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ON UNIVALENCE CRITERIA FOR MEROMORPHIC FUNCTIONS

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ABSTRACT. We propose a way of deduction of various univalence criteria for meromorphic functions on the outside of the unit circle in terms of the range of their derivatives. This is a summary of the forthcoming joint paper [15] of S. Ponnusamy and the author.

1. INTRODUCTION

Let $\mathcal{A}$ denote the set of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. The set $\mathcal{S}$ of univalent functions in $\mathcal{A}$ has been intensively studied by many authors. Let Σ denote the set of univalent functions F in the domain $\Delta = \{\zeta : |\zeta| > 1\}$ of the form

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}. \tag{1.1}$$

Note that the function $1/f(1/\zeta)$ belongs to Σ for each $f \in \mathcal{S}$. The converse is, however, not true in general. More precisely, for $F \in \Sigma$, the function $f(z) = 1/F(1/z)$ belongs to $\mathcal{S}$ if and only if F omits 0, namely, $F(\zeta) \neq 0$ for $\zeta \in \Delta$.

In parallel with the analytic case, we consider the set $\mathcal{M}$ of meromorphic functions in Δ with the expansion (1.1) around $\zeta = \infty$. For some technical reason, we also consider the sets $\mathcal{A}_n = \{f \in \mathcal{A} : f^{(m)}(0) = 0 \text{ for } m = 2, \dots, n\}$ and $\mathcal{M}_n = \{F \in \mathcal{M} : b_0 = \cdots = b_n = 0\}$. Note that $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{M}_{-1} = \mathcal{M}$.

Practically, it is an important problem to determine univalence of a given function in $\mathcal{A}_n$ or in $\mathcal{M}_n$. The best known conditions for univalence are probably those involving pre-Schwarzian or Schwarzian derivatives, which are defined by

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

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We define quantities for functions $f \in \mathcal{A}$ and $F \in \mathcal{M}$ by

$$\mathrm{B}(f) = \sup_{|z|<1}(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right|,$$
$$\mathrm{B}(F) = \sup_{|\zeta|>1}(|\zeta|^2-1)\left|\frac{\zeta F''(\zeta)}{F'(\zeta)}\right|,$$
$$\mathrm{N}(f) = \sup_{|z|<1}(1-|z|^2)^2\,|S_f(z)|,$$
$$\mathrm{N}(F) = \sup_{|\zeta|>1}(|\zeta|^2-1)^2\,|S_F(\zeta)|.$$

Note that these quantities may take ∞ as their values. For example, if F has a pole at a finite point, then $\mathrm{B}(F) = \infty$.

If $f \in \mathcal{A}$ and $F \in \mathcal{M}$ have the relation $f(z) = 1/F(1/z)$, then we can easily see that

$$(1-|z|^2)^2 S_f(z) = (|\zeta|^2-1)^2 S_F(\zeta)$$

holds for $z = 1/\zeta$. In particular, we have $\mathrm{N}(f) = \mathrm{N}(F)$.

Theorem A (Nehari [14]). *Every $f \in \mathcal{S}$ satisfies $\mathrm{N}(f) \le 6$. Conversely, if $f \in \mathcal{A}$ satisfies $\mathrm{N}(f) \le 2$ then f must be univalent. The constants 6 and 2 are best possible. The same is true for meromorphic F.*

Though $zf'(z)/f(z) = \zeta F'(\zeta)/F(\zeta)$, there is no such a simple relation between $zf''(z)/f'(z)$ and $\zeta F''(\zeta)/F'(\zeta)$, and thus, between $B(f)$ and $B(F)$ for $f(z) = 1/F(1/z)$, $\zeta = 1/z$. Nevertheless, it is rather surprising that the formally same conclusions can be deduced for f and F. Compare Theorem B with Theorem C.

Theorem B. *Every $f \in \mathcal{S}$ satisfies $\mathrm{B}(f) \le 6$. Conversely, if $f \in \mathcal{A}$ satisfies $\mathrm{B}(f) \le 1$ then $f \in \mathcal{S}$. Moreover, if $\mathrm{B}(f) \le k < 1$, then f extends to a k-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.*

Here and hereafter, a quasiconformal mapping g is called k-quasiconformal if its Beltrami coefficient $\mu = g_{\bar z}/g_z$ satisfies $\|\mu\|_\infty \le k$.

The sufficiency of univalence and quasiconformal extendibility are due to Becker [6]. The sharpness of the constant 1 is due to Becker and Pommerenke [8]. The sharp inequality $\mathrm{B}(f) \le 6$ follows from a standard argument in the coefficient estimation (see, e.g., [9, Theorem 2.4]).

