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Stationary isothermic surfaces and a new characterization of the sphere

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1 Introduction

This is based on the author's recent work with R. Magnanini and J. Prajapat [MPS]. We establish a relationship between *stationary isothermic surfaces* and *uniformly dense domains.* A stationary isothermic surface is a level surface of temperature which does not evolve with time. A domain Ω in the N-dimensional Euclidean space $\mathbb{R}^N$ $(N \geqq 3)$ is said to be uniformly dense in a surface $\Gamma \subset \mathbb{R}^N$ of codimension 1 if, for every small $r > 0$, the volume of the intersection of Ω with a ball of radius r and centered at x does not depend on x for $x \in \Gamma$. We prove that the boundary of every uniformly dense domain which is bounded (or whose complement is bounded) must be a sphere; this is a new characterization of the sphere. We then examine a uniformly dense domain with unbounded boundary $\partial\Omega$ and we show that the principal curvatures of $\partial\Omega$ satisfy certain identities. The case in which the surface Γ coincides with $\partial\Omega$ is particularly interesting. In fact, we show that, if the boundary of a uniformly dense domain is connected, then ,if $N = 3$, it must be either a sphere, a spherical cylinder or a minimal surface. We conclude with a discussion on uniformly dense domains whose boundary is a minimal surface.

In §2, we consider stationary isothermic surfaces in the Cauchy problem for the heat equation with characteristic functions of domains as initial data. The purpose of §3 is to give the problem we will consider as well as to give the definition of uniformly dense domains, its relation to the Cauchy problem, and important examples of uniformly dense domains. In §4, we show that, if Ω is uniformly dense in Γ, then Γ is smooth under some conditions. §5 gives the main theorems concerning the symmetry of Γ where Ω is uniformly dense in Γ. In §6, we give very rough outline of proofs.

2 Stationary isothermic surfaces

Consider the following Cauchy problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in } \mathbb{R}^N \times (0, +\infty), \quad \text{and} \quad u = \mathcal{X}_\Omega \quad \text{on } \mathbb{R}^N \times \{0\}, \tag{2.1}$$

where $\mathcal{X}_\Omega$ is the characteristic function of a domain Ω in $\mathbb{R}^N$ $(N \geqq 3)$. Let $\Gamma \subset \mathbb{R}^N$ be an open subset of a hypersurface. Γ is said to be a stationary isothermic surface of u if there exists a positive function $a = a(t)$ satisfying

$$u(x,t) = a(t) \quad \text{for all} \quad (x,t) \in \Gamma \times (0, +\infty). \tag{2.2}$$

In [CK], I. Chavel and L. Karp have shown that, if Ω is bounded and the solution of (2.1) has a stationary isothermic surface Γ, then Γ must extend to a whole sphere centered at $x_0 = \frac{1}{|\Omega|} \int_\Omega x \, dx$, where $|\Omega|$ denotes the $N-$dimensional Lebesgue measure of Ω (see [CK, Theorem 2, p. 275]). Moreover, by using functions

$$-x_j \frac{\partial u}{\partial x_i} + x_i \frac{\partial u}{\partial x_j}, \quad i \neq j,$$

with a little more argument, we can conclude that Ω is radially symmetric with respect to x_0.

3 Uniformly dense domains

Let $B(x,r)$ be the open ball with radius $r > 0$ and centered at $x \in \mathbb{R}^N$. If $x \in \mathbb{R}^N$ and $r > 0$, we define the (spherical) *average r-density* of Ω at x as the ratio

$$\rho(x,r) = \frac{|\Omega \cap B(x,r)|}{|B(x,r)|}. \tag{3.1}$$

(We shall use the same symbol — single bars — to denote both the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure of sets; thus, for instance, $|\Omega|$ and $|\partial\Omega|$ indicate the N-dimensional Lebesgue measure of Ω and the

$(N-1)$-dimensional Hausdorff measure of $\partial\Omega$, respectively.) If $\Gamma \subset \mathbb{R}^N$, we say that Ω is *uniformly dense in* Γ if it satisfies the following property:

$$\begin{gathered}\text{there exists } r_0 \in (\text{dist}(\Gamma, \partial\Omega), +\infty] \text{ such that, for each fixed } r \in (0, r_0), \\ \text{the function } x \mapsto \rho(x,r) \text{ is constant on } \Gamma.\end{gathered} \tag{3.2}$$

