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Optimal interface width for the Allen-Cahn equation

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1 Introduction

We revisit the parabolic problem for the Allen-Cahn equation

$$(P^\varepsilon)\quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - \varepsilon g(x,t)) & \text{in } \Omega \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where ε is a small parameter and f a bistable nonlinearity. More precisely we assume that f is smooth and has exactly three zeros $\alpha_- < a < \alpha_+$ such that

$$f'(\alpha_\pm) < 0, \quad f'(a) > 0, \tag{1.1}$$

and that

$$\int_{\alpha_-}^{\alpha_+} f(u)du = 0. \tag{1.2}$$

A typical example is the cubic nonlinearity $f(u) = u(1-u^2)$. We suppose that the perturbation term $g(x,t)$ is a smooth function, defined on $\overline{\Omega} \times [0,+\infty)$ satisfying

$$\frac{\partial g}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

and we consider rather general initial data $u_0 \in C^2(\bar{\Omega})$. The constant C_0 will stand for the following quantity:

$$C_0 := \|u_0\|_{C^0(\bar{\Omega})} + \|\nabla u_0\|_{C^0(\bar{\Omega})} + \|\Delta u_0\|_{C^0(\bar{\Omega})}. \tag{1.4}$$

Furthermore we define the "initial interface" Γ_0 by

$$\Gamma_0 := \{x \in \Omega,\ u_0(x) = a\},$$

and suppose that Γ_0 is a smooth hypersurface without boundary such that, n being the Euclidian unit normal vector exterior to Γ_0,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x) \cdot n(x) \neq 0 \quad \text{if } x \in \Gamma_0, \tag{1.5}$$

$$u_0 > a \quad \text{in} \quad \Omega_0^+, \quad u_0 < a \quad \text{in} \quad \Omega_0^-, \tag{1.6}$$

where Ω_0^- denotes the region enclosed by Γ_0 and Ω_0^+ the region enclosed between $\partial\Omega$ and Γ_0. It is standard that Problem (P^ε) has a unique smooth solution u^ε. As $\varepsilon \to 0$, studies of de Mottoni and Schatzman [10] and [11] and X. Chen [5] and [6] show the following: in the very early stage, the diffusion term is negligible compared with the reaction term $\varepsilon^{-2}(f(u) - \varepsilon g(x,t))$ so that, rescaling time by $\tau = t/\varepsilon^2$ leads to the ordinary differential equation $u_\tau = f(u)$. Hence, f being bistable, an interface is formed between the regions $\{u \approx \alpha_-\}$ and $\{u \approx \alpha_+\}$. Once such an interface is developed, the diffusion term becomes large near the interface, and comes to balance with the reaction term so that the interface starts to propagate, in a much slower time scale. To study such interfacial behavior, it is useful to consider the singular limit of (P^ε) as $\varepsilon \to 0$. Then the limit solution $\tilde{u}(x,t)$ will be a step function taking the value α_+ on one side of the interface, and α_- on the other side. This sharp interface, which we will denote by Γ_t, obeys a certain law of motion. It is well known that Γ_t evolves by the mean curvature flow:

$$(P^0) \quad \begin{cases} V_n = -(N-1)\kappa + c_0(\alpha_+ - \alpha_-)g(x,t) & \text{on } \Gamma_t \\ \Gamma_t\Big|_{t=0} = \Gamma_0, \end{cases}$$

where V_n is the normal velocity on Γ_t, κ the mean curvature at each point of Γ_t,

$$c_0 = \left[\sqrt{2}\int_{\alpha_-}^{\alpha_+} (W(s) - W(\alpha_-))^{1/2} ds\right]^{-1}, \tag{1.7}$$

$$W(s) = -\int_a^s f(r)dr.$$

It is standard that Problem (P^0) possesses locally in time a unique smooth solution $\Gamma = \bigcup_{0\le t\le T}(\Gamma_t \times \{t\})$.

Next we set $Q_T := \Omega \times (0,T)$ and for each $t \in (0,T)$, we define Ω_t^- as the region enclosed by the hypersurface Γ_t and Ω_t^+ as the region enclosed between $\partial\Omega$ and Γ_t. Then we define a function $\tilde{u}(x,t)$ by

$$\tilde{u}(x,t) = \begin{cases} \alpha_+ & \text{in } \Omega_t^+ \\ \alpha_- & \text{in } \Omega_t^- \end{cases} \quad \text{for } t \in (0,T). \tag{1.8}$$

As $\varepsilon \to 0$, the solution u^ε of Problem (P^ε) converges to that of Problem (P^0). The aim of the present note is to present an optimal estimate on the width of the transition layer, namely to show that it is of order ε. To that purpose we use new pairs of upper and lower solutions both for the generation and the propagation of interface stages.

