On sufficient conditions for Carathéodory functions

Dingong Yang, Shigeyoshi Owa, and Kyohei Ochiai

Abstract


1 Introduction

Let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_{n}z^{n}$$

which are analytic in the unit disk $\mathbb{E} = \{z ||z|<1\}$. If $p(z)$ in $\mathcal{P}$ satisfies $\text{Re}(p(z)) > 0$ for $z \in \mathbb{E}$, then we say that $p(z)$ is the Carathéodory function.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{E}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{E}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{E}$ such that $|w(z)| \leq |z|$ and $f(z) = g(w(z)) \ (z \in \mathbb{E})$. If $g(z)$ is univalent in $\mathbb{E}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n}$$

which are analytic in $\mathbb{E}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $\alpha$ in $\mathbb{E}$ if it satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \ (z \in \mathbb{E})$$

2000 Mathematics Subject Classification: Primary 30C45.

Key Words and Phrases: Carathéodory function, starlike function, subordination.
for some $\alpha (0 \leq \alpha < 1)$. We denote by $S^*(\alpha) (0 \leq \alpha < 1)$ the subclass of $A$ consisting of all starlike functions of order $\alpha$ in $E$. Also we denote by $S^*(0) = S^*$. For $-1 \leq a \leq 1$, $-1 \leq b \leq 1$ and $a \neq b$, a function $f(z)$ in $A$ is said to be in the class $S^*(a, b)$ if satisfies

$$\frac{zf'(z)}{f(z)} < \frac{1+az}{1+bz} \quad (z \in E).$$

The class $S^*(a, b)$ can be reduced to several well known classes of starlike functions by selecting special values for $a$ and $b$. In particular,

$$S^*(1-2\alpha, -1) = S^*(2\alpha-1, 1) = S^*(\alpha) \quad (0 \leq \alpha < 1).$$

For Carathéodory functions, Nunokawa et al. [3] have given the following two theorems.

**Theorem A.** If $p(z) \in \mathcal{P}$ satisfies

$$\alpha(p(z))^2 + \beta z p'(z) < \frac{2\alpha + \beta}{2} \left(\frac{1+z}{1-z}\right)^2 - \frac{\beta}{2},$$

where $\beta > 0$ and $\alpha > -\frac{\beta}{2}$, then $\text{Re}(p(z)) > 0 \quad (z \in E)$.

**Theorem B.** Let $p(z) \in \mathcal{P}$ and $w(z)$ be analytic in $E$ with $w(0) = \alpha$ and $w(z) \neq ik \quad (k \in \mathbb{R}, z \in E)$. If

$$\alpha(p(z)) + \beta \frac{z p'(z)}{p(z)} < w(z),$$

where $\alpha > 0$, $\beta > 0$ and $k^2 \geq \beta(2\alpha + \beta)$, then $\text{Re}(p(z)) > 0 \quad (z \in E)$.

For the starlikeness of functions in $A$, the following results have been proved.

**Theorem C ([4]).** If $f(z) \in A$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right) \right\} > -\frac{1}{2} \quad (z \in E),$$

then $f(z) \in S^*$.

**Theorem D ([1]).** If $f(z) \in A$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\frac{zf''(z)}{f(z)} + 1\right) \right\} > 0 \quad (z \in E),$$

then $f(z) \in S^* \left(\frac{1}{2}\right)$.

**Theorem E ([5]).** If $f(z) \in A$ satisfies $f'(z) \neq 0$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} < 2 - \frac{2}{(1-z)^2} \quad (z \in E),$$

then $f(z) \in S^*$.

**Theorem F ([5]).** If $f(z) \in A$ satisfies $f'(z) \neq 0$ and

$$\left| \frac{f(z)f''(z)}{(f'(z))^2} \right| < 2 \quad (z \in E),$$

then $f(z) \in S^*$. 
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In this paper we shall generalize or refine the above results.

To derive our results, we need the following lemma due to Miller and Mocanu [2].