We notice that (3.2) holds if and only if

$$\begin{gathered}\text{there exists } r_0 \in (\text{dist}(\Gamma, \partial\Omega), +\infty] \text{ such that,} \\ \text{for almost every fixed } r \in (0, r_0), \text{the function } x \mapsto \sigma(x,r) \text{ is constant on } \Gamma,\end{gathered} \tag{3.3}$$

where

$$\sigma(x,r) = \frac{|\Omega \cap \partial B(x,r)|}{|\partial B(x,r)|}. \tag{3.4}$$

It is clear that, if Ω is uniformly dense in Γ, then any $x \in \Gamma$ must have the same distance from $\partial\Omega$. In other words, Γ must be *parallel* to a portion of $\partial\Omega$ and, for this reason, many of the properties of $\partial\Omega$ will be inherited by Γ.

Theorem 3.1 *Let Ω be a domain in $\mathbb{R}^N$ and let u be the solution of* (2.1). *Then $\Gamma \subset \mathbb{R}^N$ is a stationary isothermic surface for u if and only if Ω is uniformly dense in Γ with $r_0 = +\infty$.*

Proof. The solution of problem (2.1) is represented by

$$u(x,t) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \mathcal{X}_\Omega(\xi)\, e^{-\frac{|x-\xi|^2}{4t}} d\xi \quad \text{for all } (x,t) \in \mathbb{R}^N \times (0,+\infty). \tag{3.5}$$

We compute that

$$\begin{aligned} u(x,t) &= (4\pi t)^{-\frac{N}{2}} \int_0^{+\infty} e^{-\frac{r^2}{4t}} \left(\int_{\partial B(x,r)} \mathcal{X}_\Omega(\xi)\, dS_\xi \right) dr \\ &= (4\pi t)^{-\frac{N}{2}} \int_0^{+\infty} e^{-\frac{r^2}{4t}} \,|\Omega \cap \partial B(x,r)|\, dr. \end{aligned} \tag{3.6}$$

Let $p,\, q \in \Gamma$ be any pair of points. Since the Laplace transform is injective, (3.6) implies that $u(p,t) = u(q,t)$ for every $t > 0$ if and only if $|\Omega \cap \partial B(p,r)| = |\Omega \cap \partial B(q,r)|$ for almost every $r > 0$. This completes the proof. □

Now, let us consider the following problem.

Problem 3.2 *When $r_0 < +\infty$, classify pairs (Ω, Γ) satisfying that Ω is uniformly dense in Γ.*

We know several examples with $r_0 = +\infty$.

Example 3.3 A smooth function $f : \mathbb{R}^N \to \mathbb{R}$ is called *isoparametric* if both $|\nabla f|^2$ and Δf are functions of f. The family of the level hypersurfaces of f is called an *isoparametric family*, and each level hypersurface of f is called an *isoparametric hypersurface.* It was shown in [LC] and [Seg] that any isoparametric family must be either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. See [No, PaTe] for surveys of isoparametric hypersurfaces. If Ω is either a strip or a half space, and if Γ is a hyperplane parallel to $\partial\Omega$, then Ω is uniformly dense in Γ. If Ω is either a ball, an annulus, or the exterior of a ball, and if Γ is a sphere having the same center as Ω, then Ω is uniformly dense in Γ. Since any spherical cylinder is the Cartesian product of a lower dimensional Euclidean space and a lower dimensional sphere, the similar proposition holds if Γ is a spherical cylinder.

Example 3.4 Let $N = 3$. A right *helicoid* $\mathcal{H}$ is defined as the set

$$\mathcal{H} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 = s\cos t, \ x_2 = s\sin t, \ x_3 = at + b, \ (s,t) \in \mathbb{R}^2 \right\},$$

where $a \neq 0, b$ are real constants. $\mathcal{H}$ splits $\mathbb{R}^3$ up into two connected components, and let Ω be one of them. Then Ω is uniformly dense in $\mathcal{H}$ $(= \partial\Omega)$. Furthermore, we have that $\rho(x, r) = \frac{1}{2}$ for every $x \in \mathcal{H}$ and every $r > 0$, and the symmetry of $\mathcal{H}$ implies that the solution u of (2.1) satisfies $u = \frac{1}{2}$ on $\mathcal{H} \times (0, +\infty)$. It is evident that, when $N \geqq 4$, $\Omega \times \mathbb{R}^{N-3}$ is uniformly dense in $\mathcal{H} \times \mathbb{R}^{N-3}$.

