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The Hypergeometric Series and Trigonometric Sums

高野勝男 [Takano Katsuo] 茨城大

1. Introduction

Let us consider the following Gauss hypergeometric series,

$$F(-n,2m;2m+n+1;z)=\Sigma_{k=0}^{n}\frac{(2m)_k(-n)_kz^k}{(2m+n+1)_kk!},$$

where m is a positive constant and n is an positive integer. Here $(a)_k=a(a+1)(a+2)\cdots(a+k-1)$ denotes the Pochhammer symbol. Consider the unit circle $C:\ z=e^{it}\ (0\leq t\leq 2\pi)$. Let

$$R(-n,2m;2m+n+1;t)=\Re\{F(-n,2m;2m+n+1;e^{it})\}.$$

We have

$$R(-n,2m;2m+n+1;t)=\Sigma_{k=0}^{n}\frac{(-n)_k(2m)_k}{(2m+n+1)_k}\frac{\cos kt}{k!}. \tag{1}$$

Let

$$I(-n,2m;2m+n+1;t)=\Im\{F(-n,2m;2m+n+1;e^{it})\}.$$

We have

$$I(-n,2m;2m+n+1;t)=\Sigma_{k=1}^{n}\frac{(-n)_k(2m)_k}{(2m+n+1)_k}\frac{\sin kt}{k!}. \tag{2}$$

From the Gauss hypergeometric series $F(-n,2m;2m+n+1;e^{it})$ we can deduce some formulas of trigonometric sums. These results are listed in the following form.

Formula 1. *We obtain*

$$\begin{aligned}&|F(-n,2m;2m+n+1;e^{it})|^2\\ =&\ \frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=0}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_sy^s}{(-2n)_s(2m+n+1)_{n-s}s!}\\ =&\ \Sigma_{s=0}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!y^s, \quad (3)\end{aligned}$$

where $y = 2(1-\cos t)$.

Formula 2. *We obtain*

$$\begin{aligned} & I(-n+1,2m;2m+n+1;t)R(-n,2m;2m+n+1;t) \\ & -R(-n+1,2m;2m+n+1;t)I(-n,2m;2m+n+1;t) \\ = \ & B_1\sin t + B_2\sin 2t + B_3 \sin 3t + \cdots \\ + \ & B_{n-2}\sin(n-2)t + B_{n-1}\sin(n-1)t + B_n \sin nt, \end{aligned} \tag{4}$$

where B_{n-j} *are given by the following formula,*

$$\begin{aligned} B_{n-r} = \ & (-1)^{n-r-1}\Sigma_{k=0}^{r}\frac{(2m)_{k+n-r}(2m)_k}{(2m+n+1)_{k+n-r}(2m+n+1)_k} \\ & \cdot \frac{(n-1)!n!(n-r)}{(n+k-r)!(r-k)!k!(n-k)!}, \quad (r=0,1,2,...,n-1). \end{aligned} \tag{5}$$

Formula 3. *We obtain*

$$\begin{aligned} & n\{I(-n+1,2m;2m+n+1;t)R(-n,2m;2m+n+1;t) \\ & -R(-n+1,2m;2m+n+1;t)I(-n,2m;2m+n+1;t)\} \\ = \ & \frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=1}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_s s y^{s-1}}{(-2n)_s(2m+n+1)_{n-s}s!}\sin t \\ = \ & \Sigma_{s=1}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!sy^{s-1} \\ & \cdot \sin t. \end{aligned} \tag{6}$$

Formula 4. *We obtain the following combinatorial equality.*

$$\begin{aligned} & \frac{(2m)_{n-r+1}}{(2m+n+1)_n\,(2m+n+1)_{r-1}}\binom{n-1}{n-r}\binom{n+r-1}{n}(2(r-1))! \\ = \ & \Sigma_{k=0}^{r-1}\Sigma_{j=0}^{k}(-1)^{r+k+1}\frac{(2m)_{j+n-k}(2m)_j}{(2m+n+1)_{j+n-k}(2m+n+1)_j} \\ & \cdot \frac{(n-1)!n!(n-k)}{(n+j-k)!(k-j)!j!(n-j)!}\cdot\binom{2n-r-k}{r-k-1}. \end{aligned} \tag{7}$$