**Lemma.** Let $g(z)$ be analytic and univalent in $E$, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $g(E)$, with $\phi(w) \neq 0$ when $w \in g(E)$. Set

$$Q(z) = zg'(z)\phi(g(z)), h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) $Q(z)$ is univalent and starlike in $E$, and

(ii) $\text{Re} \left\{ \frac{zg'(z)}{Q(z)} \right\} = \text{Re} \left\{ \frac{\theta'(g(z)) + zQ'(z)}{\phi(g(z))} \right\} > 0 \quad (z \in E)$.

If $p(z)$ is analytic in $E$, with $p(0) = g(0)$, $p(E) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(g(z)) + zg'(z)\phi(g(z)) = h(z),$$

then $p(z) < g(z)$ is the best dominant of the subordination.

## 2 Main results

Our main result is contained in

**Theorem 1.** Let $a$, $b$, $\lambda$ and $\mu$ satisfy either

(i) $0 < a = -b \leq 1$, $\lambda > -\frac{1}{2}$, $\mu \in \mathbb{C}$ and $\text{Re}(\mu) \geq 0$, or

(ii) $-1 \leq b < a \leq 1$, $\lambda \geq 0$, $\mu \in \mathbb{C}$ and $\text{Re}(\mu) \geq -2\lambda \frac{(1-a)}{(1-b)}$.

If $p(z) \in P$ and

(1) $\lambda(p(z))^2 + \mu p(z) + zp'(z) < h(z),$

where

(2) $h(z) = \frac{a(\lambda a + \mu b)z^2 + (2\lambda a + \mu(a + b) + a - b)z + \lambda + \mu}{(1 + bz)^2}$

then $p(z) < \frac{1 + az}{1 + bz}$ and $\frac{1 + az}{1 + bz}$ is the best dominant of (1).

**Proof.** Set

(3) $g(z) = \frac{1 + az}{1 + bz}, \theta(w) = \lambda w^2 + \mu w, \phi(w) = 1.$

Then $g(z)$ is analytic and univalent in $E$, $g(0) = p(0) = 1$, $\theta(w)$ and $\phi(w)$ are analytic with $\phi(w) \neq 0$ in the $w$-plane. The function

(4) $Q(z) = zg'(z)\phi(g(z)) = \frac{(a - b)z}{(1 + bz)^2}$
is univalent and starlike in $\mathbb{E}$ because
\[
\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \text{Re} \left( \frac{1-bz}{1+bz} \right) > 0 \quad (z \in \mathbb{E}).
\]

Further, we have
\[
\theta(g(z)) + Q(z) = \lambda \left( \frac{1+az}{1+bz} \right)^2 + \mu \frac{1+az}{1+bz} + \frac{(a-b)z}{(1+bz)^2}
\]
\[
= \frac{a(\lambda a + \mu b)z^2 + (2\lambda a + \mu(a+b) + a-b)z + \lambda + \mu}{(1+bz)^2} = h(z)
\]
and
\[
= \frac{zh'(z)}{Q(z)} = 2\lambda \frac{1+az}{1+bz} + \mu + \frac{1-bz}{1+bz}.
\]

Therefore
\[
(5) \quad \theta(g(z)) + Q(z) = \lambda \left( \frac{1+az}{1+bz} \right)^2 + \mu \frac{1+az}{1+bz} + \frac{(a-b)z}{(1+bz)^2}
\]
\[
= \frac{a(\lambda a + \mu b)z^2 + (2\lambda a + \mu(a+b) + a-b)z + \lambda + \mu}{(1+bz)^2} = h(z)
\]
and
\[
= \frac{zh'(z)}{Q(z)} = 2\lambda \frac{1+az}{1+bz} + \mu + \frac{1-bz}{1+bz}.
\]

Therefore
\[
(i) \quad \text{For } 0 < a = -b \leq 1, \lambda > -\frac{1}{2}, \text{ and } \text{Re}(\mu) \geq 0, \text{ it follows from (6) that}
\]
\[
\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = (2\lambda + 1)\text{Re} \left( \frac{1-bz}{1+bz} \right) + \text{Re}(\mu) > 0 \quad (z \in \mathbb{E}).
\]