In [Ni] J. Nitsche settled a conjecture of G. Cimmino [Cim]. He showed that the plane and the right helicoid are the only (smooth) hypersurfaces in $\mathbb{R}^3$ such that, for each point x on the surface, every sufficiently small sphere centered at x has its area bisected by the surface. This result was derived by computing the Taylor's formula for $\sigma(x, r)$ near $r = 0$ up to the relevant degrees, where Ω is one part of a neighborhood of x split up by the surface. Since J. Nitsche assumes that $\sigma(x, r) \equiv \frac{1}{2}$ for sufficiently small $r > 0$, his result does not rule out the existence of hypersurfaces Γ in $\mathbb{R}^3$ other than the helicoid, the plane, the sphere, or the spherical cylinder such that Ω is uniformly dense in Γ for some domain Ω.

4 Regularity of Γ where Ω is uniformly dense in Γ

The purpose of this section is to show that, if Ω is uniformly dense in Γ, then Γ is smooth under some conditions. We consider the case where $r_0 < +\infty$. The first theorem takes care of the case where $\Gamma \subset \partial\Omega$, and the second one takes care of the case where $\Gamma \cap \partial\Omega = \emptyset$.

Theorem 4.1 *Let Ω be an open set in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^0, and let Γ be an open subset of $\partial\Omega$. If Ω is uniformly dense in Γ, then Γ must be smooth.*

Proof. Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ such that supp $(\psi) \subset B(0, r_0)$ and $\psi(x) = \eta(|x|)$. The convolution $\psi \star \mathcal{X}_\Omega$ belongs to $C^\infty(\mathbb{R}^N)$ and we have that

$$\psi \star \mathcal{X}_\Omega(x) = \int_{B(0,r_0)} \psi(\xi)\ \mathcal{X}_\Omega(x-\xi)\ d\xi = \int_0^{r_0} \eta(r) \left(\int_{\Omega\cap\partial B(x,r)} dS_\xi \right) dr. \tag{4.1}$$

Since Ω satisfies (3.3), we infer that $\psi \star \mathcal{X}_\Omega$ must be constant on Γ and hence Γ is the level surface of a smooth function. If we prove that, for every $x \in \Gamma$, there is a smooth function ψ such that $\psi \star \mathcal{X}_\Omega$ has non-vanishing gradient at x, then, by the implicit function theorem, we can conclude that Γ is smooth.

Assume, by contradiction, that there exists a point $x_0 \in \Gamma$ such that $\nabla(\psi\star\mathcal{X}_\Omega)(x_0) = 0$ for every function ψ with the properties stated above. Since

$$\begin{aligned}\nabla(\psi \star \mathcal{X}_\Omega)(x_0) &= \int_{B(x_0,r_0)} \mathcal{X}_\Omega(\xi)\ \eta'(|x_0-\xi|)\ \frac{x_0-\xi}{|x_0-\xi|}\ d\xi = \\ & \int_0^{r_0} \eta'(r) \left(\int_{\Omega\cap\partial B(x_0,r)} \frac{x_0-\xi}{|x_0-\xi|}\ dS_\xi \right) dr,\end{aligned}$$

then

$$\int_0^{r_0} \eta'(r)\ M(r)\ dr = 0, \tag{4.2}$$

where

$$M(r) = \int_{\Omega\cap\partial B(x_0,r)} \frac{x_0-\xi}{|x_0-\xi|}\ dS_\xi.$$

Equation (4.2) implies that the distributional derivative of the bounded function $M(r)$ equals zero on $(0, r_0)$. Therefore, by observing that $\lim\limits_{r\to0^+} M(r) = 0$, we conclude that $M(r)$ equals zero for almost every $r \in (0, r_0)$, and hence

$$\int_{\Omega\cap\partial B(x_0,r)} (\xi - x_0)\ dS_\xi = 0 \quad \text{for almost every } \ r \in (0, r_0).$$

Thus, by integrating this equation in r, we see that

$$\int\limits_{\Omega\cap B(x_0,r)} (\xi - x_0)\, d\xi = 0 \quad \text{for every } \ r \in (0, r_0). \tag{4.3}$$

Hence, x_0 must be the center of mass of the set $\Omega \cap B(x_0, r)$ for every $r \in (0, r_0)$.