2. On Formula 1

If m is a positive integer it is known in [5]. In this report m is assumed to be a positive constant. The curve of the function $F(-n, 2m; 2m+n+1; e^{it})$ of the variable t is symmetric with respect to the real axis in the complex plane. We make use of Watson's formula

$$\begin{aligned} &F(-n, b; c; z)F(-n, b; c; Z) \\ = &\frac{(c-b)_n}{(c)_n} F_4[-n, b; c, 1-n+b-c; zZ, (1-z)(1-Z)] \end{aligned}$$

in Slater's book (cf. [3. (8.4.2)]). Let $b = 2m,\ c = 2m+n+1$. Then

$$\begin{aligned} &F_4[-n, 2m; 2m+n+1, -2n; zZ, (1-z)(1-Z)] \\ = &\Sigma_{r=0}^{n}\Sigma_{s=0}^{n}\frac{(-n)_{r+s}(2m)_{r+s}(zZ)^r[(1-z)(1-Z)]^s}{(2m+n+1)_r(-2n)_s r!s!}. \end{aligned}$$

Let $z = e^{it},\ Z = e^{-it}$. Then $zZ = 1$ and $(1-z)(1-Z) = 2(1-\cos t)$. Let $x = 1$ and $y = 2(1-\cos t)$. From the above F_4 we obtain

$$\begin{aligned} &|F(-n, 2m; 2m+n+1; e^{it})|^2 \\ = &\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{r=0}^{n}\Sigma_{s=0}^{n}\frac{(-n)_{r+s}(2m)_{r+s}x^r y^s}{(2m+n+1)_r(-2n)_s r!s!}. \end{aligned} \tag{8}$$

It holds that

$$\begin{aligned} &|F(-n, 2m; 2m+n+1; e^{it})|^2 \\ = &\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=0}^{n}\frac{(-n)_s(2m)_s y^s}{(-2n)_s s!}\left\{\Sigma_{r=0}^{n-s}\frac{(-n+s)_r(2m+s)_r x^r}{(2m+n+1)_r r!}\right\}. \end{aligned}$$

We see that the right hand side of the above equality is strictly positive for all t in the interval $[0, \pi]$ since it holds that

$$\begin{aligned} &\Sigma_{r=0}^{n-s}\frac{(-n+s)_r(2m+s)_r x^r}{(2m+n+1)_r r!} \\ = &\frac{\Gamma(2m+n+1)}{\Gamma(2m+s)\Gamma(n-s+1)}\int_0^1 \tau^{2m+s-1}(1-\tau)^{n-s}(1-x\tau)^{n-s}d\tau. \end{aligned}$$

When $x = 1$ we see by Vandermonde's formula that

$$\begin{aligned} &|F(-n, 2m; 2m+n+1; e^{it})|^2 \\ = &\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=0}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_s y^s}{(-2n)_s(2m+n+1)_{n-s}s!} \\ = &\Sigma_{s=0}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!y^s, \end{aligned}$$

where $y = 2(1-\cos t)$.

3. On Formula 2

From the following relation

$$aF(a+1,b;c;z) = aF(a,b;c;z) + z\frac{d}{dz}F(a,b;c;z)$$

we have

$$\begin{aligned}
& \frac{d}{dt}F(-n,2m;2m+n+1;e^{it}) \\
= & \frac{d}{dz}F(-n,2m;2m+n+1;z)\ ie^{it} \\
= & \{F(-n+1,2m;2m+n+1;e^{it}) - F(-n,2m;2m+n+1;e^{it})\}(-n)i
\end{aligned}$$