We will state our main results in the next section. For the complete proofs we refer to [1] where we study the more general case of the Allen-Cahn type equation $u_t = \Delta u + \varepsilon^{-2}(f(u) - \varepsilon g(x,t,u))$, where the perturbation function g also depends on the unknown function u.

The singular limit of Allen-Cahn equations has been studied in a large number of articles: Let us mention for instance the results of Bronsard and Kohn [4] in the case of spherical symmetry, the articles of de Mottoni and Schatzman [10, 11] and those of Xinfu Chen [5, 6]. These results prove convergence to the limit interface equation in a classical framework; that is, under the assumption that the limit problem has a classical solution Γ_t for $0 \le t \le T$. As for the case where Γ_t is a viscosity or a weak solution of the limit interface equation, we refer to the work of Barles, Soner and Souganidis [2], Evans, Soner and Souganidis [8], Ilmanen [9] and Barles and Souganidis [3].

2 The main results

Our results deal with the limiting behavior of the solution u^ε of Problem (P^ε) as $\varepsilon \to 0$. Our first main result, Theorem 2.1, describes the profile of the solution after a very short initial period. It asserts that: given a virtually arbitrary initial data u_0, the solution u^ε quickly becomes close to $\alpha_\pm$, except in a small neighborhood of the initial interface Γ_0, creating a steep transition layer around Γ_0 (generation of interface). The time needed to develop such a transition layer, which we will denote by t^ε, is of order $\varepsilon^2|\ln\varepsilon|$. The theorem then states that the solution u^ε remains close to the step function $\tilde{u}$ on the time interval (t^ε, T) (motion of interface). Moreover, as is clear from the estimates in the theorem, the "thickness" of the transition layer is of order ε.

Theorem 2.1 (Generation and motion of interface). *Let $\eta \in (0, \min(a-\alpha_-, \alpha_+ - a))$ be arbitrary and set*

$$t_\varepsilon = \frac{\varepsilon^2|\ln\varepsilon|}{f'(a)}.$$

Then there exist positive constants ε_0 and C such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $t^\varepsilon \leq t \leq T$, we have

$$u^\varepsilon(x,t) \in \begin{cases} [\alpha_- - \eta, \alpha_+ + \eta] & \text{if} \quad x \in \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [\alpha_- - \eta, \alpha_- + \eta] & \text{if} \quad x \in \Omega_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [\alpha_+ - \eta, \alpha_+ + \eta] & \text{if} \quad x \in \Omega_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \end{cases} \tag{2.1}$$

where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega, dist(x, \Gamma_t) < r\}$ denotes the r-neighborhood of Γ_t.

Corollary 2.2 (Convergence). *As $\varepsilon \to 0$, u^ε converges to $\tilde{u}$ everywhere in $\bigcup_{0<t<T}(\Omega_t^\pm \times \{t\})$.*

The next theorem is concerned with the relation between the actual interface $\Gamma_t^\varepsilon := \{x \in \Omega, u^\varepsilon(x,t) = a\}$ and the solution Γ_t of Problem (P^0).

Theorem 2.3 (Error estimate). *There exists $C > 0$ such that*

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{for} \quad 0 \leq t \leq T. \tag{2.2}$$

Corollary 2.4 (Convergence of interface). *There exists $C > 0$ such that*

$$d_{\mathcal{H}}(\Gamma_t^\varepsilon, \Gamma_t) \leq C\varepsilon \quad \text{for} \quad 0 \leq t \leq T, \tag{2.3}$$

where $d_{\mathcal{H}}(A,B) := \max\{\sup_{a\in A} d(a,B); \sup_{b\in B} d(b,A)\}$ denotes the Hausdorff distance between two compact sets A and B.