Now, by choosing $r > 0$ sufficiently small and by eventually translating and rotating the axes, we can suppose that $x_0 = 0$ and that $\partial\Omega$ be represented, in a neighborhood of $x_0 = 0$, by the graph of a continuous function $\varphi : U(0) \to \mathbb{R}$, where $U(0) \subset \mathbb{R}^{N-1}$ is a suitable neighborhood of 0 and $\varphi(0) = 0$. Let $\varphi_\pm(y) = \max[\varphi(y), \pm\sqrt{r^2 - |y|^2}]$ for $y \in B' = \{y \in \mathbb{R}^{N-1} : |y| < r\} \subset U(0)$; the set $\Omega \cap B(x, r)$ can be represented as $\{(y, y_N) \in B' \times \mathbb{R} : \varphi_-(y) < y_N < \varphi_+(y)\}$. Therefore, we can infer that

$$\int\limits_{\Omega\cap B(x,r)} (\xi_N - x_N)\, d\xi = \int\limits_{\Omega\cap B(x,r)} y_N\, dy_N dy =$$

$$\int\limits_{B'} \left(\int_{\varphi_-(y)}^{\varphi_+(y)} y_N\, dy_N \right) dy = \frac{1}{2} \int\limits_{B'} \left[\varphi_+(y)^2 - \varphi_-(y)^2\right]\, dy > 0,$$

which contradicts (4.3). □

Theorem 4.2 *Let Ω be an open set in $\mathbb{R}^N$ satisfying the interior sphere condition and suppose that D is a domain satisfying the interior cone condition and such that $\overline{D} \cap \overline{\Omega} = \emptyset$. Let $\Gamma = \partial D$. If Ω is uniformly dense in Γ, then there exists an open subset Λ of $\partial\Omega$ satisfying the following properties:*

(i) *both Γ and Λ are smooth;*

(ii) *Γ and Λ are parallel;*

(iii) *each principal curvature of Λ with respect to the exterior normal direction to $\partial\Omega$ is smaller than the number $\frac{1}{R}$, where $R = \operatorname{dist}(\Gamma, \partial\Omega)$.*

Proof. We proceed as in the proof of Theorem 4.1 and calculate $\psi \star \mathcal{X}_\Omega(x)$ and $\nabla(\psi \star \mathcal{X}_\Omega)(x)$. By supposing that there exists a point $x_0 \in \Gamma$ such that $\nabla(\psi \star \mathcal{X}_\Omega)(x_0) = 0$ for every function ψ with the properties stated in the beginning of the proof of Theorem 4.1, we conclude that (4.3) holds for every $r \in (0, r_0)$. Define the function $d = d(x)$ by

$$d(x) = \operatorname{dist}(x, \partial\Omega) \ \text{ for every } \ x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{4.4}$$

Since Ω is uniformly dense in Γ, as is observed in §3 we have

$$d(x) = R \text{ for every } x \in \Gamma, \tag{4.5}$$

where $R = \mathrm{dist}(\Gamma, \partial\Omega)$. Since D satisfies the interior cone condition, there exists a finite right spherical cone K with vertex at x_0 such that $K \subset \overline{D}$ and $\overline{K} \cap \partial D = \{x_0\}$. By translating and rotating if needed, we can suppose that $x_0 = 0$ and that K is the set $\{x \in B(0,\rho) : x_N < -|x|\cos\theta\}$, for some choice of $\rho \in (0, R)$ and $\theta \in (0, \frac{\pi}{2})$.

