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On certain subclasses of analytic functions involving a linear operator

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Abstract

A certain linear operator defined by a Hadamard product or convolution for functions which are analytic in the open unit disk is introduced. The object of the present paper is to derive some properties of this linear operator.

1 Introduction

A_p : the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in N) \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$, and $A_1 = A$.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$,

we define the Hadamard product (or convolution)

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$$

<u>Def. 1.1</u> Let $\phi_p(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}$ (1.2)

($c \neq 0, -1, -2, \cdots$; $z \in U$), where $(x)_n$ is the Pochhammer symbol.

<u>Remark 1.1</u> $\phi_p(a, c; z) = z^p \cdot {}_2F_1(1, a; c; z)$, where

$${}_2F_1 = \sum_{n=0}^{\infty} \frac{(1)_n (a)_n}{(c)_n} \cdot \frac{z^n}{n!}$$

<u>Def. 1.2.</u> $$\mathcal{L}_p(a,c)f(z) = \phi_p(a,c;z) * f(z) \qquad (f(z)\in A_p) \qquad (1.3)$$

<u>Def. 1.3.</u> A function $f(z)\in A_p$ is said to be p-valently starlike

$$\Longleftrightarrow \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z\in U). \qquad (1.4)$$

A function $f(z)\in A_p$ is said to be p-valently convex

$$\Longleftrightarrow \quad 1+\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z\in U). \qquad (1.5)$$

These subclasses are denoted by S_p^* and K_p, respectively.

<u>Remark 1.2</u> $$\mathcal{L}_p(\nu+p,1)f(z) = \frac{z^p}{(1-z)^{\nu+p}} * f(z) = D^{\nu+p-1}f(z), \qquad (1.6)$$

$(f(z)\in A_p,\ \nu>-p)$

$$\mathcal{L}_p(\nu+p,\nu+p+1)f(z) = \frac{\nu+p}{z^\nu}\int_0^z t^{\nu-1}f(t)\,dt \quad (\nu+p>0). \qquad (1.7)$$

2. Some properties of certain analytic functions involving the operator $\mathcal{L}_p(a,c)$.

<u>Lemma 2.1</u> For $f(z)\in A_p$, we have

$$z(\mathcal{L}_p(a,c)f(z))' = a\,\mathcal{L}_p(a+1,c)f(z) - (a-p)\,\mathcal{L}_p(a,c)f(z), \qquad (2.1)$$

where $c \neq 0, -1, -2, \cdots$.

<u>Lemma 2.2.</u> Let $\varphi(z)$ be analytic in U and satisfies $|\varphi(z)|\leqq 1$, then

$$|\varphi'(z)| \leqq \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z\in U). \qquad (2.2)$$

<u>Theorem 2.1.</u> Let $f(z) \in A_p$ and $a > 0$. If

$$\left| \frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1 \right| < 1 \qquad (z \in U),$$

then
$$\left| \frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} - 1 \right| \leqq \frac{1}{a} \cdot \frac{|z|}{1-|z|}.$$

<Proof> We put $F(z) = \frac{\mathcal{L}_p(a,c)f(z)}{z^p} - 1$. We can see that $F(z)$ is analytic in U and $|F(z)| < 1$, $F(0) = 0$. Thus $F(z) = z\varphi(z)$ where $\varphi(z)$ is analytic in U and $|\varphi(z)| \leqq 1$. For such functions we have (by Lemma 2.2.)

$$|\varphi'(z)| \leqq \frac{1-|\varphi(z)|^2}{1-|z|^2}$$

Since $\mathcal{L}_p(a,c)f(z) = z^p + z^{p+1}\varphi(z)$, by using Lemma 2.1 we have

$$\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} - 1 = \frac{z}{a} \cdot \frac{\varphi(z) + z\varphi'(z)}{1 + z\varphi(z)}.$$