Note that the estimates (2.2) and (2.3) follow from Theorem 2.1 in the range $t^\varepsilon \leq t \leq T$, but the range $0 \leq t \leq t^\varepsilon$ has to be treated by a separate argument. In fact, this is the time range in which a clear transition layer is formed rapidly from an arbitrarily given initial data, therefore the behavior of the solution is quite different from the one in the later time range $t^\varepsilon \leq t \leq T$, where things move more slowly.

The estimate (2.1) in Theorem 2.1 implies that, once a transition layer is formed, its thickness remains of order ε for the rest of the time. The best estimate, so far, was of order $\varepsilon|\ln\varepsilon|$ (see [5]), except that Xinfu Chen has recently obtained an order ε estimate for the case $N=1$ by a different method (private communication). Here, by "thickness of interface" we mean the smallest $r>0$ satisfying

$$\{\, x\in\Omega,\ u(x,t)\notin[\alpha_- - \eta,\ \alpha_- + \eta]\cup[\alpha_+ - \eta,\ \alpha_+ + \eta]\,\}\subset\mathcal{N}_r(\Gamma_t^\varepsilon).$$

Naturally this quantity depends on η, but the estimate (2.1) asserts that it always remains within $O(\varepsilon)$ regardless of the choice of $\eta>0$.

Remark 2.5 *(Optimality of the thickness estimate).* The above $O(\varepsilon)$ estimate is optimal, *i.e.*, the interface cannot be thinner than this order. In fact, rescaling time and space as $\tau:=t/\varepsilon^2$, $y:=x/\varepsilon$, we get

$$u_\tau=\Delta_y u+f(u)-\varepsilon\, g.$$

Thus, by the uniform boundedness of u and by standard parabolic estimates, we have $|\nabla_y u|\le M$ for some constant $M>0$, which implies

$$|\nabla_x u(x,t)|\le\frac{M}{\varepsilon}.$$

From this bound it is clear that the thickness of interface cannot be smaller than $M^{-1}(\alpha_+-\alpha_-)\,\varepsilon$, hence, by (2.1), it has to be exactly of order ε. $\square$

Remark 2.6 *(Optimality of the generation time).* The estimate (2.1) also implies that the generation of interface takes place within the time span of t^ε. This estimate is optimal. In other words, a well-developed interface cannot form much earlier, as the following proposition shows. $\square$

Proposition 2.7. *Denote by $\tilde t^\varepsilon$ the smallest time such that* (2.1) *holds for all $t\in[\,\tilde t^\varepsilon,\,T\,]$. Then there exists a constant $L>0$ such that*

$$\tilde t^\varepsilon\ge\mu^{-1}\varepsilon^2(|\ln\varepsilon|-L)$$

for all $\varepsilon\in(0,\varepsilon_0)$.

3 Generation of interface

The result below shows that within a very short time interval of order $\varepsilon^2|\ln\varepsilon|$ an interface is formed in a neighborhood of $\Gamma_0=\{x\in\Omega, u_0(x)=a\}$. In the sequel, η_0 will stand for the following quantity:

$$\eta_0:=\min(a-\alpha_-,\alpha_+-a).$$

Theorem 3.1. *Let $\eta \in (0, \eta_0)$ be arbitrary and set*

$$t_\varepsilon = \frac{\varepsilon^2 |\ln \varepsilon|}{f'(a)}. \tag{3.1}$$

Then there exist positive constants ε_0 and M_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$,

(i) *for all $x \in \Omega$,*

$$\alpha_- - \eta \leq u^\varepsilon(x, t_\varepsilon) \leq \alpha_+ + \eta; \tag{3.2}$$

(ii) *for all $x \in \Omega$ such that $|u_0(x) - a| \geq M_0\varepsilon$, we have that*

$$\text{if} \quad u_0(x) \geq a + M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, t_\varepsilon) \geq \alpha_+ - \eta, \tag{3.3}$$

$$\text{if} \quad u_0(x) \leq a - M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, t_\varepsilon) \leq \alpha_- + \eta. \tag{3.4}$$

As we will see below, the above theorem is proved by constructing a suitable pair of sub and super-solutions.

3.1 The perturbed bistable ordinary differential equation

We first consider a slightly perturbed nonlinearity,

$$f_\delta(u) = f(u) + \delta,$$

where δ is any constant. For $|\delta|$ small enough, this function is still bistable, and more precisely it has the following properties.