and we see that

$$\begin{aligned}
& \frac{d}{dt}|F(-n,2m;2m+n+1;e^{it})|^2 \\
= & \{\frac{d}{dt}F(-n,2m;2m+n+1;e^{it})\}F(-n,2m;2m+n+1;e^{-it}) \\
& +F(-n,2m;2m+n+1;e^{it})\{\frac{d}{dt}F(-n,2m;2m+n+1;e^{-it})\} \\
= & \Big[\{F(-n+1,2m;2m+n+1;e^{it}) - F(-n,2m;2m+n+1;e^{it})\} \\
\cdot & (-n)i\ F(-n,2m;2m+n+1;e^{-it}) \\
+ & F(-n,2m;2m+n+1;e^{it}) \\
\cdot & \{F(-n+1,2m;2m+n+1;e^{-it}) - F(-n,2m;2m+n+1;e^{-it})\}ni\Big] \\
= & \Big[F(-n+1,2m;2m+n+1;e^{it})F(-n,2m;2m+n+1;e^{-it})(-n)i \\
+ & F(-n,2m;2m+n+1;e^{it})F(-n+1,2m;2m+n+1;e^{-it})ni)\Big] \\
= & 2\Re\Big[(-in)F(-n+1,2m;2m+n+1;e^{it}) \\
& \cdot F(-n,2m;2m+n+1;e^{-it})\Big] \\
= & 2(-in)i\Big\{-I(-n,2m;2m+n+1;t)R(-n+1,2m;2m+n+1;t) \\
& +R(-n,2m;2m+n+1;t)I(-n+1,2m;2m+n+1;t)\Big\} \\
= & 2n\Big\{I(-n+1,2m;2m+n+1;t)R(-n,2m;2m+n+1;t) \\
& -R(-n+1,2m;2m+n+1;t)I(-n,2m;2m+n+1;t)\Big\}. \qquad (9)
\end{aligned}$$

Let us denote the derivative of $|F(-n,2m;2m+n+1;e^{it})|^2$ by $2d(t)$. We obtain

$$d(t) = n\Big\{I(-n+1,2m;2m+n+1;t)R(-n,2m;2m+n+1;t)$$

$$-R(-n+1,2m;2m+n+1;t)I(-n,2m;2m+n+1;t)\Big\}.\quad(10)$$

We will show Formula 2. We have

$$\begin{aligned}&d(t)\\ =\ &\Re[(-in)F(-n+1,2m;2m+n+1;e^{it})F(-n,2m;2m+n+1;e^{-it})].\end{aligned}\quad(11)$$

We see that

$$\begin{aligned}
&F(-n+1,2m;2m+n+1;e^{it})F(-n,2m;2m+n+1;e^{-it})\\
=\ &\Big\{\Sigma_{k=0}^{n-1}\frac{(2m)_k(-n+1)_k e^{ikt}}{(2m+n+1)_k k!}\Big\}\Big\{\Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j e^{-ijt}}{(2m+n+1)_j j!}\Big\}\\
=\ &\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j e^{-ijt}}{(2m+n+1)_j j!}\Big\}\\
+\ &\frac{(2m)_1(-n+1)_1}{(2m+n+1)_1 1!}\cdot\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j e^{-i(j-1)t}}{(2m+n+1)_j j!}\Big\}\\
+\ &\frac{(2m)_2(-n+1)_2}{(2m+n+1)_2 2!}\cdot\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j e^{-i(j-2)t}}{(2m+n+1)_j j!}\Big\}\\
+\ &\cdots\\
+\ &\frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!}\cdot\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j e^{-i(j-n+1)t}}{(2m+n+1)_j j!}\Big\}.
\end{aligned}\quad(12)$$

From the above we obtain

$$\begin{aligned}
&\Re\{i\ F(-n+1,2m;2m+n+1;e^{it})F(-n,2m;2m+n+1;e^{-it})\}\\
=\ &\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j\sin(jt)}{(2m+n+1)_j j!}\Big\}\\
+\ &\frac{(2m)_1(-n+1)_1}{(2m+n+1)_1 1!}\cdot\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j\sin(j-1)t}{(2m+n+1)_j j!}\Big\}\\
+\ &\frac{(2m)_2(-n+1)_2}{(2m+n+1)_2 2!}\cdot\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j\sin(j-2)t}{(2m+n+1)_j j!}\Big\}\\
+\ &\cdots\\
+\ &\frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!}\cdot\Big\{\ \Sigma_{j=0}^{n}\frac{(2m)_j(-n)_j\sin(j-n+1)t}{(2m+n+1)_j j!}\Big\}
\end{aligned}\quad(13)$$