Lemma 2.2 and the triangle inequalities yield

$$\left| \frac{\varphi(z) + z\varphi'(z)}{1 + z\varphi(z)} \right| \leqq \frac{|\varphi(z)| + |z| \cdot \frac{1-|\varphi(z)|^2}{1-|z|^2}}{1 - |z||\varphi(z)|}$$

$$= \frac{|\varphi(z)| + |z|}{1-|z|^2} \leqq \frac{1+|z|}{1-|z|^2} = \frac{1}{1-|z|}. \tag{2.3}$$

Therefore, we have

$$\left| \frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} - 1 \right| \leqq \frac{1}{a} \cdot \frac{|z|}{1-|z|}.$$

Q.E.D.

<u>Corollary 2.1</u> Let $f(z) \in A_p$. If $\left| \frac{f(z)}{z^p} - 1 \right| < 1$ $(z \in U)$, then $f(z)$ is p-valently starlike in $|z| < \frac{p}{p+1}$.

<proof> Since $\mathcal{L}_p(p,p)f(z)=f(z)$ and $\mathcal{L}_p(p+1,p)f(z)=zf'(z)/p$, we have

$$\left|\frac{zf'(z)}{f(z)}-p\right| \leqq \frac{|z|}{1-|z|}, \quad \text{i.e.,}$$

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \quad \text{in} \quad |z|<\frac{p}{p+1}.$$

Q.E.D.

Corollary 2.2. (MacGregor) If $f(z)=z+a_2z^2+\cdots$ is analytic and satisfies $\left|\frac{f(z)}{z}-1\right|<1$ in U, then $f(z)$ is starlike in $|z|<\frac{1}{2}$.

Corollary 2.3. Let $f(z)\in A_p$. If $\left|\frac{f'(z)}{pz^{p-1}}-1\right|<1$ $(z\in U)$, then $f(z)$ is p-valently convex in $|z|<\frac{p}{p+1}$.

<proof> Since $\mathcal{L}_p(p+1,p)f(z)=zf'(z)/p$ and

$$\mathcal{L}_p(p+2,p)f(z)=\frac{2zf'(z)+z^2f''(z)}{p(p+1)},$$ we have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| \leqq \frac{|z|}{1-|z|}, \quad \text{i.e.,}$$

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}>0 \quad \text{in} \quad |z|<\frac{p}{p+1}.$$

Q.E.D.

Corollary 2.4. (MacGregor) If $f(z)=z+a_2z^2+\cdots$ is analytic and satisfies $|f'(z)-1|<1$ in U, then $f(z)$ is convex in $|z|<\frac{1}{2}$.

Theorem 2.2. Let $f(z)\in A_p$ and $a>0$. Let $g(z)\in A_p$ satisfies $\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)g(z)}{\mathcal{L}_p(a,c)g(z)}\right\}>0$. $(z\in U)$. If $\left|\frac{\mathcal{L}_p(a,c)f(z)}{\mathcal{L}_p(a,c)g(z)}-1\right|<1$ $(z\in U)$,

then

$$\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right\} \geqq \frac{a-(2a+1)|z|+(a-1)|z|^2}{a(1-|z|^2)}.$$

<Proof> We put $\frac{\mathcal{L}_p(a,c)f(z)}{\mathcal{L}_p(a,c)g(z)} - 1 = z\varphi(z)$. Then $\varphi(z)$ is analytic and $|\varphi(z)| \leqq 1$ in U. Using Lemma 2.1 and Lemma 2.2, we have

$$\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} = \frac{\mathcal{L}_p(a+1,c)g(z)}{\mathcal{L}_p(a,c)g(z)} + \frac{z}{a}\cdot\frac{\varphi(z)+z\varphi'(z)}{1+z\varphi(z)} \quad (2.4)$$

Since $\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)g(z)}{\mathcal{L}_p(a,c)g(z)}\right\} > 0$ in U, therefore

$$\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)g(z)}{\mathcal{L}_p(a,c)g(z)}\right\} \geqq \frac{1-|z|}{1+|z|}. \quad (2.5)$$

Applying (2.3) and (2.5) to (2.4), we can see

$$\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right\} \geqq \frac{1-|z|}{1+|z|} - \frac{1}{a}\cdot\frac{|z|}{1-|z|} = \frac{a-(2a+1)|z|+(a-1)|z|^2}{a(1-|z|^2)}.$$

Q.E.D.

