On a subclass of $n$-starlike functions

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ABSTRACT. In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$ and $w$ is a fixed point in $U$. In [1] the authors introduced the classes of functions close to convex and $\alpha$-convex, which are normalized in the same way. All these definitions are somewhat similar to the ones for the uniformly type functions and is easy to see that for $w = 0$ are obtained the well-known classes of starlike, convex, close to convex and $\alpha$-convex functions. In this paper we continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where $w$ is a fixed point in $U$.

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1 Introduction

Let $H(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in H(U) : f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

We recall here the definitions of the well-known classes of starlike and convex functions:

$$S^* = \left\{ f \in A : Re \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\},$$

$$S^c = \left\{ f \in A : Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ z \in U \right\}$$

Let $w$ be a fixed point in $U$ and $A(w) = \{f \in H(U) : f(w) = f'(w) - 1 = 0\}$. In [3] S. Kanas and F. Ronning introduced the following classes:

$$S(w) = \{f \in A(w) : f$ is univalent in $U\}$$

$$ST(w) = S^*(w) = \left\{ f \in S(w) : Re \frac{(z-w)f'(z)}{f(z)} > 0, \ z \in U \right\}$$

$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + \frac{(z-w)f''(z)}{f'(z)} > 0, \ z \in U \right\}.$$

It is obvious that exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

$g \in S^c(w)$ if and only if $f(z) = (z-w)g'(z) \in S^*(w)$. 
Let denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$ that are regular in $U$ and satisfy $p(w) = 1$ and $Re\, p(z) > 0$ for $z \in U$.

2 Preliminary results

If is easy to see that a function $f(z) \in A(w)$ have the series expansions:

$$f(z) = (z - w) + a_2(z - w)^2 + ...$$

In [8] J. K. Wald gives the sharp bounds for the coefficients $B_n$ of the function $p \in \mathcal{P}(w)$:

**Teorema 2.1** If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z - w)^n$, then

$$|B_n| \leq \frac{2}{(1+d)(1-d)^n}, \text{ where } d = |w| \text{ and } n \geq 1.$$  \hspace{1cm} (1)

Using the above result, S. Kanas and F. Ronning obtain in [3]:

**Teorema 2.2** Let $f \in S^*(w)$ and $f(z) = (z - w) + b_2(z - w)^2 + ...$ Then

$$|b_2| \leq \frac{2}{1-d^2}, \quad |b_3| \leq \frac{3 + d}{(1-d^2)^2},$$

$$|b_4| \leq \frac{2}{3} \cdot \frac{(2 + d)(3 + d)}{(1-d^2)^3}, \quad |b_5| \leq \frac{1}{6} \cdot \frac{(2 + d)(3 + d)(3d + 5)}{(1-d^2)^4}$$

(2)

where $d = |w|$.

**Remark 2.1** It is clear that the above theorem also provides bounds for the coefficients of functions in $S^*(w)$, due to the relation between $S^*(w)$ and $S^*(w)$.

In [1] are also defined the following sets:

$$D(w) = \left\{ z \in U : Re\left[\frac{w}{z}\right] < 1 \text{ and } Re\left[\frac{z(1+z)}{(z-w)(1-z)}\right] > 0 \right\} \text{ for } w \neq 0 \text{ and } D(0) = U;$$

$$s(w) = \{ f : D(w) \to \mathbb{C}\} \cap S(w) ; s^*(w) = S^*(w) \cap s(w)$$

where $w$ is a fixed point in $U$. 

The authors consider the integral operator \( L_a : A(w) \rightarrow A(w) \) defined by

\[
(3) \quad f(z) = L_a F(z) = \frac{1 + a}{(z - w)^a} \cdot \int_{w}^{z} F(t) \cdot (t - w)^{a-1} dt, \quad a \in \mathbb{R}, \quad a \geq 0.
\]

The next theorem is results of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [3], [4], [5]).