Summing the terms up with respect to the same sine functions in the above we obtain

$$\Re\{i\ F(-n+1,2m;2m+n+1;e^{it})F(-n,2m;2m+n+1;e^{-it})\}$$

$$
\begin{aligned}
&= \Big\{\Sigma_{k=0}^{n-1} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+1}(-n)_{k+1}}{(2m+n+1)_{k+1}(k+1)!}\Big\} \sin t \\
&+ \Big\{\Sigma_{k=0}^{n-2} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+2}(-n)_{k+2}}{(2m+n+1)_{k+2}(k+2)!}\Big\} \sin 2t \\
&+ \Big\{\Sigma_{k=0}^{n-3} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+3}(-n)_{k+3}}{(2m+n+1)_{k+3}(k+3)!}\Big\} \sin 3t \\
&+ \cdots \\
&+ \Big\{\Sigma_{k=0}^{3} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-3}(-n)_{k+n-3}}{(2m+n+1)_{k+n-3}(k+n-3)!}\Big\} \\
&\quad \cdot \sin(n-3)t \\
&+ \Big\{\Sigma_{k=0}^{2} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-2}(-n)_{k+n-2}}{(2m+n+1)_{k+n-2}(k+n-2)!}\Big\} \\
&\quad \cdot \sin(n-2)t \\
&+ \Big\{\Sigma_{k=0}^{1} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-1}(-n)_{k+n-1}}{(2m+n+1)_{k+n-1}(k+n-1)!}\Big\} \\
&\quad \cdot \sin(n-1)t \\
&+ \frac{(2m)_n(-n)_n}{(2m+n+1)_n n!} \sin nt \\
&- \Big\{\Sigma_{j=0}^{n-2} \frac{(2m)_{j+1}(-n+1)_{j+1}}{(2m+n+1)_j(j+1)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\Big\} \sin t \\
&- \Big\{\Sigma_{j=0}^{n-3} \frac{(2m)_{j+2}(-n+1)_{j+2}}{(2m+n+1)_k k!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\Big\} \sin 2t \\
&- \Big\{\Sigma_{j=0}^{n-4} \frac{(2m)_{j+3}(-n+1)_{j+3}}{(2m+n+1)_{j+3}(j+3)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\Big\} \sin 3t \\
&- \cdots \\
&- \Big\{\Sigma_{j=0}^{2} \frac{(2m)_{j+n-3}(-n+1)_{j+n-3}}{(2m+n+1)_{j+n-3}(j+n-3)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\Big\} \\
&\quad \cdot \sin(n-3)t \\
&- \Big\{\Sigma_{j=0}^{1} \frac{(2m)_{j+n-2}(-n+1)_{j+n-2}}{(2m+n+1)_{j+n-2}(j+n-2)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\Big\} \\
&\quad \cdot \sin(n-2)t \\
&- \frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} \sin(n-1)t. \qquad (14)
\end{aligned}
$$

After all we obtain

$$
\begin{aligned}
&d(t) \\
&= n\Big[\Big\{\Big(\Sigma_{j=0}^{n-2} \frac{(2m)_{j+1}(-n+1)_{j+1}}{(2m+n+1)_j(j+1)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\Big)
\end{aligned}
$$

$$
\begin{aligned}
- & \left(\Sigma_{k=0}^{n-1} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+1}(-n)_{k+1}}{(2m+n+1)_{k+1}(k+1)!}\right)\Big\} \sin t \\
+ & \Big\{\left(\Sigma_{j=0}^{n-3} \frac{(2m)_{j+2}(-n+1)_{j+2}}{(2m+n+1)_k k!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\right) \\
- & \left(\Sigma_{k=0}^{n-2} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+2}(-n)_{k+2}}{(2m+n+1)_{k+2}(k+2)!}\right)\Big\} \sin 2t \\
+ & \Big\{\left(\Sigma_{j=0}^{n-4} \frac{(2m)_{j+3}(-n+1)_{j+3}}{(2m+n+1)_{j+3}(j+3)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\right) \\
- & \left(\Sigma_{k=0}^{n-3} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+3}(-n)_{k+3}}{(2m+n+1)_{k+3}(k+3)!}\right)\Big\} \sin 3t \\
+ & \cdots \\
+ & \Big\{\left(\Sigma_{j=0}^{2} \frac{(2m)_{j+n-3}(-n+1)_{j+n-3}}{(2m+n+1)_{j+n-3}(j+n-3)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\right) \\
- & \left(\Sigma_{k=0}^{3} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-3}(-n)_{k+n-3}}{(2m+n+1)_{k+n-3}(k+n-3)!}\right)\Big\} \\
& \cdot \sin(n-3)t \\
+ & \Big\{\left(\Sigma_{j=0}^{1} \frac{(2m)_{j+n-2}(-n+1)_{j+n-2}}{(2m+n+1)_{j+n-2}(j+n-2)!} \frac{(2m)_j(-n)_j}{(2m+n+1)_j j!}\right) \\
- & \left(\Sigma_{k=0}^{2} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-2}(-n)_{k+n-2}}{(2m+n+1)_{k+n-2}(k+n-2)!}\right)\Big\} \\
& \cdot \sin(n-2)t \\
+ & \Big\{\frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} \\
- & \left(\Sigma_{k=0}^{1} \frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!} \frac{(2m)_{k+n-1}(-n)_{k+n-1}}{(2m+n+1)_{k+n-1}(k+n-1)!}\right)\Big\} \\
& \cdot \sin(n-1)t \\
- & \frac{(2m)_n(-n)_n}{(2m+n+1)_n n!} \sin nt\Big]. \qquad (15)
\end{aligned}
$$