Lemma 3.2. *For $|\delta| < \delta_0$ small enough,*

(i) *f_δ has exactly three zero, namely $\alpha_-(\delta)$, $a(\delta)$ and $\alpha_+(\delta)$ and we can find a positive constant C such that*

$$|\alpha_-(\delta) - \alpha_-| + |a(\delta) - a| + |\alpha_+(\delta) - \alpha_+| \leq C|\delta|. \tag{3.5}$$

(ii) *We have that*

$$\begin{array}{lll} f_\delta & \text{is strictly positive in} & (-\infty, \alpha_-(\delta)) \cup (a(\delta), \alpha_+(\delta)), \\ f_\delta & \text{is strictly negative in} & (\alpha_-(\delta), a(\delta)) \cup (\alpha_+(\delta), +\infty). \end{array} \tag{3.6}$$

(iii) *Set*

$$\mu(\delta) := f'_\delta(a(\delta)) = f'(a(\delta)),$$

then we can find a positive constant, which we denote again by C, such that

$$|\mu(\delta) - \mu| \leq C|\delta|. \tag{3.7}$$

In order to construct a pair of sub and super-solutions for Problem (P^ε) we define $Y(\tau,\xi;\delta)$ as the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau,\xi;\delta) & = f_\delta(Y(\tau,\xi;\delta)) \quad \text{for} \quad \tau > 0 \\ Y(0,\xi;\delta) & = \xi, \end{cases} \tag{3.8}$$

for $\delta \in (-\delta_0, \delta_0)$ and $\xi \in (-2C_0, 2C_0)$. In [1], we present several useful estimates on the growth of Y and its derivatives.

3.2 Construction of sub and super-solutions

We set

$$w_\varepsilon^\pm(x,t) = Y\Big(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 r(\pm\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}); \pm\varepsilon\mathcal{G}\Big),$$

where the constant $\mathcal{G}$ is defined by

$$\mathcal{G} = \sup_{(x,t)\in\overline{\Omega}\times[0,T]} |g(x,t)|,$$

and the function $r(\delta,\tau)$ is given by

$$r(\delta,\tau) = C_6(e^{\mu(\delta)\tau} - 1).$$

Lemma 3.3. *There exist positive constants ε_0 and C_6 such that for all $\varepsilon \in (0,\varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub and super-solutions for Problem (P^ε).*

Proof. We define the operator

$$Lu = u_t - \Delta u - \frac{1}{\varepsilon^2}(f(u) - \varepsilon g(x,t)). \tag{3.9}$$

Then

$$Lw_\varepsilon^+ = \frac{1}{\varepsilon}[\mathcal{G} + g(x,t)] + Y_\xi\big[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_\xi}|\nabla u_0|^2\big].$$

By the definition of $\mathcal{G}$ the first term is positive, and one can show that, for a positive constant C_5 independent of ε, there holds

$$\begin{aligned} Lw_\varepsilon^+ &\geq Y_\xi\big[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - |\Delta u_0| - C_5(e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - 1)|\nabla u_0|^2\big] \\ &\geq Y_\xi\big[(C_6\mu(\varepsilon\mathcal{G}) - C_5(|\nabla u_0|^2)e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - |\Delta u_0| + C_5|\nabla u_0|^2\big]. \end{aligned}$$

In view of (3.7), this inequality implies that, for $\varepsilon \in (0, \varepsilon_0)$, with ε_0 small enough, and for C_6 large enough,

$$Lw_\varepsilon^+ \geq [\frac{\mu C_6}{2} - C_5{C_0}^2 - C_0] \geq 0,$$

which completes the proof of the lemma. □

Hence the comparison principle can be applied to deduce that

$$w_\varepsilon^- \leq u^\varepsilon \leq w_\varepsilon^+ \text{ in } \overline{\Omega} \times [0, T], \tag{3.10}$$

which in turn yields the result of Theorem 3.1.

4 Motion of interface

We consider below Problem (P^ε) with an ε-dependent initial function u_0^ε which converges to $\alpha_\pm$ in $\Omega_0^\pm$ as $\varepsilon \to 0$. The precise hypotheses on u_0^ε will clearly appear in Corollary 4.3.

In this section we sketch the proof of the following convergence result.