Remark 2.1 $\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right\} > 0$ for $|z| < \frac{2a}{2a+1+\sqrt{8a+1}}$.

Corollary 2.5. Let $f(z) \in A_p$ and $g(z) \in S_p^*$. If $\left|\frac{f(z)}{g(z)} - 1\right| < 1$ $(z \in U)$, then $f(z)$ is p-valently starlike in $|z| < \frac{2p}{2p+1+\sqrt{8p+1}}$.

Corollary 2.6. (MacGregor) Let $f(z) \in A$ and $g(z) \in S^*$. If $\left|\frac{f(z)}{g(z)} - 1\right| < 1$ $(z \in U)$, then $f(z)$ is starlike in $|z| < \frac{1}{3}$.

Corollary 2.7. Let $f(z) \in A_p$, $g(z) \in A_p$ s.t. $\mathrm{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} > -1$ $(z \in U)$. If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ $(z \in U)$, then $f(z)$ is p-valently convex in $|z| < \frac{2p+3-\sqrt{8p+9}}{2(p+1)}$.

Corollary 2.8. Let $f(z) \in A$, $g(z) \in A$ s.t. $\mathrm{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} > -1$ $(z \in U)$.
If $\left|\frac{f'(z)}{g'(z)}-1\right| < 1$ $(z \in U)$, then $f(z)$ is convex in $|z| < \frac{5-\sqrt{17}}{4}$

Theorem 2.3 Let $f(z) \in A_p$ and $g(z) \in A_p$ satisfies

$$\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)g(z)}{\mathcal{L}_p(a,c)g(z)}\right\} > \frac{a}{a+c} \quad (z \in U), \text{ where } a \geqq 1 \text{ and } c \geqq 1.$$

If $\left|\frac{\mathcal{L}_p(a,c)f(z)}{\mathcal{L}_p(a,c)g(z)}-1\right| < 1$ $(z \in U)$, then

$$\mathrm{Re}\left\{\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right\} \geqq \frac{a^2-a(a+1)|z|-c|z|^2}{a(a+c|z|)(1-|z|)}.$$

Corollary 2.9. Let $f(z) \in A_p$ and $g(z) \in S_p^*(\frac{p}{2})$, i.e., $\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \frac{p}{2}$ $(z \in U)$. If $\left|\frac{f(z)}{g(z)}-1\right| < 1$ $(z \in U)$, then $f(z)$ is p-valently starlike in $|z| < \left(\sqrt{p^2+6p+1}-(p+1)\right)/2$.

Corollary 2.10. (MacGregor) Let $f(z) \in A$ and $g(z) \in S^*(\frac{1}{2})$. If $\left|\frac{f(z)}{g(z)}-1\right| < 1$ $(z \in U)$, then $f(z)$ is starlike in $|z| < \sqrt{2}-1$.

Lemma 2.3. Suppose that $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic and satisfies $\mathrm{Re}\{h(z)\} > 0$ in U. Then we have

$$|h'(z)| \leqq \frac{2\,\mathrm{Re}\{h(z)\}}{1-|z|^2}.$$

Lemma 2.3 is obtained by using Lemma 2.2.

Applying Lemma 2.1 and Lemma 2.3, we have next theorem.