From the above we obtain the following form,

$$
\begin{aligned}
d(t) \;=\; & n\Big\{B_1 \sin t + B_2 \sin 2t + B_3 \sin 3t + \cdots + B_{n-2} \sin(n-2)t \\
& B_{n-1} \sin(n-1)t + B_n \sin nt\Big\}. \qquad (16)
\end{aligned}
$$

We will determine the coefficients $B_k,\ k = 1, \cdots, n$. In fact we have

$$
B_n = -\frac{(2m)_n(-n)_n}{(2m+n+1)_n n!} = (-1)^{n-1}\frac{(2m)_n}{(2m+n+1)_n}. \qquad (17)
$$

And we have

$$\begin{aligned}
B_{n-1} &= \frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} \\
&\quad -\Sigma_{k=0}^{1}\frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!}\frac{(2m)_{k+n-1}(-n)_{k+n-1}}{(2m+n+1)_{k+n-1}(k+n-1)!} \\
&= \Big\{\frac{(2m)_{n-1}(-n+1)_{n-1}}{(2m+n+1)_{n-1}(n-1)!} - \frac{(2m)_{n-1}(-n)_{n-1}}{(2m+n+1)_{n-1}(n-1)!}\Big\} \\
&\quad -\frac{(2m)(-n+1)}{(2m+n+1)1!}\frac{(2m)_n(-n)_n}{(2m+n+1)_n n!} \\
&= (-1)^{n-1}\Big\{\frac{(2m)_{n-1}}{(2m+n+1)_{n-1}} + \frac{(2m)_{n-1}(-n)}{(2m+n+1)_{n-1}}\Big\} \\
&\quad +\frac{(2m)}{(2m+n+1)}\frac{(2m)_n(n-1)}{(2m+n+1)_n} \\
&= (-1)^{n-2}\Big\{\frac{(2m)_{n-1}(n-1)}{(2m+n+1)_{n-1}} + \frac{(2m)}{(2m+n+1)}\frac{(2m)_n(n-1)}{(2m+n+1)_n}\Big\} \\
&= (-1)^{n-2}\Sigma_{k=0}^{1}\frac{(2m)_{k+n-1}(2m)_k}{(2m+n+1)_{k+n-1}(2m+n+1)_k} \\
&\quad \cdot\frac{(n-1)!n!(n-1)}{(n+k-1)!(1-k)!(n-k)!}. \qquad (18)
\end{aligned}$$