**Theorem 2.3** Let \( h \) convex in \( U \) and \( \Re[\beta h(z) + \gamma] > 0, \ z \in U \). If \( p \in \mathcal{H}(U) \) with \( p(0) = h(0) \) and \( p \) satisfied the Briot - Bouquet differential subordination

\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then} \quad p(z) \prec h(z).
\]

### 3 Main results

**Definition 3.1** Let \( w \) be a fixed point in \( U, \ n \in \mathbb{N} \). We denote by \( D^n_w \) the differential operator:

\[
D^n_w : A(w) \rightarrow A(w) \text{ with:}
\]

\[
D^0_w f(z) = f(z)
\]

\[
D^1_w f(z) = D_w f(z) = (z - w) \cdot f'(z)
\]

\[
D^n_w f(z) = D_w (D^{n-1}_w f(z)).
\]

**Remark 3.1** For \( f \in A(w), \ f(w) = (z - w) + \sum_{j=2}^{\infty} a_j (z - w)^j, \) we have

\[
D^n_w f(z) = (z - w) + \sum_{j=2}^{\infty} j^n \cdot a_j \cdot (z - w)^j.
\]

It easy to see that if we take \( w = 0 \) we obtain the Sălăgean differential operator (see [7]).

**Definition 3.2** Let \( w \) be a fixed point in \( U, \ n \in \mathbb{N} \) and \( f \in S(w) \). We say that \( f \) is a \( n-w \)-starlike functions if

\[
\Re \frac{D^{n+1}_w f(z)}{D^n_w f(z)} > 0, \ z \in U.
\]

We denote the class of all this functions by \( S_n^*(w) \).
Remark 3.2 1. \( S^*_0(w) = S^*(w) \) and \( S^*_0(0) = S^*_n \), where \( S^*_n \) is the class of \( n \)-starlike functions introduced by Sălăgean in [7].

2. If \( f(z) \in S^*_n(w) \) and we denote \( D^*_w f(z) = g(z) \), we obtain \( g(z) \in S^*(w) \).

3. Using the class \( s(w) \), we obtain \( s^*_n(w) = S^*_n(w) \cap s(w) \).

\textbf{Teorema 3.1}  Let \( w \) be a fixed point in \( U \) and \( n \in \mathbb{N} \). If \( f(z) \in s^*_n+1(w) \) then \( f(z) \in s^*_n(w) \). This means

\[ s^*_n+1(w) \subset s^*_n(w). \]

\textbf{Proof.} From \( f(z) \in s^*_n+1(w) \) we have \( Re \frac{D^{n+2}_w f(z)}{D^{n+1}_w f(z)} > 0 \), \( z \in U \).

We denote \( p(z) = \frac{D^{n+1}_w f(z)}{D^n_w f(z)} \), where \( p(0) = 1 \) and \( p(z) \in \mathcal{H}(U) \).

We obtain:

\[
\frac{D^{n+2}_w f(z)}{D^{n+1}_w f(z)} = \frac{D_w (D^{n+1}_w f(z))}{D_w (D^n_w f(z))} = \frac{(z-w)(D^{n+1}_w f(z))'}{(z-w)(D^n_w f(z))'} = \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'}
\]

\[
p'(z) = \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'} \cdot \frac{(D^n_w f(z))'}{(D^n_w f(z))} - p(z) \cdot \frac{(D^n_w f(z))'}{(D^n_w f(z))}.
\]

Thus we have:

\[
(z-w) \cdot p'(z) = \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'} \cdot (z-w) \cdot (D^n_w f(z))' - p(z) \cdot (z-w) \cdot (D^n_w f(z))' =
\]

\[
(z-w) \cdot p'(z) = \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))} \cdot p(z) - [p(z)]^2
\]

and

\[
\frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))} = p(z) + \frac{1}{p(z)} \cdot (z-w) \cdot p'(z).
\]

From \( Re \frac{D^{n+2}_w f(z)}{D^{n+1}_w f(z)} > 0 \) we obtain \( p(z) + \frac{1}{p(z)} \cdot (z-w) \cdot p'(z) < \frac{1+z}{1-z} \)

or

\[
p(z) + \frac{zp'(z)}{1-w} < \frac{1+z}{1-z} = h(z), \text{ with } h(0) = 1.
\]

From hypothesis we have \( Re \left[ \frac{1}{1-w} \cdot h(z) \right] > 0 \), and thus from Theorem 2.3 we obtain \( p(z) < h(z) \) or \( Re p(z) > 0 \). This means \( f \in s^*_n(w) \).
Remark 3.3 From Theorem 3.1 we obtain \( s_n^*(w) \subseteq s_0^*(w) \subseteq S^*(w) \), \( n \in \mathbb{N} \).

