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Normal forms of some vector fields by transformations with Borel summable functions

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Abstract

Let $L = \sum_{i=1}^{d} X_i(z)\partial_{z_i}$ be a holomorphic vector field degenerating at $z = 0$ such that Jacobi matrix $(\frac{\partial X_i}{\partial z_j}(0))$ has a zero eigenvalue. We study finding normal forms of L and try to simplify L by transformations with functions with asymptotic expansion in strong sense, that is, called Borel summable functions.

Key words: Normal forms of vector fields, Borel summable functions, Asymptotic expansion

0 Introduction

Let $L = \sum_{i=1}^{d} X_i(z)\partial_{z_i}$ be a holomorphic vector fields in a neighborhood of the origin such that $X_i(0) = 0$ for all $1 \le i \le d$. It is an important and classical problem to simplify L, that is, to find a normal form of L by suitable transformations (see [1]). Let $(\dfrac{\partial X_i(z)}{\partial z_j})$ be Jacobi matrix of the coefficients and $\{\lambda_i\}_{i=1}^{d}$ be eigenvalues of $(\dfrac{\partial X_i(0)}{\partial z_j})$. The typical problem is whether we represent L in the form $\sum_{i=1}^{d} \lambda_i w_i \partial_{w_i}$ by finding an invertible transformation $w_i = \phi_i(z)$ with $\phi_i(0) = 0$, $i = 1, 2, \cdots, d$. In general it is not true. If we assume that the eigenvalues $\{\lambda_i\}_{i=1}^{d}$ are distinct and non resonance condition, that is, for $k = 1, \cdots, d$

$$\sum_{i=1}^{d} m_i\lambda_i - \lambda_k \neq 0 \quad \text{for } |m| \ge 2, \tag{0.1}$$

then we can find a formal transformation $w_i = \phi_i(z) \in \mathbb{C}[[z]]$ $(1 \le i \le d)$ such that L is formally transformed to $\sum_{i=1}^{d} \lambda_i w_i \partial_{w_i}$. It was studied whether the formal transformation converges. If we assume the Poincaré's condition, that is, *the convex hull of* $\{\lambda_1, \lambda_2, \cdots, \lambda_d\}$ *in the complex plane does not contain the origin,* then the above transformation converges, hence, there exists a holomorphic coordinates system $w = (w_1, w_2, \cdots, w_d)$ such that L is of the form $\sum_{i=1}^{d} \lambda_i w_i \partial_{w_i}$. The convergence are also valid under Siegel's condition [8] or Bruno's one [4], which are arithmetic. In this paper we study L such that $(\dfrac{\partial X_i(0)}{\partial z_j})$ has one zero eigenvalue. The purpose of this paper is to try to transform L to a normal form, by using not only transformations with holomorphic functions in a full neighborhood of the origin but also ones with holomorphic functions in a sectorial region with asymptotic expansions. The main results are Theorems 2.1 and 2.2.

1 Function Spaces

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$. A series $\tilde{f}(x) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha$, $f_\alpha \in \mathbb{C}$, is called formal series in x and the set of all such formal series is denoted by $\mathbb{C}[[x]]$. The totality of all convergent series in x, that is, all holomorphic functions in a neighborhood of $x = 0$ is denoted by $\mathbb{C}\{x\}$. Let U be an open set in $\mathbb{C}^n$. $\mathcal{O}(U)$ is the set of all holomorphic functions on U. The set of all formal series $\tilde{f}(x,t) = \sum_{m=0}^{\infty} f_m(x) t^m$, $f_m(x) \in \mathcal{O}(U)$, with coefficients in $\mathcal{O}(U)$ is denoted by $\mathcal{O}(U)[[t]]$.

Definition 1.1. *We say that $\tilde{f}(x,t) = \sum_{m=0}^{\infty} f_m(x) t^m \in \mathcal{O}(U)[[t]]$ has Gevrey order s in t, if there are A and B such that*

$$\sup_{x \in U} |f_m(x)| \le AB^m \Gamma(sm+1). \tag{1.1}$$

The set of all such formal series is denoted by $\mathcal{O}(U)[[t]]_s$.

Let us introduce spaces of holomorphic functions on sectorial regions with asymptotic expansion. Set $S(\theta, \delta, r) = \{t \in \mathbb{C}; 0 < |t| < r, |\arg t - \theta| < \delta\}$. The set of all polydisks centered at $x = 0$ is denoted by $\mathfrak{U}_0$.