For B_{n-r}, $r = 2, \cdots, n-1$ we have

$$\begin{aligned}
B_{n-r} &= \Sigma_{j=0}^{r-1}\frac{(2m)_{j+n-r}(-n+1)_{j+n-r}}{(2m+n+1)_{j+n-r}(j+n-r)!}\frac{(2m)_j(-n)_j}{(2m+n+1)_j j!} \\
&\quad -\Sigma_{k=0}^{r}\frac{(2m)_k(-n+1)_k}{(2m+n+1)_k k!}\frac{(2m)_{k+n-r}(-n)_{k+n-r}}{(2m+n+1)_{k+n-r}(k+n-r)!} \\
&= \Sigma_{k=0}^{r-1}\frac{(2m)_{k+n-r}(-1)^{k+n-r}(n-1)!}{(2m+n+1)_{k+n-r}(k+n-r)!(r-k-1)!} \\
&\quad \cdot\frac{(2m)_k(-1)^k n!}{(2m+n+1)_k k!(n-k)!} \\
&\quad -\Sigma_{k=0}^{r-1}\frac{(2m)_k(-1)^k(n-1)!}{(2m+n+1)_k k!(n-k-1)!} \\
&\quad \cdot\frac{(2m)_{k+n-r}(-1)^{k+n-r}n!}{(2m+n+1)_{k+n-r}(k+n-r)!(r-k)!} \\
&\quad -\frac{(2m)_r(-1)^r(n-1)!}{(2m+n+1)_r r!(n-r-1)!}\frac{(2m)_n(-1)^n n!}{(2m+n+1)_n n!}. \qquad (19)
\end{aligned}$$

And we see that

$$B_{n-r} = (-1)^{n-r}\Sigma_{k=0}^{r-1}$$

$$\frac{(2m)_{k+n-r}(2m)_k(n-1)!n!}{(2m+n+1)_{k+n-r}(2m+n+1)_k(k+n-r)!(r-1-k)!k!(n-k-1)!}$$
$$\cdot\left\{\frac{1}{n-k}-\frac{1}{r-k}\right\}$$
$$+\quad(-1)^{n-r-1}\frac{(2m)_r(2m)_n(n-1)!}{(2m+n+1)_r(2m+n+1)_n r!(n-r-1)!}$$
$$=\quad(-1)^{n-r-1}\Sigma_{k=0}^{r-1}$$
$$\frac{(2m)_{k+n-r}(2m)_k n!(n-1)!(n-r)}{(2m+n+1)_{k+n-r}(2m+n+1)_k(k+n-r)!(r-k)!k!(n-k)!}$$
$$+\quad(-1)^{n-r-1}\frac{(2m)_n(2m)_r(n-1)!}{(2m+n+1)_n(2m+n+1)_r r!(n-r-1)!}$$
$$=\quad(-1)^{n-r-1}\Sigma_{k=0}^{r}$$
$$\frac{(2m)_{k+n-r}(2m)_k(n-1)!n!(n-r)}{(2m+n+1)_{k+n-r}(2m+n+1)_k(k+n-r)!(r-k)!k!(n-k)!}\qquad(20)$$

and in particular

$$B_1=\Sigma_{k=0}^{n-1}$$
$$\frac{(2m)_{k+1}(2m)_k(n-1)!n!}{(2m+n+1)_{k+1}(2m+n+1)_k(k+1)!(n-k-1)!k!(n-k)!}.\qquad(21)$$

4. On Formula 3

By Theorem 1 it follows

$$|F(-n,2m;2m+n+1;e^{it})|^2$$
$$=\quad\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=0}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_s y^s}{(-2n)_s(2m+n+1)_{n-s}s!}$$
$$=\quad\Sigma_{s=0}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!y^s,$$

where $y=2(1-\cos t)$. Therefore we see that

$$2d(t)$$
$$=\quad\frac{d}{dt}|F(-n,2m;2m+n+1;e^{it})|^2$$
$$=\quad\frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=1}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_s s y^{s-1}2\sin t}{(-2n)_s(2m+n+1)_{n-s}s!}$$
$$=\quad\Sigma_{s=1}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!s y^{s-1}$$
$$\cdot 2\sin t\qquad(22)$$

and after all we obtain

$$\begin{aligned} & d(t) \\ = \quad & \Sigma_{s=1}^{n} \frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! s y^{s-1} \\ & \cdot \sin t. \end{aligned} \tag{23}$$

From the above we can write $d(t)$ such as the following form,

$$\begin{aligned} d(t) \quad = \quad & n \sin\, t \Big[A_0 + A_1(2x)^2 + A_2(2x)^4 + \cdots + A_{n-2}(2x)^{2(n-2)} \\ & + A_{n-1}(2x)^{2(n-1)} \Big], \end{aligned} \tag{24}$$

where let $x = \sin(t/2)$ and we see that

$$A_{n-r} = \frac{(2m)_{n-r+1}}{(2m+n+1)_n\ (2m+n+1)_{r-1}} \binom{n-1}{n-r} \binom{n+r-1}{n} (2(r-1))!. \tag{25}$$