(ii) \text{For } -1 \leq b < a \leq 1, \lambda \geq 0, \text{ and } \text{Re}(\mu) \geq -2\lambda \frac{(1-a)}{(1-b)}, \text{ from (6) we get}
\[
\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 2\lambda \frac{1-a}{1-b} + \text{Re}(\mu) \geq 0 \quad (z \in \mathbb{E}).
\]

Thus the function $h(z)$ in (5) is close-to-convex and univalent in $\mathbb{E}$. From (1) to (5), we see that
\[
\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(g(z)) + zg'(z)\phi(g(z)) = h(z).
\]

Therefore, by applying the lemma, we conclude that $p(z) < g(z)$ and $g(z)$ is the best dominant of (1). The proof of the theorem is complete.

\[
\square
\]

Remark 1. For $a = -b = 1, \lambda = \frac{\alpha}{\beta}, \beta > 0$, $\alpha > -\frac{\beta}{2}$ and $\mu = 0$, Theorem 1 (i) coincides with Theorem A by Nunokawa et al [3].

Corollary 1. If $f(z) \in A$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and
\[
(7) \quad \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{(a-b)z}{(1+bz)^2} \quad (z \in \mathbb{E})
\]

for some $a$ and $b \ (-1 \leq b < a \leq 1)$, then $f(z) \in S^*(a,b)$. 

Proof. Let \( p(z) = \frac{zf'(z)}{f(z)} \). Then \( p(z) \in \mathcal{P} \) and (7) can be written as

\[
(8) \quad z f'(z) < \frac{(a - b)z}{(1 + bz)^2}
\]

Putting \( \lambda = \mu = 0 \) in Theorem 1 (ii) and using (8), the desired result follows at once.

\[ \square \]

Remark 2. Corollary 1 with \( a = 1 - 2 \alpha \ (0 \leq \alpha < 1) \) and \( b = -1 \) implies that if \( f(z) \in \mathcal{A} \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[
\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)}\right) < 2(1 - \alpha) \frac{z}{(1 - z)^2},
\]

then \( f(z) \in S^*(\alpha) \) and the order \( \alpha \) is sharp for \( f(z) = \frac{z}{(1 - z)^2(1 - \alpha)} \). When \( \alpha = 0 \), this result improves Theorem C by Owa and Obradović [4].

Corollary 2. If \( f(z) \in \mathcal{A} \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[
(9) \quad \frac{zf''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1) \left(\frac{zf'(z)}{f(z)}\right)^2 < \frac{\lambda + z}{(1 - z)^2} \quad (z \in \mathbb{E})
\]

for some \( \lambda \ (\lambda \geq 0) \), then \( f(z) \in S^*\left(\frac{1}{2}\right) \) and the order \( \frac{1}{2} \) is sharp.

Proof. If we let \( p(z) = \frac{zf'(z)}{f(z)} \), then \( p(z) \in \mathcal{P} \) and it follows from (9) that

\[
(10) \quad \lambda(p(z))^2 + z f'(z) = \frac{zf''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1) \left(\frac{zf'(z)}{f(z)}\right)^2
\]

\[
< \frac{\lambda + z}{(1 - z)^2}.
\]

Taking \( a = 0 \), \( b = -1 \), \( \lambda \geq 0 \) and \( \mu = 0 \) in Theorem 1 (ii) and using (10), we know that \( f(z) \in S^*\left(\frac{1}{2}\right) \).

For \( f(z) = \frac{z}{(1 - z)} \), we have

\[
\frac{zf''(z)}{f(z)} + \frac{zf'(z)}{f(z)} + (\lambda - 1) \left(\frac{zf'(z)}{f(z)}\right)^2 = \frac{\lambda + z}{(1 - z)^2}
\]

and

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} \rightarrow \frac{1}{2} \quad \text{as} \quad z \to -1.
\]

Hence the corollary is proved.

\[ \square \]
Remark 3. If we put $h(z) = \frac{\lambda + z}{(1-z)^2}$ ($\lambda > 0$), then

$$h(e^{i\theta}) = -\frac{1 + \lambda \cos \theta - i \lambda \sin \theta}{2(1 - \cos \theta)} \quad (0 < \theta < 2\pi)$$

and hence

$$h(\mathbb{E}) = \left\{ w = u + iv : v^2 > -\frac{\lambda^2}{1 + \lambda} \left( u - \frac{\lambda - 1}{4} \right) \right\},$$

which properly contains the half plane $\text{Re}(w) > \frac{\lambda - 1}{4}$. Thus Corollary 2 with $\lambda = 1$ improves Theorem D by Li and Owa [1].