Definition 1.2. *Let $\gamma > 0$ and $U \in \mathfrak{U}_0$. Let $f(x,t) \in \mathcal{O}(U \times S(\theta, \delta, r))$ with asymptotic expansion $f(x,t) \sim \sum_{m=0}^{\infty} f_m(x) t^m$ in the following sense. There* *exist constants A and B such that for any $N \in \mathbb{N}$*

$$\sup_{x \in U} |f(x,t) - \sum_{m=0}^{N-1} f_m(x) t^m| \leq AB^N \Gamma(\frac{N}{\gamma}+1)|t|^N \tag{1.2}$$

holds for $t \in S(\theta,\delta,r)$. The set of all such holomorphic functions is denoted by $\mathscr{A}^{\{\gamma\}}(U \times S(\theta,\delta,r))$.

Set

$$\mathbb{C}\{x\}[[t]]_s := \cup_{U \in \mathfrak{U}_0} \mathcal{O}(U)[[t]]_s, \tag{1.3}$$

$$\mathcal{O}(U)\{t\}_{\gamma,\theta} := \cup_{\delta > \pi/2\gamma} \cup_{r>0} \mathscr{A}^{\{\gamma\}}(U \times S(\theta,\delta,r)), \tag{1.4}$$

$$\mathbb{C}\{x\}\{t\}_{\gamma,\theta} := \cup_{U \in \mathfrak{U}_0} \mathcal{O}(U)\{t\}_{\gamma,\theta}. \tag{1.5}$$

We can define a homomorphism $\mathfrak{J} : \mathbb{C}\{x\}\{t\}_{\gamma,\theta} \Rightarrow \mathbb{C}\{x\}[[t]]_{1/\gamma}$, for $f(x,t) \in \mathbb{C}\{x\}\{t\}_{\gamma,\theta}$

$$\mathfrak{J}f = \sum_{m=0}^{\infty} f_m(x) t^m \in \mathbb{C}\{x\}[[t]]_{1/\gamma}. \tag{1.6}$$

Since $\delta > \frac{\pi}{2\gamma}$, $\mathfrak{J}$ is not surjective but injective (see [2]). Therefore, we can identify $\tilde{f}(x,t) = (\mathfrak{J}f)(x,t) \in \mathfrak{J}(\mathbb{C}\{x\}\{t\}_{\gamma,\theta}) \subset \mathbb{C}\{x\}[[t]]_{1/\gamma}$ with $f(x,t) \in \mathbb{C}\{x\}\{t\}_{\gamma,\theta}$.

Definition 1.3. *Let $\tilde{f}(x,t) \in \mathbb{C}\{x\}[[t]]_{1/\gamma}$. If there exists $f(x,t) \in \mathbb{C}\{x\}\{t\}_{\gamma,\theta}$ such that $\tilde{f} = \mathfrak{J}f$, then we say that $\tilde{f}(x,t)$ is γ-Borel summable in the direction θ and $f(x,t)$ is γ-Borel sum of $\tilde{f}(x,t)$. We also say that $f(x,t) \in \mathbb{C}\{x\}\{t\}_{\gamma,\theta}$ is γ-Borel summable in the direction θ.*

As for functions with asymptotic expansion, in particular, Borel summable functions, more generally, multisummable functions we refer to [2].

2 Vector fields

Let $z=(z_1,z_2,\cdots,z_d)\in\mathbb{C}^d$, $\partial_{z_i}=\frac{\partial}{\partial z_i}$, $\partial_z=(\partial_{z_1},\cdots,\partial_{z_d})$ and L be a holomorphic vector field in a neighborhood W of the origin,

$$L:=L(z,\partial_z)=\sum_{i=1}^{d}X_i(z)\partial_{z_i}. \tag{2.1}$$

L is singular at $z=0$, that is, $X_i(0)=0$ for all $1\le i\le d$. Set

$$\Sigma=\{z\in W;X_i(z)=0 \quad \text{for } 1\le i\le d\}. \tag{2.2}$$

Then Σ contains the origin. We denote the Jacobi matrix of the coefficients $(X_1(z),X_2(z),\cdots,X_d(z))$ by $(\dfrac{\partial X_i(z)}{\partial z_j})$. We assume L satisfies the following C.1, C.2 and C.3.

C.1 $\Sigma=\{0\}$.

C.2 The Jordan canonical form of $(\dfrac{\partial X_i(0)}{\partial z_j})$ is diagonal

$$\begin{pmatrix}
\lambda_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
0 & 0 & \lambda_3 & 0 & \cdots & \cdots & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & 0\\
\vdots & \ddots & \ddots & & \ddots & & & \vdots\\
\vdots & \ddots & \ddots & & 0 & \lambda_{d-2} & 0 & 0\\
\vdots & \ddots & \ddots & & \ddots & 0 & \lambda_{d-1} & 0\\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0
\end{pmatrix}, \tag{2.3}$$

where $\lambda_i\neq 0$ and distinct.

