ON UNIVALENCE CRITERIA FOR MEROMORPHIC FUNCTIONS

TOSHIYUKI SUGAWA

HIROSHIMA UNIVERSITY

ABSTRACT. We propose a way of deduction of various univalence criteria for meromorphic functions on the outside of the unit circle in terms of the range of their derivatives. This is a summary of the forthcoming joint paper [15] of S. Ponnusamy and the author.

1. INTRODUCTION

Let $\mathcal{A}$ denote the set of analytic functions $f$ in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. The set $\mathcal{S}$ of univalent functions in $\mathcal{A}$ has been intensively studied by many authors. Let $\Sigma$ denote the set of univalent functions $F$ in the domain $\Delta = \{\zeta : |\zeta| > 1\}$ of the form

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}. \quad (1.1)$$

Note that the function $1/f(1/\zeta)$ belongs to $\Sigma$ for each $f \in \mathcal{S}$. The converse is, however, not true in general. More precisely, for $F \in \Sigma$, the function $f(z) = 1/F(1/z)$ belongs to $\mathcal{S}$ if and only if $F$ omits 0, namely, $F(\zeta) \neq 0$ for $\zeta \in \Delta$.

In parallel with the analytic case, we consider the set $\mathcal{M}$ of meromorphic functions in $\Delta$ with the expansion (1.1) around $\zeta = \infty$. For some technical reason, we also consider the sets $\mathcal{A}_n = \{f \in \mathcal{A} : f^{(m)}(0) = 0 \text{ for } m = 2, \ldots, n\}$ and $\mathcal{M}_n = \{F \in \mathcal{M} : b_0 = \cdots = b_n = 0\}$. Note that $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{M}_{-1} = \mathcal{M}$.

Practically, it is an important problem to determine univalence of a given function in $\mathcal{A}_n$ or in $\mathcal{M}_n$. The best known conditions for univalence are probably those involving pre-Schwarzian or Schwarzian derivatives, which are defined by

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2.$$

1991 Mathematics Subject Classification. Primary 30C55; Secondary 33C45.

Key words and phrases. univalent criterion, pre-Schwarzian derivative.

The second author was partially supported NBHM (DAE, India) grant.
We define quantities for functions $f \in \mathcal{A}$ and $F \in \mathcal{M}$ by

$$
B(f) = \sup_{|z|<1} (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right|,
$$

$$
B(F) = \sup_{|\zeta|>1} \left| \frac{\zeta F''(\zeta)}{F'(\zeta)} \right|,
$$

$$
N(f) = \sup_{|z|<1} (1 - |z|^2)^2 |S_f(z)|,
$$

$$
N(F) = \sup_{|\zeta|>1} (1 - |\zeta|^2)^2 |S_F(\zeta)|.
$$

Note that these quantities may take $\infty$ as their values. For example, if $F$ has a pole at a finite point, then $B(F) = \infty$.

If $f \in \mathcal{A}$ and $F \in \mathcal{M}$ have the relation $f(z) = 1/F(1/z)$, then we can easily see that

$$(1 - |z|^2)^2 S_f(z) = (|\zeta|^2 - 1)^2 S_F(\zeta)$$

holds for $z = 1/\zeta$. In particular, we have $N(f) = N(F)$.

**Theorem A** (Nehari [14]). Every $f \in \mathcal{S}$ satisfies $N(f) \leq 6$. Conversely, if $f \in \mathcal{A}$ satisfies $N(f) \leq 2$ then $f$ must be univalent. The constants 6 and 2 are best possible. The same is true for meromorphic $F$.

Though $zf'(z)/f(z) = \zeta F'(\zeta)/F(\zeta)$, there is no such a simple relation between $zf''(z)/f'(z)$ and $\zeta F''(\zeta)/F'(\zeta)$, and thus, between $B(f)$ and $B(F)$ for $f(z) = 1/F(1/z)$, $\zeta = 1/z$. Nevertheless, it is rather surprising that the formally same conclusions can be deduced for $f$ and $F$. Compare Theorem B with Theorem C.

**Theorem B.** Every $f \in \mathcal{S}$ satisfies $B(f) \leq 6$. Conversely, if $f \in \mathcal{A}$ satisfies $B(f) \leq 1$ then $f \in \mathcal{S}$. Moreover, if $B(f) \leq k < 1$, then $f$ extends to a $k$-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.

Here and hereafter, a quasiconformal mapping $g$ is called $k$-quasiconformal if its Beltrami coefficient $\mu = g_z/g_\overline{z}$ satisfies $\|\mu\|_{\infty} \leq k$.

The sufficiency of univalence and quasiconformal extendibility are due to Becker [6]. The sharpness of the constant 1 is due to Becker and Pommerenke [8]. The sharp inequality $B(f) \leq 6$ follows from a standard argument in the coefficient estimation (see, e.g., [9, Theorem 2.4]).