Theorem C. *Every $F \in \Sigma$ satisfies $\mathrm{B}(F) \le 6$. Conversely, if $F \in \mathcal{M}$ satisfies $\mathrm{B}(F) \le 1$ then $F \in \Sigma$. Moreover, if $\mathrm{B}(F) \le k < 1$, then F extends to a k-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.*

The sufficiency of univalence and quasiconformal extendibility are due to Becker [7]. The sharpness of the constant 1 is also due to Becker and Pommerenke [8]. On the other hand, the estimate $\mathrm{B}(F) \le 6$ lies deeper. Avhadiev [3] first showed the sharp inequality $\mathrm{B}(F) \le 6$ by appealing to Goluzin's inequality (see [10, p. 139]).

Note that many authors use a different norm for the pre-Schwarzian derivative of $f \in \mathcal{A}$, namely,

$$\|T_f\| = \sup_{|z|<1}(1-|z|^2)|T_f(z)|.$$

By definition, we observe $\mathrm{B}(f) \le \|T_f\|$.

Recall that a plane domain $\Omega \subset \mathbb{C}$ is called *hyperbolic* if $\partial\Omega$ contains at least two points. Let Ω be a hyperbolic plain domain such that $1 \in \Omega$ but $0 \notin \Omega$ and set

$$\Pi(\Omega) = \{F \in \mathcal{M} : F'(\zeta) \in \Omega \text{ for all } \zeta \in \Delta\}.$$

Set also $\Pi_n(\Omega) = \Pi(\Omega) \cap \mathcal{M}_n$ for $n = -1, 0, 2, \ldots$. One of our main results in the present paper is an estimate of $\mathrm{B}(F)$ for $F \in \Pi(\Omega)$. The proof is given in [15].

Theorem 1. *Let Ω be a domain such that $1 \in \Omega$ but $0 \notin \Omega$. For every $F \in \Pi_n(\Omega)$, $n \ge 0$, the inequality*

$$\mathrm{B}(F) \le C_n W(\Omega)$$

holds, where C_n is the constant given by

$$C_n = \sup_{0<r<1} \frac{(n+2)(1-r^2)r^n}{1-r^{2n+4}} \tag{1.2}$$

and $W(\Omega)$ is the circular width of Ω with respect to the origin, namely,

$$W(\Omega) = \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{p'(z)}{p(z)}\right|$$

for an analytic universal covering projection p of $\mathbb{D}$ onto Ω.

Note that $W(\Omega)$ does not depend on the particular choice of p. For more details on circular width, see [12]. As one sees easily, $C_0 = 2$ and $1 \le C_n \le (n+2)/(n+1)$. If we write $F \in \Pi(\Omega)$ in the form $F = F_0 + b_0$, where $F_0 \in \Pi_0(\Omega)$, the relation $\mathrm{B}(F) = \mathrm{B}(F_0)$ holds. Therefore, the above theorem can be applicable to the whole family $\Pi(\Omega)$. We note that the analytic counterpart of this theorem is known and much simpler to prove (see [11, Theorem 4.1]); $\mathrm{B}(f) \le \|T_f\| \le W(\Omega)$ *holds for $f \in \mathcal{A}$ with $f'(\mathbb{D}) \subset \Omega$.*

As is well known, if $f \in \mathcal{A}$ satisfies $\operatorname{Re} f' > 0$ then f is necessarily univalent (cf. [9, Theorem 2.16]). However, the meromorphic counterpart does not hold (see, for instance, the example given in Section 3). The following univalence criterion is due to Aksent'ev [1] (see also [5, Theorem 11]). Later, Krzyż [13] gave quasiconformal extensions for the functions.

Theorem D (Aksent'ev, Krzyż). *Let $0 \le k \le 1$. If $F \in \mathcal{M}$ satisfies the inequality*

$$|F'(\zeta) - 1| \le k, \quad |\zeta| > 1, \tag{1.3}$$

then F is univalent. Furthermore, if $k < 1$, then F extends to a k-quasiconformal mapping of the extended plane. The radii 1 and k are best possible.

Note that the range of F' cannot be enlarged to $\{w : |w-1| < a\}$, $a > 1$, for univalence [2].

2. Examples

The following examples can be found in [12].

Example 1 (sectors). For $S(\beta)=\{w:|\arg w|<\pi\beta/2\}$, $0<\beta\le 2$, we have $W(S(\beta))=2\beta$.

Example 2 (annuli). For the annulus $A(r,R)=\{w:r<|w|<R\}$, $0<r<R<\infty$, we have $W(A(r,R))=(2/\pi)\log(R/r)$.