Theorem 4.1. *Let $\Gamma_0 = \partial\Omega_0$ be a given smooth interface in Ω. Let $\Gamma := \bigcup_{0 \leq t \leq T}(\Gamma_t \times \{t\})$ be the smooth solution of the free boundary problem (P^0) on $(0, T)$. Then there exists a family of continuous functions $\{u_0^\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$, with ε_0 small enough, such that the solution u^ε of Problem (P^ε) with initial data u_0^ε satisfies:*

$$\lim_{\varepsilon \to 0} u^\varepsilon(x, t) = \begin{cases} \alpha_+ & \text{for all } x \in \Omega_t^+ \\ \alpha_- & \text{for all } x \in \Omega_t^-. \end{cases}$$

The idea is to construct sub and super-solutions u_ε^- and u_ε^+ for Problem (P^ε) which are such that

$$u_\varepsilon^- \leq u^\varepsilon \leq u_\varepsilon^+ \quad \text{on } Q_T,$$

and such that, for all $t \in (0, T)$,

$$u_\varepsilon^-(t), u_\varepsilon^+(t) \to \begin{cases} \alpha_+ & \text{in } \Omega_t^+ \\ \alpha_- & \text{in } \Omega_t^- \end{cases}$$

as $\varepsilon \to 0$. As a consequence the same property will hold as well for u^ε.

To begin with we present mathematical tools which are essential for the construction of sub and super-solutions.

4.1 A modified signed distance function

Let u^ε be the solution of (P^ε). We recall that $\Gamma_t^\varepsilon := \{x \in \Omega, u^\varepsilon(x,t) = a\}$ is the interface at time t and call $\Gamma^\varepsilon := \bigcup_{t\geq 0}(\Gamma_t^\varepsilon \times \{t\})$ the interface. Let $\Gamma = \bigcup_{0\leq t\leq T}(\Gamma_t \times \{t\})$ be the unique solution of the limit geometric motion Problem (P^0) and let $\widetilde{d}$ be the signed distance function to Γ defined by:

$$\widetilde{d}(x,t) = \begin{cases} \quad \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^+ \\ - \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases} \tag{4.1}$$

where $\operatorname{dist}(x,\Gamma_t)$ is the distance from x to the hypersurface Γ_t in Ω. We remark that $\widetilde{d} = 0$ on Γ and that $|\nabla\widetilde{d}| = 1$ in a neighborhood of Γ. Rather than working with the signed distance function, we define a cut-off signed distance function d as follows. Let $t \in [0,T]$ for some $T > 0$. Let d_0 a positive number such that $\widetilde{d}(\cdot,\cdot)$ is smooth in the tubular neighborhood of Γ

$$\{(x,t) \in \overline{Q_T}, |\widetilde{d}(x,t)| < 3d_0\}$$

and that

$$dist(\Gamma_t, \partial\Omega) > 3d_0 \text{ for all } t \in [0,T]. \tag{4.2}$$

We define d as a smooth modification of $\widetilde{d}$ such that $d\widetilde{d} \geq 0$ and:

$$\begin{cases} d = \widetilde{d} & \text{if } |\widetilde{d}| < d_0 \\ d_0 \leq |d| < 2d_0 & \text{if } d_0 \leq |\widetilde{d} < 2d_0 \\ |d| = 2d_0 & \text{if } |\widetilde{d}| \geq 2d_0. \end{cases}$$

Note that $|\nabla d| = 1$ in $\{(x,t) \in \overline{Q_T}, |\widetilde{d}(x,t)| < d_0\}$ and that, in view of (4.2), $\nabla d = 0$ in a neighborhood of $\partial\Omega$. Furthermore, since the moving interface Γ satisfies Problem (P^0), an alternative equation for Γ is given by

$$d_t = \Delta d - c_0(\alpha_+ - \alpha_-)g(x,t) \quad \text{on } \Gamma_t. \tag{4.3}$$

4.2 Construction of sub and super-solutions

First we define $U_0(z)$ as the unique solution of the stationary problem

$$\begin{cases} {U_0}'' + f(U_0) = 0 \\ U_0(-\infty) = \alpha_-, \quad U_0(0) = a, \quad U_0(+\infty) = \alpha_+, \end{cases} \tag{4.4}$$

and $U_1(x,t,z)$ as the unique solution of the problem

$$\begin{cases} U_{1zz} + f'(U_0(z))U_1 = g(x,t) - \gamma_0(x,t){U_0}'(z), \\ U_1(x,t,0) = 0, \qquad U_1(x,t,\cdot) \in L^\infty(\mathbb{R}) \end{cases} \tag{4.5}$$