C.3 The convex hull of $(d-1)$ points $\{\lambda_1,\lambda_2,\cdots,\lambda_{d-1}\}$ in the complex plane does not contain the origin.

It follows from C.1 and C.2 that codim $\Sigma=d$ and $(\dfrac{\partial X_i(0)}{\partial z_j})$ has one zero simple eigenvalue, so its rank is $(d-1)$. The assumption C.3 is equivalent to that nonzero $(d-1)$ points $\{\lambda_1,\lambda_2,\cdots,\lambda_{d-1}\}$ lie in the one side divided by a line through the origin. The main theorems are the following.

Theorem 2.1. *Assume C.1, C.2, C.3 and*

$$\sum_{i=1}^{d-1} m_i\lambda_i - \lambda_k \neq 0 \quad k = 1, 2, \cdots, d-1 \tag{2.4}$$

hold for all $m = (m_1, m_2, \cdots, m_{d-1}) \in \mathbb{N}^{d-1}$ *with* $|m| \geq 2$. *Then there exist an integer* $\sigma \geq 2$ *and a holomorphic coordinates system* $(x(z), t(z)) \in \mathbb{C}^{d-1} \times \mathbb{C}$ *with* $(x(0), t(0)) = (0, 0)$ *such that the following holds.*

There exist $\zeta_1(x,t), \cdots, \zeta_{d-1}(x,t), \eta(x,t) \in \mathbb{C}\{x\}\{t\}_{\sigma-1,\theta}$ *for some* θ *with*

$$\begin{cases} \zeta_1(0,0) = \cdots = \zeta_{d-1}(0,0) = 0, \ \ \eta(x,0) = 0 \\ (\dfrac{\partial \zeta_i}{\partial x_j}(0,0)) = \delta_{i,j}, \ \dfrac{\partial \eta}{\partial t}(0,0) \neq 0, \end{cases} \tag{2.5}$$

and by transformation

$$\begin{cases} \zeta_i = \zeta_i(x,t) \quad i = 1, \cdots, d-1, \\ \eta = \eta(x,t) \end{cases} \tag{2.6}$$

L is represented in the form

$$\sum_{i=1}^{d-1} \lambda_i(\eta)\zeta_i \frac{\partial}{\partial \zeta_i} + \eta^\sigma c(\eta) \frac{\partial}{\partial \eta}, \tag{2.7}$$

where $\{\lambda_i(\eta)\}_{i=1}^{d-1}$ *and* $c(\eta)$ *are polynomials* η *with degree* $\leq \sigma - 1$, $\lambda_i(0) = \lambda_i$ *and* $c(0) = 1$.

If we admit multiplications of nonvanishing functions to vector fields in the process to find normal forms of vector fields, we have

Theorem 2.2. *Suppose that the same assumptions as those in Theorem 2.1 hold. Then there exist an integer* $\sigma \geq 2$, *a holomorphic function* $h(z)$ *in a neighborhood of the origin with* $h(0) \neq 0$ *and a holomorphic coordinates system* $(x(z), t(z)) \in \mathbb{C}^{d-1} \times \mathbb{C}$ *with* $(x(0), t(0)) = (0, 0)$ *such that the following holds.*

Set $L_h := h(z)L$. *There exist* $\zeta_1(x,t), \cdots, \zeta_{d-1}(x,t) \in \mathbb{C}\{x\}\{t\}_{\sigma-1,\theta}$ *for some* θ *such that*

$$\zeta_1(0,0) = \cdots = \zeta_{d-1}(0,0) = 0, \ (\frac{\partial \zeta_i}{\partial x_j}(0,0)) = \delta_{i,j}, \tag{2.8}$$

and by transformation

$$\begin{cases} \zeta_i = \zeta_i(x,t) & i = 1, \cdots, d-1, \\ \eta = t \end{cases} \tag{2.9}$$

L_h *is represented in the form*

$$\sum_{i=1}^{d-1} \lambda_i(\eta)\zeta_i \frac{\partial}{\partial \zeta_i} + \eta^{\sigma} \frac{\partial}{\partial \eta}, \tag{2.10}$$

where $\{\lambda_i(\eta)\}_{i=1}^{d-1}$ *are polynomials* η *with degree* $\leq \sigma - 1$ *and* $\lambda_i(0) = \lambda_i$.

