-1} K^\mu\,(2\mu)!\,\Gamma\big(1 + s_+(\ell + n - \mu)\big)\right] \qquad \forall\, n \geq 0\,,\; t \in G\,.$$

In the same fashion as in the proof of Theorem 1 one can see that this implies $|u_{2n}^{(\ell)}(t)| \leq C\,K^n\,(2n)!\,\Gamma(1 + s_+\ell)$, for constants C, K that are not necessarily the same as above. Analogously one can estimate $u_{2n+1}(t)$, and this completes the proof. □

Remark 3: In applications, the summability properties of the series $\hat{f}(t,z)$ may usually be known – in fact, in most situations this series shall converge. Hence, to apply Theorem 2 we are left with showing k-summability of $\hat{u}_0(t)$, $\hat{u}_1(t)$. As was already mentioned in Remark 1, the fact that $\hat{u}(t,z)$ is a formal solution of the heat equation reflects in (4.1), which in turn implies that, instead of infinitely many series $\hat{u}_n(t)$ it suffices to show k-summability of $\hat{u}_0(t)$, $\hat{u}_1(t)$ only. Nonetheless, we are still left with the task of *computing* those two series, and this question is addressed below. To simplify this task, we shall first rephrase the problem of verification of summability of $\hat{u}_0(t)$, $\hat{u}_1(t)$: □

Theorem 3 *For $s > 0$, $k = 1/s$, and $d \in \mathbb{R}$, the following statements are equivalent:*

(a) *The power series $\hat{u}_0(t)$ and $\hat{u}_1(t)$ both are k-summable in the direction d.*

(b) *The power series*

$$\hat{\psi}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+j/2)} \sum_{\substack{\nu,\mu\geq 0\\ 2\nu+\mu=j}} f_{\nu\mu} = \sum_{\nu,\mu=0}^{\infty} f_{\nu\mu} \frac{t^{2\nu+\mu}}{\Gamma(1+\nu+\mu/2)}$$

is $(2k)$-summable in the directions $d/2$ and $\pi + d/2$.

(c) *The power series*

$$w(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+s_+ j/2)} \sum_{\substack{\nu,\mu\geq 0\\ 2\nu+\mu=j}} f_{\nu\mu} = \sum_{\nu,\mu=0}^{\infty} f_{\nu\mu} \frac{t^{2\nu+\mu}}{\Gamma(1+s_+(\nu+\mu/2))}$$

has positive radius of convergence. Moreover, for sufficiently small $\delta > 0$, the function $w(t)$ can be holomorphically continued into the union of the sectors $S_{d/2,\delta}$ and $S_{\pi+d/2,\delta}$ and is of exponential growth at most of order $2\,k$ in both sectors.

Proof: Equivalence of (b) and (c) is clear by definition of $(2k)$-summability. To prove equivalence of (a) and (b), set

$$a_j = \sum_{\substack{\nu,\mu\geq 0\\ 2\nu+\mu=j}} f_{\nu\mu} \qquad \forall\, j \geq 0.$$

For even (odd) j, we conclude that the corresponding sum only contains terms $f_{\nu\mu}$ with even (odd) values of μ, and hence we conclude from (3.1) that $a_{2j} = u_{j0}$, $a_{2j+1} = u_{j1}$, for $j \geq 0$. General results that can be found in [2] then imply the following: The series $\hat{\psi}(t)$ is $(2k)$-summable in the directions $d/2$ and $\pi + d/2$ if, and only if, its odd and even part both are so summable as well. This, in turn, is equivalent to k summability in the direction d of the series $\sum_j t^j\, a_{2j}/\Gamma(1+j)$ and $\sum_j t^j\, a_{2j+1}/\Gamma(3/2+j)$, which then is equivalent to (a). □

