Movement of Hot Spots on the Exterior Domain of a Ball

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1 Introduction

We consider the initial-boundary value problems of the heat equation in the exterior domain of a ball,

\begin{align}
\begin{cases}
\partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\
\partial_\nu u = 0 & \text{on } \partial\Omega \times (0, \infty), \\
u(x, 0) = \phi(x) & \text{in } \Omega,
\end{cases}
\end{align}

and

\begin{align}
\begin{cases}
\partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\
u(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\
u(x, 0) = \phi(x) & \text{in } \Omega,
\end{cases}
\end{align}

where

\[ \Omega = \mathbb{R}^N \setminus \overline{B(0, L)}, \quad L > 0, \quad N \geq 2. \]

Here \( \partial_t = \partial/\partial t, \partial_\nu = \partial/\partial \nu, \nu = \nu(x) \) is the outer unit normal vector to \( \partial\Omega \) at \( x \in \partial\Omega \), and \( B(0, L) = \{ x \in \mathbb{R}^N : |x| < L \} \). Throughout this paper we assume that \( \phi \in L^2(\Omega, e^{\lambda|x|^2} dx) \) for some \( \lambda > 0 \). For any \( t > 0 \), we may denote by \( H(t) \) the set of the maximum points of \( u(\cdot, t) \), that is,

\[ H(t) = \left\{ x \in \partial \Omega : u(x, t) = \max_{y \in \partial \Omega} u(y, t) \right\}, \]
and call $H(t)$ the set of hot spots of the solution $u$ at the time $t$. In this paper we study the movement of hot spots $H(t)$ of the solution $u$ of (1.1) or (1.2) as $t \to \infty$.

Chavel and Karp [3] studied the heat equation $\partial_t u = \Delta u$ in several Riemannian manifolds, and obtained some asymptotic properties of solutions concerning the movement of hot spots of the solution. In particular, for the Euclidean space $\mathbb{R}^N$, they proved that, for any nonzero, nonnegative initial data $\phi \in L^\infty_c(\mathbb{R}^N)$, the hot spots $H(t)$ of the solution at each time $t > 0$ are contained in the closed convex hull of the support of $\phi$, and $H(t)$ tends to the center of mass of $\phi$ as $t \to \infty$. Subsequently, Jimbo and Sakaguchi [11] studied the movement of hot spots of the solution of the heat equation in the half space $\mathbb{R}^N_+$ and in the exterior domain of a ball, under boundary conditions. In particular, for the Cauchy-Neumann problem (1.1) in the exterior domain $\Omega = \mathbb{R}^N \setminus B(0, L)$ with the nonzero, nonnegative, radially symmetric initial data $\phi \in L^\infty_c(\Omega)$, they proved that the hot spots $H(t)$ satisfies

$$H(t) \subset \partial \Omega = \partial B(0, L)$$

for all sufficiently large $t$. Furthermore, for the Cauchy-Dirichlet problem in the exterior domain $\Omega = \mathbb{R}^3 \setminus B(0, L)$ with the nonzero, nonnegative, radially symmetric initial data $\phi \in L^\infty_c(\Omega)$, they proved that there exist a positive constant $T$ and a continuous function $r = r(t) \in C([T, \infty) : (L, \infty))$ such that

$$\lim_{t \to \infty} r(t)^3 t^{-1} = 2$$

and

$$H(t) = \{ x \in \mathbb{R}^N : |x| = r(t) \}, \quad t \geq T.$$

Their proofs of (1.3) and (1.4) heavily depend on the radially symmetry of the solutions and the properties of zero sets of the heat equation in $\mathbb{R}$, and it seems so difficult to apply their proofs to the solutions without the radially symmetry. (For the movement of hot spots of the solution for the Cauchy-Neumann problem in bounded domains, see [1], [2], [10], [12], and [14]. )