**Theorem C.** Every $F \in \Sigma$ satisfies $B(F) \leq 6$. Conversely, if $F \in \mathcal{M}$ satisfies $B(F) \leq 1$ then $F \in \Sigma$. Moreover, if $B(F) \leq k < 1$, then $F$ extends to a $k$-quasiconformal mapping of the extended plane. The constants 6 and 1 are best possible.

The sufficiency of univalence and quasiconformal extendibility are due to Becker [7]. The sharpness of the constant 1 is also due to Becker and Pommerenke [8]. On the other hand, the estimate $B(F) \leq 6$ lies deeper. Avhadiev [3] first showed the sharp inequality $B(F) \leq 6$ by appealing to Goluzin's inequality (see [10, p. 139]).
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Note that many authors use a different norm for the pre-Schwarzian derivative of \( f \in \mathcal{A} \), namely,
\[
||T_f|| = \sup_{|z|<1} (1 - |z|^2) |T_f(z)|.
\]
By definition, we observe \( \mathcal{B}(f) \leq ||T_f|| \).

Recall that a plane domain \( \Omega \subset \mathbb{C} \) is called hyperbolic if \( \partial \Omega \) contains at least two points. Let \( \Omega \) be a hyperbolic plain domain such that \( 1 \in \Omega \) but \( 0 \notin \Omega \) and set
\[
\Pi(\Omega) = \{ F \in \mathcal{M} : F'(\zeta) \in \Omega \text{ for all } \zeta \in \Delta \}.
\]
Set also \( \Pi_n(\Omega) = \Pi(\Omega) \cap \mathcal{M}_n \) for \( n = -1, 0, 2, \ldots \). One of our main results in the present paper is an estimate of \( \mathcal{B}(F) \) for \( F \in \Pi(\Omega) \). The proof is given in [15].

**Theorem 1.** Let \( \Omega \) be a domain such that \( 1 \in \Omega \) but \( 0 \notin \Omega \). For every \( F \in \Pi_n(\Omega) \), \( n \geq 0 \), the inequality
\[
\mathcal{B}(F) \leq C_n W(\Omega)
\]
holds, where \( C_n \) is the constant given by
\[
(1.2) \quad C_n = \sup_{0<r<1} \frac{(n+2)(1-r^2)r^n}{1-r^{2n+4}}
\]
and \( W(\Omega) \) is the circular width of \( \Omega \) with respect to the origin, namely,
\[
W(\Omega) = \sup_{z \in \mathbb{D}} (1-|z|^2) \left| \frac{p'(z)}{p(z)} \right|
\]
for an analytic universal covering projection \( p \) of \( \mathbb{D} \) onto \( \Omega \).

Note that \( W(\Omega) \) does not depend on the particular choice of \( p \). For more details on circular width, see [12]. As one sees easily, \( C_0 = 2 \) and \( 1 \leq C_n \leq (n+2)/(n+1) \). If we write \( F \in \Pi(\Omega) \) in the form \( F = F_0 + b_0 \), where \( F_0 \in \Pi_0(\Omega) \), the relation \( \mathcal{B}(F) = \mathcal{B}(F_0) \) holds. Therefore, the above theorem can be applicable to the whole family \( \Pi(\Omega) \). We note that the analytic counterpart of this theorem is known and much simpler to prove (see [11, Theorem 4.1]); \( \mathcal{B}(f) \leq ||T_f|| \leq W(\Omega) \) holds for \( f \in \mathcal{A} \) with \( f'(\mathbb{D}) \subset \Omega \).

As is well known, if \( f \in \mathcal{A} \) satisfies \( \text{Re} f' > 0 \) then \( f \) is necessarily univalent (cf. [9, Theorem 2.16]). However, the meromorphic counterpart does not hold (see, for instance, the example given in Section 3). The following univalence criterion is due to Aksent’ev [1] (see also [5, Theorem 11]). Later, Krzyż [13] gave quasiconformal extensions for the functions.

**Theorem D** (Aksent’ev, Krzyż). Let \( 0 \leq k \leq 1 \). If \( F \in \mathcal{M} \) satisfies the inequality
\[
(1.3) \quad |F'(\zeta) - 1| \leq k, \quad |\zeta| > 1,
\]
then \( F \) is univalent. Furthermore, if \( k < 1 \), then \( F \) extends to a \( k \)-quasiconformal mapping of the extended plane. The radii 1 and \( k \) are best possible.

Note that the range of \( F' \) cannot be enlarged to \( \{ w : |w - 1| < a \} \), \( a > 1 \), for univalence [2].
The following examples can be found in [12].

**Example 1** (sectors). For $S(\beta) = \{w : |\arg w| < \pi \beta/2\}, \ 0 < \beta \leq 2$, we have $W(S(\beta)) = 2\beta$.

**Example 2** (annuli). For the annulus $A(r, R) = \{w : r < |w| < R\}, \ 0 < r < R < \infty$, we have $W(A(r, R)) = (2/\pi) \log(R/r)$.