Remark 4: According to the last result, to verify k-summability of $\hat{u}(t,z)$ requires information upon the function $w(t)$. As we shall indicate now, this function can, in principle, be computed from the series $\hat{f}(t,z)$, or rather from its associated function

$$g(t,z) = \sum_{j=0} \frac{t^j}{\Gamma(1+s_+ j)} f_j(z),$$

which converges on polydisc about the origin of $\mathbb{C}^2$, provided that $\hat{f}(t,z)$ has Gevrey order s. Applying *Ecalle's deceleration operator* with indices $1/s_+$ and $1/2$ (see [2, Ex. 3, p. 177] for a definition) to the power series expansion of $g(t,z)$, we obtain the function

$$h(t,z) = \sum_{j=0} \frac{t^j}{\Gamma(1+2j)} f_j(z) = \sum_{j,n=0}^{\infty} \frac{t^j\, z^n}{\Gamma(1+2j)\, n!} f_{jn},$$

which is equal to $g(t,z)$ for $s=1$, resp. is an entire function of t for $s<1$. By termwise integration of its expansion, one can verify that

$$\phi(t) \;:=\; \sum_{\nu,\mu=0}^{\infty} \frac{t^{2\nu+\mu}}{\Gamma(1+2\,\nu+\mu)}\, f_{\nu\mu} \;=\; \partial_t \int_0^t h(t-\tau,\tau)\, d\tau\,. \tag{4.2}$$

For $s_+=2$, i. e., $s=1$, we have $\phi(t)=w(t)$, while for the other cases, $w(t)$ is obtained from $\phi(t)$ by an application of the *acceleration operator* with the same pair of indices $1/s_+$ and $1/2$ as above. Observing these formulae, it is theoretically possible to verify condition (c) in terms of the functions $f_j(z)$. In the special case of $f_j(z)\equiv 0$ for $j\geq 1$, i. e., of a homogeneous Cauchy problem, we have $g(t,z)=f_0(z)$, from which we conclude that $w(t)=\phi(t)=f_0(t)$ as well. If, in addition, $s=1$, then Theorem 3 coincides with a result proven by *Lutz, Miyake, and Schäfke* [14]. On the other hand, if $0<s<1$, then it was shown in Remark 2 that $\phi(t)$ must be an entire function of exponential growth (in every sector) at most of order $2/(1-s)$, and the above theorem is a special case of Theorem 1 in [1], since then $\hat f(t,z)=f_0(z)$ (hence summability holds trivially), while $\hat u_0(t)$, $\hat u_1(t)$ coincide with the series $\hat\psi_0$, $\hat\psi_1$ introduced there. The reader should note that even in this simple situation, verification of the necessary and sufficient condition for k-summability, in case of $k>1$, cannot be done in terms of $f_0(z)$ itself, but involves its Laplace transform of order $(1-s)^{-1}$. □

5 Additional remarks

In the previous section we have restricted ourselves to the situation of $s\leq 1$, and we wish to emphasize here that for $s>1$ the proof of Theorem 2 breaks down: While (4.1) still guarantees $(k=1/s)$-summability of all $\hat u_n(t)$ provided that the first two are so summable, the estimates derived later in the proof become too weak to imply k-summability of $\hat u(t,z)$. In fact, the example we shall give below shows that for $s>1$ we are naturally led to series $\hat x(t,z)$ that are not k-summable, for any value of $k>0$, but multisummable of type $k=(k_1,k_2)$, with $k_1=1$ and $k_2=1/s$.

Example: Assume that $s>1$ and $a\in\mathbb{C}\setminus\{0\}$, with $\arg a\not\equiv 0$ modulo 2π, are given. Let $f_j(z)=\Gamma(1+s_+j)\,(a-z)^{-1}$ for every $j\geq 0$ and $z\neq a$. Then

$$\hat f(t,z) \;=\; \frac{1}{a-z}\sum_{j=0}^{\infty}\frac{t^j}{j!}\,\Gamma(1+s_+j)\,, \qquad \hat x(t,z) \;=\; \sum_{\nu,\mu=0}^{\infty}\frac{t^{\nu+\mu}}{(\nu+\mu)!}\,\frac{(2\mu)!\,\Gamma(1+s_+\nu)}{(a-z)^{2\mu+1}}\,.$$