Theorem 2.4. Let $f(z)\in A_p$ and let $g(z)\in A_p$ satisfies
$$\operatorname{Re}\left\{\frac{\mathcal{L}_p(a+1,c)g(z)}{\mathcal{L}_p(a,c)g(z)}\right\}>\frac{a}{a+c} \text{ in } U, \text{ where } a\geqq 1 \text{ and } c\geqq 1.$$

If $\operatorname{Re}\left\{\frac{\mathcal{L}_p(a,c)f(z)}{\mathcal{L}_p(a,c)g(z)}\right\}>0$ in U, then we have
$$\operatorname{Re}\left\{\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a+1,c)g(z)}\right\}>0 \quad \text{in } |z|<\frac{a(\sqrt{a^2+2c+1}-1)}{a^2+2c}, \text{ and}$$
$$\operatorname{Re}\left\{\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right\}>0 \quad \text{in } |z|<\frac{a(\sqrt{a^2+2c+1}-1)}{a^2+2c}.$$

Corollary 2.11. Let $f(z)\in A_p$ and $g(z)\in S_p^*(\frac{p}{2})$.
If $\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\}>0$ in U, then $\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>0$ in $|z|<\frac{p}{p+2}$,
and $f(z)$ is p-valently starlike in $|z|<\frac{p}{p+2}$.

Corollary 2.12 Let $f(z)\in A$ and $g(z)\in S^*(\frac{1}{2})$. If $\operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\}>0$ in U,
then $\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>0$ in $|z|<\frac{1}{3}$, and $f(z)$ is starlike in $|z|<\frac{1}{3}$.

Corollary 2.13 Let $f(z)\in A_p$ and $g(z)\in K_p(\frac{p^2}{2p+1})$.
If $\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>0$, then $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>-1$
in $|z|<\frac{(p+1)(\sqrt{p^2+4p+2}-1)}{p^2+4p+1}$.

Corollary 2.14 Let $f(z)\in A$ and $g(z)\in K(\frac{1}{3})$.
If $\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>0$, then $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>-1$ in $|z|<\frac{\sqrt{7}-1}{3}$.

Theorem 2.5. Let $f(z) \in A_p$ and $F(z) = \mathcal{L}_p(a, a+1) f(z) \in S_p^*$, where $a \geqq p+1$ with $p \in \mathbb{N}$. Then $f(z)$ is p-valently starlike in $|z| < r_0$, where

$$\begin{cases} r_0 = \dfrac{-2+\sqrt{(a-p)^2+3}}{a-p-1} & (a > p+1) \\ r_0 = \dfrac{1}{2} & (a = p+1) \end{cases}$$

Remark 2.2. $F(z) = \dfrac{a}{z^{a-p}} \displaystyle\int_0^z t^{a-p-1} f(t)\, dt$. (See (1.7))

Corollary 2.15. Let $f(z) \in A$ and $F(z) = \mathcal{L}(a, a+1) f(z) \in S^*$ with $a \geqq 2$. Then $f(z)$ is starlike in $|z| < r_0$, where

$$\begin{cases} r_0 = \dfrac{-2+\sqrt{(a-1)^2+3}}{a-2} & (a > 2) \\ r_0 = \dfrac{1}{2} & (a = 2) \end{cases}$$

Corollary 2.16. (Bernardi) Let $f(z) \in A$ and

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \in S^* \quad \text{with } c \in \mathbb{N}.$$

Then $f(z)$ is starlike in $|z| < r_0$, where

$$\begin{cases} r_0 = \dfrac{-2+\sqrt{3+c^2}}{c-1} & (c > 1) \\ r_0 = \dfrac{1}{2} & (c = 1) \end{cases}$$

Corollary 2.17 Let $f(z) \in A$ and $F(z) = \dfrac{2}{z}\displaystyle\int_0^z f(t)\, dt \in S^*$, then $f(z)$ is starlike in $|z| < \frac{1}{2}$.