**Example 3** (disks). Let $\mathbb{D}(a, r) = \{w : |w - a| < r\}$ for $0 < r \leq a$. Then

$$W(\mathbb{D}(a, r)) = \frac{2r/a}{1 + \sqrt{1 - (r/a)^2}}.$$

**Example 4** (parallel strips). Let $P(a, b) = \{w : a < \Re w < b\}$ for $0 \leq a < b < \infty$. Then

$$W(P(a, b)) = \max \frac{2t \cos \theta}{1 - t \theta},$$

where $t$ is a number with $0 < t \leq 2/\pi$ determined by

$$\frac{\pi t}{2} = \frac{b - a}{b + a}.$$

**Example 5** (truncated wedges). Let $S(\beta, r, R) = \{w : |\arg w| < \pi \beta/2, r < |w| < R\}, \ 0 < \beta \leq 2, 0 < r < R < \infty$. Then

$$W(S(\beta, r, R)) = \frac{\log(R/r)}{(1 + t)K(t)},$$

where

$$K(t) = \int_{0}^{1} \frac{dx}{\sqrt{(1 - x^2)(1 - t^2 x^2)}}$$

is the complete elliptic integral of the first kind and $0 < t < 1$ is a number such that

$$\frac{K(\sqrt{1 - t^2})}{K(t)} = \frac{2\pi \beta}{\log(R/r)}.$$

3. **Applications**

We apply Theorem 1 and Theorem C to the above examples to obtain several results on univalence of meromorphic functions. As samples, we state a few theorems. Note that the univalence criteria in Theorems 2 and 3 were first given by Avhadiev and Aksent’ev [4].

Let $x_2 \approx 0.4198$ denote the unique zero of the equation

$$\sqrt{x} \log((1 + \sqrt{x})/(1 - \sqrt{x})) = 1$$

in $0 < x < 1$.

**Theorem 2.** Let $0 \leq k \leq 1$. Suppose that a function $F \in \mathcal{M}$ satisfies the condition

$$|\arg F'(\zeta)| \leq \frac{k\pi}{8}, \quad |\zeta| > 1,$$
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then $F$ must be univalent. Furthermore, if $k < 1$, then $F$ extends to a $k$-quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/8$ cannot be replaced by any smaller number than $(4/\pi) \arctan x_2$.

Note that $(4/\pi) \arctan x_2 \approx 0.506057 \approx 1.28866(\pi/8)$. The number $x_2$ appears in the following example.

We consider the function $F_n \in \mathcal{M}$ given by

$$F_n(\zeta) = \zeta - 2 \sum_{j=1}^{\infty} \frac{\zeta^{1-nj}}{nj-1}$$

for each integer $n \geq 2$, where $2F_1(a, b; c; x)$ stands for the hypergeometric function. Note that $F_n$ has the $n$-fold symmetry

$$F_n(e^{2\pi i/n} \zeta) = e^{2\pi i/n} F_n(\zeta)$$

and belongs to the class $\mathcal{M}_{n-2}$. Since the function $h_n$ defined by

$$h_n(x) = 2F_1(1, -\frac{1}{n}; 1 - \frac{1}{n}; x) - 1 \quad (x \in (0, 1))$$

has the properties that $h_n$ is monotone decreasing, $h_n(0) = 1$ and $\lim_{x \to 1} h_n(x) = -\infty$, there is the unique point $x_n$ such that $h(x_n) = 0$ in the interval $0 < x < 1$. Hence, the function $F_n$ has the $n$ zeros $e^{2\pi i j/n} x_n^{-1/n}$, $j = 0, 1, \ldots, n - 1$, in $\Delta$ and, in particular, is not univalent in $\Delta$. On the other hand, we have

$$F_n'(\zeta) = 1 + 2 \sum_{j=1}^{\infty} \zeta^{-nj} = p(\zeta^{-n}),$$

where $p(z)$ is the function given by $p(z) = (1 + z)/(1 - z)$. It is a standard fact that $p$ maps the unit disk onto the right half-plane $\mathbb{H} = \{w \in \mathbb{C} : \text{Re} w > 0\}$. Therefore, $F_n'$ maps $\Delta$ onto $\mathbb{H}$ in an $n$-to-1 way and thus $\text{Re} F_n' > 0$ holds.

In the next criterion, $F'$ may take values with negative real part.

**Theorem 3.** Let $0 \leq k \leq 1$. Suppose that a function $F \in \mathcal{M}$ satisfies the condition

$$|\log |F'(\zeta)|| \leq \frac{k\pi}{8}, \quad |\zeta| > 1,$$

then $F$ must be univalent. Furthermore, if $k < 1$, then $F$ extends to a $k$-quasiconformal mapping of the extended plane. As for univalence, the constant $\pi/8$ cannot be replaced by any smaller number than $\log((1 + x_2)/(1 - x_2))$.

Note that $\log((1 + x_2)/(1 - x_2)) \approx 0.894894 \approx 2.27883(\pi/8)$. In these results, if we assume $F$ to be in $\mathcal{M}_n$ for larger $n$, then we can make the involved constants better.

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T. SUGAWA


DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY, HIGASHIHIROSHIMA, 739-8526 JAPAN
E-mail address: sugawa@math.sci.hiroshima-u.ac.jp