Since $K \subset \overline{D}$ and $\overline{K} \cap \partial D = \{0\}$, (4.5) implies that

$$d(x) > R \text{ for every } x \in K. \tag{4.6}$$

The set defined by

$$V = \{x \in \partial B(0, R) : x_N \geq R\sin\theta\}, \tag{4.7}$$

is such that

$$\partial\Omega \cap \partial B(0, R) \subset V, \tag{4.8}$$

because, otherwise, there would be a point in K contradicting (4.6). Thus, from (4.8) it follows that there exists a number $\delta > 0$ such that

$$d(x) \geq \delta \text{ for every } x \in \partial B(0, R) \cap \{x_N \leq R\sin\frac{\theta}{2}\}. \tag{4.9}$$

Choose $r \in (R, \min\{R + \delta, r_0\})$. Then (4.9) yields that

$$\Omega \cap B(0, r) \subset B(0, r) \cap \{x_N \geq R\sin\frac{\theta}{2}\}. \tag{4.10}$$

This contradicts the fact that (4.3) holds for $x_0 = 0$.

Therefore, it follows that Γ must be smooth, and we can complete the proof by following that of [MS, Lemma 3.1]. □

5 Symmetry of Γ where Ω is uniformly dense in Γ

Under the hypotheses either of Theorem 4.1 or of Theorem 4.2, we know that a part of $\partial\Omega$ corresponding to Γ is a smooth hypersurface. Let $\kappa_j(x)$, $j = 1, \cdots, N-1$ be the principal curvatures of $\partial\Omega$ at $x \in \partial\Omega$ with respect to the exterior normal direction to

$\partial\Omega$. For each $j \in \{1, \ldots, N-1\}$, $K_j(x)$ denotes the j-th *symmetric invariant* $K_j(x)$ of the surface $\partial\Omega$ evaluated at x, that is

$$K_j(x) = \sum_{i_1 < \cdots < i_j} \kappa_{i_1}(x) \cdots \kappa_{i_j}(x), \quad j = 1, \cdots, N-1.$$

With this definition, $H = K_1/(N-1)$ and $K = K_{N-1}$ are the mean and the Gauss curvature of $\partial\Omega$, respectively. We will use the notation $\omega_N = |\partial B(0,1)|$, where $B(0,1) \subset \mathbb{R}^N$. Let us first consider the case where $\Gamma \subset \partial\Omega$.

Theorem 5.1 *Let Ω be a domain in $\mathbb{R}^N$ $(N \geqq 3)$, and suppose Γ is an open subset of the boundary $\partial\Omega$ which is of class C^0. If Ω is uniformly dense in Γ, then*

$$\sigma(x,r) = \frac{1}{2} + \sigma_1(x)\, r + \sigma_3(x)\, r^3 + O(r^5) \quad \textit{as} \quad r \to 0, \ x \in \Gamma, \tag{5.1}$$

where

$$\sigma_1(x) = \frac{\omega_{N-1}}{2\omega_N} H(x) \tag{5.2}$$

and

$$\sigma_3(x) = \begin{cases} \frac{1}{256}\left[K_1(x)^3 - 4K_1(x)K_2(x)\right] & \textit{if } N = 3, \\ \\ \frac{\omega_{N-1}}{16\omega_N(N^2-1)}\left[K_1(x)^3 - 4K_1(x)K_2(x) + 4K_3(x)\right] & \textit{if } N \geqq 4, \end{cases} \tag{5.3}$$

are constant on Γ.

Corollary 5.2 *Let Ω be a domain in $\mathbb{R}^N$ $(N \geqq 3)$, and suppose that Γ is an open subset of the boundary $\partial\Omega$ which is of class C^0. If Ω is uniformly dense in Γ, then Γ is analytic and the following hold:*

(i) *If $N \geqq 3$, $\Gamma = \partial\Omega$, and $\partial\Omega$ is bounded, then $\partial\Omega$ must be a sphere.*

(ii) *If $N = 3$ and Γ is connected, then Γ must be either a portion of a sphere, of a spherical cylinder or of a minimal surface.*

Proof of Corollary 5.2. (i) follows from Theorem 5.1 and Aleksandrov's *Soap Bubble Theorem* (see [Alek]), and Theorem 5.1 implies (ii) directly. □

Remark 5.3 In [Ni] J. Nitsche computed $\sigma_5(x)$ when $N = 3$ for the series expansion $\sigma(x.r) = \frac{1}{2} + \sum_{n=1}^{\infty} \sigma_n(x) r^n$. He showed that, if $H(x) = \sigma_5(x) = 0$ for every $x \in \Gamma$, then Γ must be either a portion of a plane or of a right helicoid.

Let us consider the case where $\Gamma \cap \partial\Omega = \emptyset$.