**Theorem 3.2** If \( F(z) \in s_n^*(w) \) then \( f(z) = L_\alpha F(z) \in S_n^*(w) \), where \( L_\alpha \) is the integral operator defined by (3).

**Proof.** From (3) we obtain

\[
(1 + a) \cdot F(z) = a \cdot f(z) + (z - w) \cdot f'(z).
\]

By means of the application of the operator \( D_w^{n+1} \) we obtain

\[
(1 + a) \cdot D_w^{n+1}F(z) = a \cdot D_w^{n+1}f(z) + D_w^{n+1}[(z - w) \cdot f'(z)]
\]

or

\[
(1 + a) \cdot D_w^{n+1}F(z) = a \cdot D_w^{n+1}f(z) + D_w^{n+2}f(z).
\]

Similarly, by means of the application of the operator \( D_w^n \) we obtain

\[
(1 + a) \cdot D_w^nF(z) = a \cdot D_w^n f(z) + D_w^{n+1}f(z).
\]

Thus

\[
\frac{D_w^{n+1}F(z)}{D_w^nF(z)} = \frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} \cdot \frac{D_w^{n+1}f(z)}{D_w^n f(z)} + a \cdot \frac{D_w^{n+1}f(z)}{D_w^n f(z)}.
\]

Using the notation \( \frac{D_w^{n+1}f(z)}{D_w^n f(z)} = p(z) \), with \( p(0) = 1 \), we have

\[
\frac{(z - w) \cdot p'(z)}{p(z)} = \frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} - p(z)
\]

or

\[
\frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} = p(z) + \frac{(z - w) \cdot p'(z)}{p(z)}.
\]

Thus

\[
\frac{D_w^{n+1}F(z)}{D_w^nF(z)} = \frac{p(z)}{p(z) + a} \left[ \frac{p(z) + (z - w)p'(z)}{p(z)} + a \right] = p(z) + \frac{zp'(z)}{1 - \frac{w}{z}p(z) + \frac{a}{1 - \frac{w}{z}}}
\]
From $F(z) \in s_n^*(w)$ we obtain
\[ \frac{D_{w}^{n+1}F(z)}{D_{w}^{n}F(z)} \prec \frac{1+z}{1-z} \equiv h(z) \text{ or } p(z) + \frac{zp'(z)}{1-wp(z)} + \frac{a}{1-w} \prec h(z). \]

From hypothesis we have $Re \left[ \frac{1}{1-w} \cdot h(z) + \frac{a}{1-w} \right] > 0$ and from Theorem 2.3 we obtain $p(z) \prec h(z)$ or $Re \left\{ \frac{D_{w}^{n+1}f(z)}{D_{w}^{n}f(z)} \right\} > 0$, $z \in U$. This means $f(z) = L_a F(z) \in S_n^*(w)$.

**Remark 3.4** If we consider $w = 0$ in Theorem 3.2 we obtain that the integral operator defined by (3) preserve the class of $n$-starlike functions, and if we consider $w = 0$ and $n = 0$ in the above Theorem we obtain that the integral operator defined by (3) preserve the well-known class of starlike functions.

**Theorema 3.3** Let $w$ be a fixed point in $U$ and $f \in S_n^*(w)$ with $f(z) = (z - w) + \sum_{j=2}^{\infty} a_j (z - w)^j$. Then we have:

\[ |a_2| \leq \frac{1}{2^{n-1} \cdot (1 - d^2)} ; \]
\[ |a_3| \leq \frac{3 + d}{3^n \cdot (1 - d^2)^2} ; \]
\[ |a_4| \leq \frac{(2 + d)(3 + d)}{2^{2n-1} \cdot 3 \cdot (1 - d^2)^3} ; \]
\[ |a_5| \leq \frac{(2 + d)(3 + d)(3d + 5)}{5^n \cdot 6 \cdot (1 - d^2)^4} , \]

where $d = |w|$.

**Proof.** From Remark 3.2 for $f \in S_n^*(w)$ we obtain
\[ D_w^n f(z) = g(z) \in S^*(w) . \]

If we consider $g(z) = (z - w) + \sum_{j=2}^{\infty} b_j (z - w)^j$, using Remark 3.1, from (4) we obtain
\[ j^n \cdot a_j = b_j , j = 2, 3, ... . \]

Thus we have $a_j = \frac{1}{j^n} \cdot b_j , j = 2, 3, ...$, and from the estimates (2) we get the result.
References


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