Corollary 3. Let $-1 \leq b < a \leq 1$ and $\text{Re}(\mu) \geq 0$. If $f(z) \in A$ satisfies $f'(z) \neq 0$ and

$$(11) \quad (1 - \mu)\frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} < h(z) \quad (z \in \mathbb{E}),$$

where

$$(12) \quad h(z) = \frac{b(b - \mu a)z^2 + (3b - a - \mu(a+b))z + 1 - \mu}{(1+bz)^2},$$

then $f(z) \in S^*(b,a)$.

Proof. Let us define $p(z)$ in $\mathbb{E}$ by

$$(13) \quad p(z) = \frac{f(z)}{zf'(z)}.$$ 

Then $p(z) \in \mathcal{P}$ and it follows from (11), (12) and (13) that

$$\mu p(z) + zp'(z) = 1 + (\mu - 1)\frac{f(z)}{zf'(z)} - \frac{f(z)f''(z)}{(f'(z))^2} < \frac{\mu abz^2 + (\mu(a+b) + a - b)z + \mu}{(1+bz)^2} \quad (z \in \mathbb{E}).$$

Therefore, by applying Theorem 1 (ii) with $\lambda = 0$ and $\text{Re}(\mu) \geq 0$, we have

$$p(z) = \frac{f(z)}{zf'(z)} < \frac{1 + az}{1+bz}.$$ 

This implies that $f(z) \in S^*(b,a)$.

Remark 4. Letting $a = 1$, $b = -1$ and $\mu = 1$ in Corollary 3, we get Theorem E by Tuneski [5].

For $a = 1$, $b = 0$ and $\mu = 1$, Corollary 3 lead to

Corollary 4. If $f(z) \in A$ satisfies $f'(z) \neq 0$ and

$$\left| \frac{f(z)f''(z)}{(f'(z))^2} \right| < 2 \quad (z \in \mathbb{E}),$$
then $f(z) \in S^*\left(\frac{1}{2}\right)$ and the order $\frac{1}{2}$ is sharp for the function $f(z) = \frac{z}{1 - z}$.

**Remark 5.** Corollary 4 refines Theorem F by Tuneski [5].

Taking $a = 0$, $b = -c$ and $\mu = 1$ in Corollary 3, we have

**Corollary 5.** If $f(z) \in A$ satisfies $f'(z) \neq 0$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} < 1 - \frac{1}{(1 - cz)^2} \quad (z \in \mathbb{E})$$

for some $c (0 < c \leq 1)$, then

(14) $$\left| \frac{zf'(z)}{f(z)} - 1 \right| < c \quad (z \in \mathbb{E}).$$

The bound $c$ in (14) is sharp for the function $f(z) = ze^{-cz}$.

Next we derive

**Theorem 2.** Let $-1 \leq b < a \leq 1$, $\lambda \geq 0$ and $\mu \geq -\frac{1-a}{1-b}$. If $p(z) \in \mathcal{P}$ with

$$p(z) \neq -\mu \quad (z \in \mathbb{E})$$

and

(15) $$\lambda p(z) + \frac{zp'(z)}{p(z) + \mu} < h(z) \quad (z \in \mathbb{E}),$$

where

$$h(z) = \frac{\lambda acz^2 + (\lambda(a + c) + c - b)z + \lambda}{(1+bz)(1+cz)}, c = \frac{a + \mu b}{1 + \mu},$$

then $p(z) < \frac{1+az}{1+bz}$ and $\frac{1+az}{1+bz}$ is the best dominant of (15).