where

$$\gamma_0(x,t) = c_0(\alpha_+ - \alpha_-)g(x,t). \tag{4.6}$$

We look for a pair of sub and super-solutions $u_\varepsilon^\pm$ for (P^ε) of the form

$$u_\varepsilon^\pm(x,t) = U_0\Big(\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\Big) + \varepsilon U_1\Big(x,t,\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\Big) \pm q(t) \tag{4.7}$$

where

$$\begin{aligned} A(t) &= e^{-\beta \frac{t}{\varepsilon^2}} \\ p(t) &= -A(t) + e^{Lt} + K \\ q(t) &= \sigma A(t) + \varepsilon^2 \widetilde{\gamma} L e^{Lt}. \end{aligned}$$

We prove below the following result.

Lemma 4.2. *There exist positive constants β and σ such that for any $K > 1$, we can find positive constants ε_0, L, and $\widetilde{\gamma}$ such that, if $\varepsilon \in (0,\varepsilon_0)$, $(u_\varepsilon^-, u_\varepsilon^+)$ is a pair of sub and super-solutions for Problem (P^ε).*

We postpone the proof of Lemma 4.2 and remark that Theorem 4.1 directly follows from the above lemma. More precisely, since for $t \in (0,T)$,

$$\lim_{\varepsilon \to 0} u_\varepsilon^\pm(x,t) = \begin{cases} \alpha_+ & \text{for all } x \in \Omega_t^+ \\ \alpha_- & \text{for all } x \in \Omega_t^-, \end{cases} \tag{4.8}$$

we have the following result.

Corollary 4.3. *The conclusion of Theorem 4.1 holds for any initial condition u_0^ε which satisfies*

$$\begin{aligned} U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) &+ \varepsilon U_1\big(x,0,\frac{d_0(x)}{\varepsilon} - K\big) - \sigma - \varepsilon^2\widetilde{\gamma}L \\ &\le u_0^\varepsilon(x) \le U_0\big(\frac{d_0(x)}{\varepsilon} + K\big) + \varepsilon U_1\big(x,0,\frac{d_0(x)}{\varepsilon} + K\big) + \sigma + \varepsilon^2\widetilde{\gamma}L \end{aligned}$$

where $d_0(x) = d(x,0)$.

Indeed, in this case, since $u_\varepsilon^-(x,0) \le u_0^\varepsilon(x) \le u_\varepsilon^+(x,0)$, the comparison principle asserts that, for all $(x,t) \in Q_T$,

$$u_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le u_\varepsilon^+(x,t).$$

Note that, for ε small enough, such functions u_0^ε exist because U_0 is increasing and U_1 is bounded.

4.3 Proof of Lemma 4.2

First, using that $\nabla d = 0$ in a neighborhood of $\partial\Omega$ and the fact that the function g satisfies the homogeneous Neumann boundary condition (1.3), one can show that $\dfrac{\partial u_\varepsilon^\pm}{\partial \nu} = 0$ on $\partial\Omega \times [0,T]$. Furthermore we prove in [1] that

$$Lu_\varepsilon^+ := (u_\varepsilon^+)_t - \Delta u_\varepsilon^+ - \frac{1}{\varepsilon^2}(f(u_\varepsilon^+) - \varepsilon g(x,t,u_\varepsilon^+)) \geq 0,$$

and a similar result for u_ε^-.

5 Proof of Theorem 2.1

Let $\eta \in (0,\eta_0)$ be arbitrary. Choose β and σ such that Lemma 4.2 holds. Moreover, we can assume that

$$\sigma \leq \frac{\eta}{3}. \tag{5.1}$$

By the generation of interface Theorem 3.1, there exist positive constants ε_0 and M_0 such that (3.2), (3.3) and (3.4) hold with $\dfrac{\sigma}{2}$ instead of η. Since $\nabla u_0 \cdot n \neq 0$ everywhere on Γ_0 and since Γ_0 is a compact hypersurface, we can find a positive constant M such that

$$\begin{array}{llll} \text{if} & d_0(x) \geq \ \ M\varepsilon & \text{then} & u_0(x) \geq a + M_0\varepsilon \\ \text{if} & d_0(x) \leq -M\varepsilon & \text{then} & u_0(x) \leq a - M_0\varepsilon. \end{array} \tag{5.2}$$