Theorem 5.4 *Under the hypotheses and the situation of Theorem 4.2, for every $x \in \Gamma$, there exists a unique point $y \in \Lambda$ such that $\overline{B(x,R)} \cap \overline{\Omega} = \{y\}$, and furthermore we have that, for each $x \in \Gamma$, as $r \to R+0$,*

$$\rho(x,r) = \frac{2^{\frac{N+1}{2}} N\omega_{N-1}}{(N^2-1)\omega_N R^N} \left\{ \prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y) \right) \right\}^{-\frac{1}{2}} (r-R)^{\frac{N+1}{2}} + o\left((r-R)^{\frac{N+1}{2}} \right). \tag{5.4}$$

In particular, both Γ and Λ are analytic, and for some constant $c > 0$

$$\prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(y) \right) = c \quad \textit{for every} \quad y \in \Lambda. \tag{5.5}$$

Moreover, if $\partial\Omega$ is bounded and connected, then $\partial\Omega$ must be a sphere.

Remark 5.5 As in [MS], by Aleksandrov's uniqueness theorem in [Alek] we see that equality (5.5) implies that, if $\partial\Omega$ is bounded and connected, then $\partial\Omega$ must be a sphere.

Remark 5.6 In Example 3.4, by using Theorem 5.4, we see that $\mathcal{H}$ is an isolated isothermic surface. Indeed, if there exists another isothermic surface sufficiently close to $\mathcal{H}$, then by Theorem 5.4

$$\left(\frac{1}{R} - \kappa_1(y) \right) \left(\frac{1}{R} - \kappa_2(y) \right) = c \quad \text{for every} \quad y \in \mathcal{H}$$

for some positive constant c. Thus, since $\kappa_1(y) + \kappa_2(y) \equiv 0$, the Gauss curvature $\kappa_1(y)\kappa_2(y)$ must be constant on $\mathcal{H}$. This is a contradiction.

Nitsche's result [Ni] does not rule out the existence of minimal surfaces (other than the helicoid or the plane) which are boundaries of uniformly dense domains. Here, we consider the case of embedded minimal surfaces of finite total curvature. The theory of complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$ has developed recently (see [HK], [LoM], and [PeRo] for some surveys). In particular, in [Kap] N.

Kapouleas constructed large families of such minimal surfaces with symmetries, and moreover in [T] M. Traizet showed the existence of such minimal surfaces with no symmetries. Note that the catenoid and the plane are the classical examples of complete embedded minimal surfaces of finite total curvature, and the helicoid is not of finite total curvature because of its periodicity. By combining Nitsche's result [Ni] and the theory of complete embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$, we conclude our analysis of uniformly dense domains with the following result.

Theorem 5.7 *Let S be a complete embedded minimal surface of finite total curvature in $\mathbb{R}^3$, and let Ω be one of the two domains disconnected by S from $\mathbb{R}^3$. If Ω is uniformly dense in S $(= \partial\Omega)$, then S must be a plane.*

6 On proofs of theorems

In this section we give very rough outline of proofs of Theorem 5.1 and Theorem 5.7. See [MPS] for their details.

On the proof of Theorem 5.1. For $x \in \partial\Omega$, denote by $T_x(\partial\Omega)$ and ν the tangent space and the interior normal unit vector to $\partial\Omega$ at x, respectively. For fixed $v \in T_x(\partial\Omega)$ with $|v| = 1$, let $\pi_x(v, \nu)$ be the plane through x spanned by v and ν. We may assume that, for $r > 0$ sufficiently small, each point z in $\Omega \cap B(x, r)$ can be parameterized in spherical coordinates as:

$$\begin{aligned} & z = x + \rho\cos\phi\ v + \rho\sin\phi\ \nu, \\ & v \in T_x(\partial\Omega) \cap \mathbb{S}^{N-2},\ \theta(\rho, v) \leqq \phi \leqq \pi/2,\ 0 \leqq \rho \leqq r, \end{aligned} \tag{6.1}$$

where, for fixed $v \in T_x(\partial\Omega) \cap \mathbb{S}^{N-2}$, $\phi = \theta(\rho, v)$ parameterizes the curve $\partial\Omega \cap \pi_x(v, \nu)$ in polar coordinates. Expand $\theta(r, v)$ in r as

$$\theta(r, v) = \theta_1(v)\ r + \theta_2(v)\ r^2 + \theta_3(v)\ r^3 + \cdots . \tag{6.2}$$