Example 3 (disks). Let $\mathbb{D}(a,r)=\{w:|w-a|<r\}$ for $0<r\le a$. Then

$$W(\mathbb{D}(a,r))=\frac{2r/a}{1+\sqrt{1-(r/a)^2}}.$$

Example 4 (parallel strips). Let $P(a,b)=\{w:a<\operatorname{Re}w<b\}$ for $0\le a<b<\infty$. Then

$$W(P(a,b))=\max_{0\le\theta\le\pi/2}\frac{2t\cos\theta}{1-t\theta},$$

where t is a number with $0<t\le 2/\pi$ determined by

$$\frac{\pi t}{2}=\frac{b-a}{b+a}.$$

Example 5 (truncated wedges). Let $S(\beta,r,R)=\{w:|\arg w|<\pi\beta/2, r<|w|<R\}$, $0<\beta\le 2, 0<r<R<\infty$. Then

$$W(S(\beta,r,R))=\frac{\log(R/r)}{(1+t)\mathcal{K}(t)},$$

where

$$\mathcal{K}(t)=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

is the complete elliptic integral of the first kind and $0<t<1$ is a number such that

$$\frac{\mathcal{K}(\sqrt{1-t^2})}{\mathcal{K}(t)}=\frac{2\pi\beta}{\log(R/r)}.$$

3. Applications

We apply Theorem 1 and Theorem C to the above examples to obtain several results on univalence of meromorphic functions. As samples, we state a few theorems. Note that the univalence criteria in Theorems 2 and 3 were first given by Avhadiev and Aksent'ev [4].

Let $x_2\approx 0.4198$ denote the unique zero of the equation

$$\sqrt{x}\log((1+\sqrt{x})/(1-\sqrt{x}))=1$$

in $0<x<1$.

Theorem 2. *Let $0\le k\le 1$. Suppose that a function $F\in\mathcal{M}$ satisfies the condition*

$$|\arg F'(\zeta)|\le\frac{k\pi}{8},\quad |\zeta|>1,$$

then F must be univalent. Furthermore, if $k < 1$, then F extends to a k-quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/8$ cannot be replaced by any smaller number than $(4/\pi)\arctan x_2$.

Note that $(4/\pi)\arctan x_2 \approx 0.506057 \approx 1.28866(\pi/8)$. The number x_2 appears in the following example.

We consider the function $F_n \in \mathcal{M}$ given by

$$F_n(\zeta) = \zeta - 2\sum_{j=1}^{\infty} \frac{\zeta^{1-nj}}{nj-1}$$
$$= \zeta\Big(2\,{}_2F_1(1, -\tfrac{1}{n}; 1 - \tfrac{1}{n}; \zeta^{-n}) - 1\Big), \quad |\zeta| > 1,$$

for each integer $n \geq 2$, where ${}_2F_1(a, b; c; x)$ stands for the hypergeometric function. Note that F_n has the n-fold symmetry

$$F_n(e^{2\pi i/n}\zeta) = e^{2\pi i/n}F_n(\zeta)$$

and belongs to the class $\mathcal{M}_{n-2}$. Since the function h_n defined by

$$h_n(x) = 2\,{}_2F_1(1, -\tfrac{1}{n}; 1 - \tfrac{1}{n}; x) - 1 \quad (x \in (0, 1))$$

has the properties that h_n is monotone decreasing, $h_n(0) = 1$ and $\lim_{x\to 1^-} h_n(x) = -\infty$, there is the unique point x_n such that $h(x_n) = 0$ in the interval $0 < x < 1$. Hence, the function F_n has the n zeros $e^{2\pi ij/n}x_n^{-1/n}$, $j = 0, 1, \ldots, n-1$, in Δ and, in particular, is *not* univalent in Δ. On the other hand, we have

$$F_n'(\zeta) = 1 + 2\sum_{j=1}^{\infty} \zeta^{-nj} = p(\zeta^{-n}),$$

where $p(z)$ is the function given by $p(z) = (1+z)/(1-z)$. It is a standard fact that p maps the unit disk onto the right half-plane $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. Therefore, F_n' maps Δ onto $\mathbb{H}$ in an n-to-1 way and thus $\operatorname{Re} F_n' > 0$ holds.

In the next criterion, F' may take values with negative real part.

Theorem 3. *Let $0 \leq k \leq 1$. Suppose that a function $F \in \mathcal{M}$ satisfies the condition*

$$|\log|F'(\zeta)|| \leq \frac{k\pi}{8}, \quad |\zeta| > 1,$$

then F must be univalent. Furthermore, if $k < 1$, then F extends to a k-quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/8$ cannot be replaced by any smaller number than $\log((1+x_2)/(1-x_2))$.

Note that $\log((1+x_2)/(1-x_2)) \approx 0.894894 \approx 2.27883(\pi/8)$. In these results, if we assume F to be in $\mathcal{M}_n$ for larger n, then we can make the involved constants better.

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