**Proof.** We choose

$$g(z) = \frac{1+az}{1+bz}, \quad \theta(w) = \lambda w, \quad \phi(w) = \frac{1}{w + \mu}$$

and $\mathbb{D} = w : w \neq -\mu$ in the Lemma. Noting that

(16) $$\text{Re} \{g(z)\} > \frac{1-a}{1-b} \geq -\mu \quad (z \in \mathbb{E}),$$

the function $\phi(w)$ is analytic in $\mathbb{D}$ containing $g(\mathbb{E})$. From (16) we see that

$$1 + \mu > 0, \quad -1 \leq b < c = \frac{a + \mu b}{1 + \mu} \leq 1.$$ 

The function

$$Q(z) = zg'(z)\phi(g(z)) = \frac{(c-b)z}{(1+bz)(1+cz)}$$

is univalent and starlike in $\mathbb{E}$ because

$$\text{Re} \left\{ \frac{zQ'(z)}{Q(z)} \right\} = -1 + \text{Re} \left( \frac{1}{1+bz} \right) + \text{Re} \left( \frac{1}{1+cz} \right)$$
\[-1 + \frac{1}{1 + |b|} + \frac{1}{1 + |c|} = \frac{1 - |bc|}{(1 + |b|)(1 + |c|)} \geq 0\]

for \(z \in E\). Further, we have

\[
\theta(g(z)) + Q(z) = \frac{1 + az}{1 + bz} + \frac{(c - b)z}{(1 + bz)(1 + cz)} = h(z)
\]

and

\[
\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \lambda(1 + \mu)\text{Re} \left( \frac{1 + cz}{1 + bz} \right) + \text{Re} \left( \frac{zQ'(z)}{Q(z)} \right) > \lambda(1 + \mu)\frac{1 - c}{1 - b} \geq 0 \quad (z \in E)
\]

for \(\lambda \geq 0\). The other conditions of the lemma are seen to be satisfied. Hence \(p(z) < g(z)\) and \(g(z)\) is the best dominant of (15). The proof is complete.

\[\square\]

**Remark 6.** Note that the univalent function \(h(z)\) defined by

\[
h(z) = \frac{\alpha z^2 + 2(\alpha + \beta)z + \alpha}{1 - z^2} = \alpha \frac{1 + z}{1 - z} + 2\beta \frac{z}{1 - z^2} \quad (\alpha > 0, \beta > 0)
\]

maps \(E\) onto the complex plane minus the half-lines

\[l_1 = w = u + iv : u = 0, \quad v \geq \sqrt{\beta(2\alpha + \beta)}\]

and

\[l_2 = w = u + iv : u = 0, \quad v \leq -\sqrt{\beta(2\alpha + \beta)}.
\]

For \(a = 1, b = -1, \lambda = \frac{\alpha}{\beta}, \alpha > 0, \beta > 0\) and \(\mu = 0\), Theorem 2 reduces to Theorem B by Nunokawa et al [3].

Theorem 2 with \(\mu = 0\) and \(p(z) = \frac{zf'(z)}{f(z)}\) leads to the following corollary.

**Corollary 6.** Let \(-1 \leq b < a \leq 1\) and \(\lambda \geq 0\). If \(f(z) \in A\) satisfies \(f(z)f'(z) \neq 0\) in \(0 < |z| < 1\) and

\[
(\lambda - 1)\frac{zf'(z)}{f(z)} + 1 + \frac{zf''(z)}{f'(z)} < h(z) \quad (z \in E),
\]

where

\[
h(z) = \frac{\lambda a^2z^2 + (2\lambda a + a - b)z + \lambda}{(1 + az)(1 + bz)},
\]

then \(f(z) \in S^*(a, b)\).
References


Dinggong Yang  
Department of Mathematics  
Suzhou University  
Suzhou, Jiangsu 215006  
People’s Republic of China

Shigeyoshi Owa  
Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577-8502  
Japan  
e-mail: owa@math.kindai.ac.jp

Kyohei Ochiai  
Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577-8502  
Japan  
e-mail: ochiai@math.kindai.ac.jp