The Jacobian of the change of variables (6.1) is $\rho^{N-1}\cos^{N-2}\phi$, so that we can write:

$$|\Omega \cap B(x, r)| = \int_{\mathbb{S}^{N-2}} \int_0^r \rho^{N-1} \int_{\theta(\rho,v)}^{\pi/2} \cos^{N-2}\phi\ d\phi\ d\rho\ dS_v, \tag{6.3}$$

where dS_v denotes the surface element on $\mathbb{S}^{N-2}$. By differentiating this formula with respect to r and dividing by $\omega_N\ r^{N-1}$, we get:

$$\sigma(x,r) = \frac{1}{\omega_N} \int\limits_{\mathbb{S}^{N-2}} \int\limits_{\theta(r,v)}^{\pi/2} \cos^{N-2}\phi\ d\phi\ dS_v = \frac{1}{2} - \frac{1}{\omega_N} \int\limits_{\mathbb{S}^{N-2}} \int\limits_{0}^{\theta(r,v)} \cos^{N-2}\phi\ d\phi\ dS_v.$$

Here we have

$$\int\limits_0^{\theta(r,v)} \cos^{N-2}\phi\ d\phi = \theta_1(v)\ r + \theta_2(v)\ r^2 + \left[\theta_3(v) - \frac{N-2}{6}\ \theta_1(v)^3\right]\ r^3 + \cdots. \tag{6.4}$$

Without loss of generality, we suppose that x is the origin in $\mathbb{R}^N$ and $T_x(\partial\Omega)$ coincides with the hyperplane $\{(y, y_N) \in \mathbb{R}^N : y_N = 0\}$, where we use the letter y to denote an element of $\mathbb{R}^{N-1}$, that is, $y = (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$. Suppose that $\partial\Omega$ is the graph of a smooth function φ in the neighborhood of a point $x = 0 \in \partial\Omega$ and we compute the coefficients (6.2) in terms of the derivatives of φ. We may assume that the function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ then parameterizes $\partial\Omega$ in a neighborhood of $x = 0$, that is $\partial\Omega$ is represented by the equation $y_N = \varphi(y)$, where $\varphi(0) = 0$, $\nabla\varphi(0) = 0$, and $-\nabla^2\varphi(0) =$ diag $(\kappa_1, \cdots, \kappa_{N-1})$. Here κ_j, $j = 1, \cdots, N-1$ are the principal curvatures of $\partial\Omega$ at $0 \in \partial\Omega$ with respect to the exterior normal direction to $\partial\Omega$. We also use a standard multi-index notation for the derivatives of φ : if $i = (i_1, \ldots, i_{N-1})$ is a multi-index, we denote $|i| = i_1 + \cdots + i_{N-1}$, $i! = i_1! \cdots i_{N-1}!$,

$$D^i\varphi = \partial_{y_1}^{i_1} \cdots \partial_{y_{N-1}}^{i_{N-1}} \varphi,$$

and $y^i = y_1^{i_1} \cdots y_{N-1}^{i_{N-1}}$ for $y \in \mathbb{R}^{N-1}$. With these notations and assumptions, the Taylor expansion of φ in a neighborhood of $y = 0$ is

$$\varphi(y) = \sum_{n=2}^{\infty} P_n(y) \quad \text{where} \quad P_n(y) = \sum_{|i|=n} \frac{D^i\varphi(0)}{i!}\ y^i,\ n = 0, 1, \cdots. \tag{6.5}$$

Since $r \sin\theta(r,v) = \varphi(r\cos\theta(r,v)\ v)$ for sufficiently small r, we have:

$$\sin\theta(r,v) = \sum_{n=2}^{\infty} r^{n-1} \cos^n\theta(r,v)\ P_n(v). \tag{6.6}$$

By expanding both sides in r and comparing their coefficients, we can get:

$$\theta_1(v) = P_2(v),\ \ \theta_2(v) = P_3(v),\ \text{ and }\ \theta_3(v) = P_4(v) - \frac{5}{6}\ P_2(v)^3. \tag{6.7}$$