5. On Formula 4

Substituting the following identity in the above (16) in place of $\sin(jt)$,

$$\sin(jt) = \cos\frac{t}{2} \Sigma_{k=0}^{j-1} (-1)^k \binom{j+k}{j-k-1} (2\sin\frac{t}{2})^{2k+1}, \tag{26}$$

then we see that

$$\begin{aligned} & d(t) \\ = \quad & n \sin t \Big[B_1 + B_2 \frac{\sin 2t}{\sin t} + B_3 \frac{\sin 3t}{\sin t} + \cdots \\ + \quad & B_{n-3} \frac{\sin(n-3)t}{\sin t} + B_{n-2} \frac{\sin(n-2)t}{\sin t} \\ + \quad & B_{n-1} \frac{\sin(n-1)t}{\sin t} + B_n \frac{\sin nt}{\sin t} \Big] \\ = \quad & n \sin t \Big\{ B_1 \\ + \quad & B_2\, \Sigma_{k=0}^{1} (-1)^k \binom{2+k}{2-k-1} (2\sin\frac{t}{2})^{2k} \\ + \quad & B_3 \Sigma_{k=0}^{2} (-1)^k \binom{3+k}{3-k-1} (2\sin\frac{t}{2})^{2k} \\ + \quad & \cdots \end{aligned}$$

$$\begin{aligned}
&+ \quad B_{n-2}\Sigma_{k=0}^{n-3}(-1)^k\binom{n+k-2}{n-k-3}(2\sin\frac{t}{2})^{2k} \\
&+ \quad B_{n-1}\Sigma_{k=0}^{n-2}(-1)^k\binom{n+k-1}{n-k-2}(2\sin\frac{t}{2})^{2k} \\
&+ \quad B_n\Sigma_{k=0}^{n-1}(-1)^k\binom{n+k}{n-k-1}(2\sin\frac{t}{2})^{2k}\}. \qquad (27)
\end{aligned}$$

Let $x = \sin(t/2)$. Then from the above we can write $d(t)$ such as the following form,

$$\begin{aligned}
d(t) \quad = \quad & n\sin\ t\Big[A_0 + A_1(2x)^2 + A_2(2x)^4 + \cdots + A_{n-2}(2x)^{2(n-2)} \\
& +A_{n-1}(2x)^{2(n-1)}\Big]. \qquad (28)
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
A_{n-r} \quad = \quad & B_n(-1)^{n-r}\binom{2n-r}{r-1} + B_{n-1}(-1)^{n-r}\binom{2n-r-1}{r-2} \\
+ \quad & B_{n-2}(-1)^{n-r}\binom{2n-r-2}{r-3} \\
+ \quad & B_{n-3}(-1)^{n-r}\binom{2n-r-3}{r-4} \\
+ \quad & \cdots \\
+ \quad & B_{n-r+2}(-1)^{n-r}\binom{2n-2r+2}{1} \\
+ \quad & B_{n-r+1}(-1)^{n-r}\binom{2(n-r)+1}{0} \\
= \quad & \Sigma_{k=0}^{r-1}B_{n-k}(-1)^{n-r}\binom{2n-r-k}{r-k-1}. \qquad (29)
\end{aligned}$$

From the above we see that the following conbinatorial equality holds.

$$\begin{aligned}
& A_{n-r} \\
= \quad & \frac{(2m)_{n-r+1}}{(2m+n+1)_n\ (2m+n+1)_{r-1}}\binom{n-1}{n-r}\binom{n+r-1}{n}(2(r-1))! \\
= \quad & \Sigma_{k=0}^{r-1}\Sigma_{j=0}^{k}(-1)^{r+k+1}\frac{(2m)_{j+n-k}(2m)_j}{(2m+n+1)_{j+n-k}(2m+n+1)_j} \\
\cdot \quad & \frac{(n-1)!n!(n-k)}{(n+j-k)!(k-j)!j!(n-j)!}\cdot\binom{2n-r-k}{r-k-1}. \qquad (30)
\end{aligned}$$

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Katsuo Takano
Department of Mathematics
Faculty of Science, Ibaraki University
Mito, Ibaraki 310, Japan
ktaka@mx.ibaraki.ac.jp