We then fix K large enough so that

$$U_0(-M+K) \geq \alpha_+ - \frac{\sigma}{3} \quad \text{and} \quad U_0(M-K) \leq \alpha_- + \frac{\sigma}{3}. \tag{5.3}$$

For this value of K, we choose ε_0, L and $\widetilde{\gamma}$ as in Lemma 4.2. Next, we prove that

$$U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) + \varepsilon U_1\big(x,0,\frac{d_0(x)}{\varepsilon} - K\big) - \sigma - \varepsilon^2\widetilde{\gamma}L \leq u^\varepsilon(x,t_\varepsilon) \tag{5.4}$$

and that

$$u^\varepsilon(x,t_\varepsilon) \leq U_0\big(\frac{d_0(x)}{\varepsilon} + K\big) + \varepsilon U_1\big(x,0,\frac{d_0(x)}{\varepsilon} + K\big) + \sigma + \varepsilon^2\widetilde{\gamma}L. \tag{5.5}$$

We only present the proof of the inequality (5.4); the proof of the inequality (5.5) is similar and omitted. To that purpose, we distinguish two cases.

First, assume that $d_0(x) \leq M\varepsilon$. Since U_0 is increasing and since $|U_1|$ is bounded by a constant $\underline{C}$, we have that

$$\begin{aligned}
U_0(\frac{d_0(x)}{\varepsilon} - K) + \varepsilon U_1(x, 0, \frac{d_0(x)}{\varepsilon} - K) - \sigma - \varepsilon^2 \tilde{\gamma} L \\
&\leq U_0(M - K) + \varepsilon \underline{C} - \sigma - \varepsilon^2 C \\
&\leq \alpha_- + \frac{\sigma}{3} + \varepsilon \underline{C} - \sigma - \varepsilon^2 C \\
&\leq \alpha_- - \frac{\sigma}{2},
\end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0)$, with ε_0 small enough. Hence, in this case, (5.4) directly follows from (3.2).

We now assume that $d_0(x) \geq M\varepsilon$. We get

$$\begin{aligned}
U_0(\frac{d_0(x)}{\varepsilon} - K) + \varepsilon U_1(x, 0, \frac{d_0(x)}{\varepsilon} - K) - \sigma - \varepsilon^2 \tilde{\gamma} L &\leq \alpha_+ + \varepsilon \underline{C} - \sigma - \varepsilon^2 C \\
&\leq \alpha_+ - \frac{\sigma}{2},
\end{aligned}$$

for $\varepsilon \in (0, \varepsilon_0)$, with ε_0 small enough. Hence, in this case, (5.4) follows from (3.3) and (5.2).

We remark that (5.4) and (5.5) can be written as

$$u_\varepsilon^-(x, 0) \leq u^\varepsilon(x, t_\varepsilon) \leq u_\varepsilon^+(x, 0),$$

where $(u_\varepsilon^-, u_\varepsilon^+)$ is the pair of sub and super-solutions of Problem (P^ε) for the motion of interface defined in (4.7). Applying the comparison principle then leads to

$$u_\varepsilon^-(x, t) \leq u^\varepsilon(x, t + t_\varepsilon) \leq u_\varepsilon^+(x, t) \quad \text{for} \quad 0 \leq t \leq T. \tag{5.6}$$

Note that, in view of (4.8), this completes the proof of Corollary 2.2. Let now C be a positive constant such that

$$U_0(C - e^{LT} - K) \geq \alpha_+ - \frac{\eta}{2} \quad \text{and} \quad U_0(-C + e^{LT} + K) \leq \alpha_- + \frac{\eta}{2}. \tag{5.7}$$

One then easily checks, in view of (5.6) and (5.1), that, for ε_0 small enough, for $t \geq 0$, we have

$$\begin{array}{llll}
\text{if} & d(x,t) \geq \ \ C\varepsilon & \text{then} & u^\varepsilon(x, t + t_\varepsilon) \geq \alpha_+ - \eta \\
\text{if} & d(x,t) \leq -C\varepsilon & \text{then} & u^\varepsilon(x, t + t_\varepsilon) \leq \alpha_- + \eta,
\end{array} \tag{5.8}$$

and

$$u^\varepsilon(x, t + t_\varepsilon) \in [\alpha_- - \eta, \alpha_+ + \eta],$$

which completes the proof of Theorem 2.1.

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