3 Some applications of differential subordination

Def. 3.1. Let $f(z)$ and $g(z)$ be analytic in U. Then we say that $f(z)$ is subordinate to $g(z)$ (written $f(z) \prec g(z)$) if $g(z)$ is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 3.1. (Eenigenburg, Miller, Mocanu and Reade) Let $\beta, \gamma \in \mathbb{C}$ and let $h(z)$ be convex univalent in U with $h(0) = 1$ and $\mathrm{Re}\{\beta h(z) + \gamma\} > 0$ $(z \in U)$. If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in U, then

$$p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec h(z) \text{ implies that } p(z) \prec h(z).$$

Def. 3.2. Let $R_p(a,c)$ denote the class of functions $f \in A_p$ such that

$$\frac{z(\mathcal{L}_p(a,c)f(z))'}{p\,\mathcal{L}_p(a,c)f(z)} \prec h(z) \qquad (z \in U),$$

where $h(z)$ is convex in U with $h(0) = 1$ and $\mathrm{Re}\{h(z)\} > 0$.

Theorem 3.1. $R_p(a+1, c) \subset R_p(a,c)$ $(a \geqq p)$

<proof> Let $p(z) = \dfrac{z(\mathcal{L}_p(a,c)f(z))'}{p\,\mathcal{L}_p(a,c)f(z)}$. Using Lemma 2.1, we have

$$a \frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} = p \cdot p(z) + (a-p).$$

Taking logarithmic derivatives, we have

$$\frac{z(\mathcal{L}_p(a+1,c)f(z))'}{p\mathcal{L}_p(a+1,c)f(z)} = \frac{z p'(z)}{p \cdot p(z) + (a-p)} + p(z).$$

This means that if $f(z) \in R_p(a+1,c)$, then

$$\frac{z p'(z)}{p \cdot p(z) + (a-p)} + p(z) \prec h(z).$$

From Lemma 3.1, it follows that for $a \geq p$, $p(z) \prec h(z)$.
Therefore, $\dfrac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)} \prec h(z)$. which means $f(z) \in R_p(a,c)$ for all $a \geq p$. Q.E.D.

Def. 3.3. $$F(z) = \frac{\gamma+p}{z^\gamma}\int_0^z t^{\gamma-1} f(t)\,dt \quad (f(z) \in A_p)$$

Theorem 3.2. $f(z) \in R_p(a,c) \Longrightarrow F(z) \in R_p(a,c)$

<proof> From above definition, we have

$$zF'(z) + \gamma F(z) = (\gamma+p) f(z)$$

We write $\mathcal{L}_p(a,c)f(z) = \phi_p(a,c) * f(z) = (\phi_p * f)(z)$.
So we have

$$\phi_p * (zF')(z) + \gamma(\phi_p * F)(z) = (\gamma+p)(\phi_p * f)(z)$$

Using the fact $z(\phi_p * F)'(z) = (\phi_p * zF')(z)$,

we have

$$z(\phi_p * F)'(z) + \gamma(\phi_p * F)(z) = (\gamma+p)(\phi_p * f)(z).$$

Let $p(z) = \dfrac{z(\phi_p * F)'(z)}{p\cdot(\phi_p * F)(z)}$. Then we obtain

$$p\cdot p(z) + \gamma = (\gamma+p)\frac{(\phi_p * f)(z)}{(\phi_p * F)(z)}.$$

Taking logarithmic derivative, we have

$$\frac{zp'(z)}{p\cdot p(z)+\gamma} + p(z) = \frac{z(\phi_p * f)'(z)}{p\cdot(\phi_p * f)(z)}.$$

Since $f(z) \in R_p(a,c)$, it follows that

$$\frac{z(\phi_p * f)'(z)}{p\cdot(\phi_p * f)(z)} = \frac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)} \prec h(z)$$

which implies that $p(z) \prec h(z)$ by Lemma 3.1. Therefore, we have $F(z) \in R_p(a,c)$. Q.E.D.