For the function $v(t,z)$, associated to $\hat x(t,z)$, we find the following integral representation:

$$v(t,z) \;=\; \frac{1}{a-z}\left[\frac{1}{1-t} \;+\; \int_0^1 k\big((1-x)^{s_+}t(a-z)^{-2}\big)\,\frac{dx}{(1-x^{s_+}t)\,(1-x)}\right],$$

with a *kernel* $k(t)$ that is entire and of exponential growth $1/(s-1)$ and is given by the power series

$$k(t) \;=\; \sum_{\mu=1}^{\infty}\frac{(2\mu)!}{\Gamma(s_+\mu)}\,t^{\mu}\,.$$

From this integral representation we conclude that $v(t,z)$, for fixed $z\neq a$, is holomorphic for t in a plane with a cut, along the positive real axis, from 1 to infinity, and is of exponential growth $\kappa=1/(s-1)$ there. Therefore, the acceleration operator with indices $1/2$ and $1/s_+$ can be applied and transforms $v(t,z)$ into a function $h(t,z)$ that is asymptotic of Gevrey order $1/\kappa=s-1$ to the series

$$\hat h(t,z) \;=\; \sum_{\nu,\mu=0}^{\infty}\frac{t^{\nu+\mu}}{(2\nu+2\mu)!}\,\frac{(2\mu)!\,\Gamma(1+s_+\nu)}{(a-z)^{2\mu+1}}$$

(for fixed $z\neq a$) in the sector $S_{\pi,\pi\,(2+1/\kappa)}$ with bisecting direction $d=\pi$ and opening $\pi\,(2+1/\kappa)$. Since this sector is so large, the asymptotic determins the function $h(t,z)$ uniquely, and using this fact, one can obtain the following integral representation:

$$h(t,z) \;=\; \frac{1}{a-z}\left[h(t) \;+\; t\int_0^1 h\big((1-x)^{s_+}t\big)\,\frac{x^{s_+-1}\,dx}{(a-z)^2\;-\;x^{s_+}t}\right],$$

where $h(t)$ is the (unique) function that has the series $\hat{h}(t) = \sum_j \Gamma(1 + s_+ j) t^j/(2j)!$ as its Gevrey asymptotic of order $s-1$ in $S_{\pi,\pi(2+1/\kappa)}$. One can see that $h(t)$ can be obtained from the geometric series by application of the acceleration operator with indices $1/2$ and $1/s_+$, and this implies that $h(t)$ remains bounded as $t \to \infty$ in $S_{\pi,\pi(2+1/\kappa)}$. Due to the above integral representation, we see that the same holds for $h(t,z)$, except for singularities at $t = (a-z)^2$. Hence we may apply the acceleration operator with indices 1 and $1/2$ to the function $h(t,z)$, integrating along any direction that avoids this singularity. The function so obtained then is asymptotic to $\hat{x}(t,z)$ in a corresponding sector. This, with help of the general theory of multisummation and in particular [2, Chapter 10], proves that $\hat{x}(t,z)$ is $(1,1/s)$-summable in all admissible multidirections (d_1,d_2) with $d_0 \not\equiv 0$ and $d_1 \not\equiv s \arg(a-z)$ modulo 2π. If this series were $1/s$-summable *in all but finitely many directions*, then the general theory would imply absence of the singularities of $h(t,z)$ at the points $t=(a-z)^2$, which clearly is not the case. It is worth emphasizing that this is so, whereas the series $\hat{f}(t,z)$ is $1/s$-summable in every direction $d \not\equiv 0$. □

The above example shows that for $s > 1$ it is to be expected that the series $\hat{x}(t,z)$, under suitable conditions upon $\hat{f}(t,z)$, will be $(1,1/s)$-summable. We shall, however, not discuss this situation in this article.

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