Hence, combining this with (6.4) yields that in the Taylor expansion (5.1)

$$\sigma_1(x) = -\tfrac{1}{\omega_N} \int_{\mathbb{S}^{N-2}} P_2(v)\, dS_v, \quad \sigma_2(x) = -\tfrac{1}{\omega_N} \int_{\mathbb{S}^{N-2}} P_3(v)\, dS_v, \quad \text{and}$$

$$\sigma_3(x) = -\tfrac{1}{\omega_N} \int_{\mathbb{S}^{N-2}} [P_4(v) - \tfrac{N+3}{6}\, P_2(v)^3]\, dS_v. \tag{6.8}$$

Lemma 6.1 *Let $i = (i_1, \dots, i_{N-1})$ be a multi-index. We have*

$$\int_{\mathbb{S}^{N-2}} v^i\, dS_v = 0$$

if at least one entry of i is odd; otherwise,

$$\frac{1}{\omega_{N-1}} \int_{\mathbb{S}^{N-2}} v^{2i}\, dS_v = \frac{(N-3)!!\, (2i)!}{(2|i| + N - 3)!!\, 2^{|i|} i!} \tag{6.9}$$

where $n!! = \prod_{k=0}^{[\frac{n-1}{2}]} (n - 2k)$.

Consider σ_2 first. Lemma 6.1 and (6.8) directly imply that $\sigma_2 = 0$. Let us consider σ_1. Since $P_2(v) = -\frac{1}{2} \sum_{j=1}^{N-1} \kappa_j v_j^2$, we have from Lemma 6.1 and (6.8)

$$\sigma_1(x) = \frac{\omega_{N-1}}{2(N-1)\omega_N} \sum_{j=1}^{N-1} \kappa_j = \frac{\omega_{N-1}}{2\omega_N} H(x),$$

which is just (5.2). Therefore, the assumption that Ω is uniformly dense in Γ implies that $H(x) \equiv H_0$ on Γ for some constant H_0. By using this fact, we get

$$(1 + |\nabla\varphi|^2)\Delta\varphi - \sum_{k,\ell=1}^{N-1} \frac{\partial\varphi}{\partial y_k} \frac{\partial\varphi}{\partial y_\ell} \frac{\partial^2\varphi}{\partial y_k \partial y_\ell} \equiv -(N-1)H_0(1 + |\nabla\varphi|^2)^{\frac{3}{2}}.$$

This fact implies that φ is analytic in y. By differentiating this equation twice and letting $y = 0$, we can get (5.3) from Lemma 6.1 and (6.8).

Remark 6.2 It can be shown that $\sigma(x, r)$ admits the series expansions

$$\sigma(x, r) = \frac{1}{2} + \sum_{n=1}^{\infty} \sigma_n(x)\, r^n. \tag{6.10}$$

Here, for each $n \in \mathbb{N}$, the integrand in the expression for $\sigma_n(x)$ is a polynomial, without zeroth order coefficient, of the functions $P_2(v), \dots, P_{n+1}(v)$ and hence each coefficient $\sigma_n(x)$ ($n \in \mathbb{N}$) is a polynomial, without zeroth order coefficient, of $D^\beta\varphi(0, 0)$, $2 \le |\beta| \le n + 1$. In particular, we obtain that $\sigma_{2k}(x) = 0$ for any $k = 1, 2, \cdots$.

On the proof of Theorem 5.7. We recall from [PeRo, p. 18] that "a complete embedded minimal surface in $\mathbb{R}^3$ with finite total curvature, outside a big ball in space, has a nice shape: there are a finite number of parallel ends and each end is asymptotic to a plane or to a halfcatenoid" (see also [HK, Proposition 2.5, pp. 36 – 37] for a more precise description concerning complete, nonplanar, minimal surfaces with finite total curvature). On one end of S, we can see that, as x goes to the end,

$$\sigma_j(x) \to 0 \quad \text{for every} \quad j \in \mathbb{N}.$$

Hence, since Ω is uniformly dense in S, we must have

$$\sigma_j(x) \equiv 0 \quad \text{for every} \quad x \in S \quad \text{and every} \quad j \in \mathbb{N},$$

which shows that $\sigma(x,r) \equiv \frac{1}{2}$ for sufficiently small $r > 0$. Finally, by Nitsche's result [Ni], we can conclude that S must be a plane.

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