Corollary 3.1 (Libera)

(1) $f(z) \in S^* \implies F(z) \in S^*$

(2) $f(z) \in K \implies F(z) \in K$

Def. 3.4. Let $R_p^{\alpha}(a,c)$ $(\alpha > 0)$ denote the class of functions $f(z) \in A_p$ such that

$$I_p^{\alpha}(a,c)(z) = \alpha\frac{z(\mathcal{L}_p(a+1,c)f(z))'}{p\mathcal{L}_p(a+1,c)f(z)} + (1-\alpha)\frac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)} \prec h(z)$$

$(z \in U)$.

Theorem 3.3. $f(z) \in R_p^{\alpha}(a,c) \implies f(z) \in R_p(a,c)$ $(a \geqq p,\ \alpha > 0)$.

<proof> Let $p(z) = \dfrac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)}$. From Lemma 2.1, we have

$$a\,\mathcal{L}_p(a+1,c)f(z) = (p\cdot p(z) + (a-p))\,\mathcal{L}_p(a,c)f(z).$$

Taking logarithmic derivative, we obtain

$$\frac{z(\mathcal{L}_p(a+1,c)f(z))'}{p\cdot\mathcal{L}_p(a+1,c)f(z)} = \frac{zp'(z)}{p\cdot p(z)+a-p} + p(z)$$

Since $f(z) \in R_p^{\alpha}(a,c)$, it follows that

$$I_p^{\alpha}(a,c)(z) = \frac{\alpha zp'(z)}{p\cdot p(z)+a-p} + \alpha p(z) + (1-\alpha)p(z) \prec h(z),$$

that is

$$I_p^{\alpha}(a,c)(z) = \frac{zp'(z)}{\frac{p}{\alpha}p(z)+\frac{\alpha-p}{\alpha}} + p(z) \prec h(z).$$

From Lemma 3.1, we have $p(z) \prec h(z)$, that is

$$\frac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)} \prec h(z)$$

which means $f(z) \in R_p(a,c)$. Q.E.D.

<u>Corollary 3.2</u> (Miller, Mocanu and Reade)

All α-convex functions are starlike.

<u>Theorem 3.4</u> $R_p^{\alpha}(a,c) \subset R_p^{\beta}(a,c)$ $(\alpha > \beta \geqq 0)$

<proof> If $\beta=0$, this theorem means Theorem 3.3. Hence we assume that $\beta \neq 0$. Let $f(z) \in R_p^{\alpha}(a,c)$. Therefore

$$I_p^{\alpha}(a,c)(z) = \alpha\frac{z(\mathcal{L}_p(a+1,c)f(z))'}{p\cdot\mathcal{L}_p(a+1,c)f(z)} + (1-\alpha)\frac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)} \prec h(z).$$

Let z_0 be arbitrary point in U, then $I_p^{\alpha}(a,c)(z_0) \in h(U)$.

From Theorem 3.3, $f(z) \in R_p(a,c)$, that is

$$\frac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)} \prec h(z).$$ This implies

$$\frac{z_0(\mathcal{L}_p(a,c)f(z_0))'}{p\cdot\mathcal{L}_p(a,c)f(z_0)} \in h(U).$$

Also $$I_p^{\beta}(a,c)(z) = \left(1-\frac{\beta}{\alpha}\right)\frac{z(\mathcal{L}_p(a,c)f(z))'}{p\cdot\mathcal{L}_p(a,c)f(z)} + \frac{\beta}{\alpha}I_p^{\alpha}(a,c)(z).$$

Since $\frac{\beta}{\alpha} < 1$ and $h(U)$ is convex, therefore $I_p^{\beta}(a,c)(z_0) \in h(U)$. It follows that $I_p^{\beta}(a,c)(z) \prec h(z)$. That is, $f(z) \in R_p^{\beta}(a,c)$. This means $R_p^{\alpha}(a,c) \subset R_p^{\beta}(a,c)$.

Q.E.D.

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