In order to show Theorems 2.1 and 2.2 we have to find coordinates transformations. The transformations are constructed by using solutions of systems of nonlinear ordinary differential equations and those of singular semi linear first order partial differential equations. For these equations we need the existence of holomorphic solutions and that of Borel summable solutions, for which we refer to [3], [5], [6] and [7]. In particular, the existence of Borel summable solutions of systems of nonlinear ordinary differential equations is given in [3] and [7], and that of singular semi linear first order partial differential equations is given in [6]. The proofs of Theorems 2.1 and 2.2 and the details are given in [6].

3 A simple example

We give a simple example and show the process to transform it to a normal form. Let $(x,t) \in \mathbb{C}^2$ and

$$L := L(x, t, \partial_x, \partial_t) = (\lambda x + x^2 + xt + t^2)\partial_x + t^{\gamma+1}\partial_t, \tag{3.1}$$

where $\lambda > 0$ and γ is a positive integer. We have $\sigma = \gamma + 1$. Let θ be a real constant such that $0 < |\theta| < \pi/\gamma$.

(1) First consider

$$t^{\gamma+1}\varphi'(t) = \lambda\varphi(t) + \varphi(t)^2 + t\varphi(t) + t^2. \tag{3.2}$$

Then there exists $\varphi(t) \in \mathbb{C}\{t\}_{\gamma,\theta}$ with $\varphi(t) \sim \sum_{n=2}^{\infty} c_n t^n$, $c_2 = -1/\lambda$. By transformation $w = x - \varphi(t)$, $t = t$,

$$L := L(w,t,\partial_w,\partial_t) = \big((\lambda + t + 2\varphi(t))w + w^2\big)\partial_w + t^{\gamma+1}\partial_t.$$

Set $\lambda(t) = \lambda + t + 2\sum_{n=0}^{\gamma} c_n t^n$ and $A(t) = \lambda + t + 2\varphi(t) - \lambda(t)$. Then $\lambda(t)$ is a polynomial with degree $\leq \gamma$ and $A(t) \in \mathbb{C}\{t\}_{\gamma,\theta}$ with $A(t) \sim 2\sum_{n=\gamma+1}^{\infty} c_n t^n = O(t^{\gamma+1})$ and

$$L(w,t,\partial_w,\partial_t) = \big((\lambda(t) + A(t))w + w^2\big)\partial_w + t^{\gamma+1}\partial_t. \tag{3.3}$$

(2) Next consider

$$L(w,t,\partial_w,\partial_t)\phi(w,t) = \lambda(t)\phi(w,t), \tag{3.4}$$

which is a singular first order partial differential equation with coefficients in $\mathcal{O}\{w\}\{t\}_{\gamma,\theta}$. Consider an auxilliay equation to solve (3.4)

$$t^{\gamma+1}\psi_*'(t) + A(t)\psi_*(t) + A(t) = 0.$$

Since $A(t) = O(t^{\gamma+1})$. there exists a solution $\psi_*(t) \in \mathbb{C}\{t\}_{\gamma,\theta}$ with $\psi_*(0) = 0$. Set $\phi(w,t) = (1 + \psi_*(t))w + \psi(w,t)$. Then (3.4) becomes

$$L(w,t,\partial_w,\partial_t)\psi(w,t) = \lambda(t)\psi(w,t) - (1 + \psi_*(t))w^2. \tag{3.5}$$

It is not difficult to find a formal solution $\tilde{\psi}(w,t) = \sum_{n=0}^{\infty} \psi_n(w)t^n \in \mathcal{O}(U)[[t]]_{1/\gamma}$ for a neighborhood U of $w = 0$ with $\psi_n(w) = O(|w|^2)$ for all n. We can show that $\tilde{\psi}(w,t)$ is γ-Borel summable in the direction θ, that is, there exists $\psi(w,t) \in \mathbb{C}\{w\}\{t\}_{\gamma,\theta}$ with $\psi(w,t) \sim \tilde{\psi}(w,t)$. Hence $\phi(w,t) = (1 + \psi_*(t))w + \psi(w,t) \in \mathbb{C}\{w\}\{t\}_{\gamma,\theta}$ is a solution of (3.4). By transformation $\zeta(x,t) = \phi(x - \varphi(t),t)$, $\eta(x,t) = t$, the vector field L is transformed to $\lambda(\eta)\zeta\dfrac{\partial}{\partial\zeta} + \eta^{\gamma+1}\dfrac{\partial}{\partial\eta}$, which is a normal form of L.

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