In this paper we study the movement of hot spots $H(t)$ of the solutions of the Cauchy-Neumann problem (1.1) or the Cauchy-Dirichlet problem (1.2) in the exterior domain $\Omega$ of a ball, without the radially symmetry of the solutions. In Sections 2 and 3, we give the results on the movement of the set of hot spots $H(t)$ for the problems (1.1) and (1.2), respectively.
2 On the Cauchy-Neumann Problem (1.1)

In this section we assume

(2.1) \[ \phi \in L^2(\Omega, e^{\lambda|x|^2}dx), \quad \int_{\Omega} \phi(x)dx > 0, \]

and give some results on the movement of the hot spots $H(t)$ for the solution of (1.1) as $t \to \infty$. We first give a sufficient condition for the hot spots $H(t)$ to exist only on the boundary $\partial \Omega$ for all sufficiently large $t$.

**Theorem 2.1** (See Theorem 1.1 in [8].) Let $u$ be a solution of the Cauchy-Neumann problem (1.1) under the condition (2.1). Put

\[ A_{\phi}^N = \int_{\Omega} x \phi(x) \left( 1 + \frac{L^N}{N-1} |x|^{-N} \right) dx / \int_{\Omega} \phi(x)dx. \]

Assume

(2.2) \[ A_{\phi}^N \in B(0, L) = \mathbb{R}^N \setminus \overline{\Omega}. \]

Then there exists a positive constant $T$ such that

(2.3) \[ H(t) \subset \partial \Omega = L \{ x \in \mathbb{R}^N : |x| = L \} \]

for all $t \geq T$.

In particular, we see that, under the condition (2.1), the hot spots $H(t)$ of the radial solution of (1.1) exists only on the boundary of the domain $\Omega$ for all sufficiently large $t$.

**Remark 2.1** Let $u$ be a solution of the Cauchy-Neumann problem (1.1) under the condition (2.1). Let $C(t)$ a center of mass of $u(t)$, that is,

\[ C(t) = \int_{\Omega} x u(x, t)dx / \int_{\Omega} u(x, t)dx. \]

Then it does not necessarily hold that $C(t) = C(0)$ for all $t > 0$. On the other hand, we put

\[ A(t)^N = \int_{\Omega} x u(x, t) \left( 1 + \frac{L^N}{N-1} |x|^{-N} \right) dx / \int_{\Omega} u(x, t)dx, \quad t > 0. \]

Then we have $A_{\phi}^N(t) = A_{\phi}^N$ for all $t > 0$, and $\lim_{t \to \infty} C(t) = A_{\phi}^N$. 

Next we give a result on the limit of the set $H(t)$ as $t \to \infty$.

**Theorem 2.2** (See Theorem 1.2 in [8].)

Let $u$ be a solution of the Cauchy-Neumann problem (1.1) under the condition (2.1). Assume $A^N_\phi \neq 0$. Put

$$x_\infty = L \frac{A^N_\phi}{|A^N_\phi|} \quad \text{if} \quad A^N_\phi \in B(0, L) \quad \text{and} \quad x_\infty = A^N_\phi \quad \text{if} \quad A^N_\phi \in \overline{\Omega}.$$

Then

$$\lim_{t \to \infty} \sup \{|x_\infty - y| : y \in H(t)\} = 0.$$

By Theorem 2.2, we see that the hot spots $H(t)$ tends to one point $x_\infty$ as $t \to \infty$ if $A_\phi \neq 0$, and see that (1.3) does not hold if $A_\phi \in \Omega$.

Next we will explain the outline of the proofs of Theorems 2.1 and 2.2. As in stated in [11], it is difficult to know the sign of differential of the Neumann heat kernel even for the case that $\Omega$ is the exterior of a ball, and so it seems difficult to obtain Theorems 2.1 and 2.2 by using the fundamental properties of the Neumann heat kernel. We consider the following two eigenvalue problems,

\[
\begin{align*}
(P_0 \varphi) & \equiv \frac{1}{\rho} \text{div} (\rho \nabla \varphi) = -\lambda \varphi \quad \text{in} \quad \mathbb{R}^N, \\
\varphi & \in H^1(\mathbb{R}^N, \rho dy), \quad \rho(y) = \exp \left( \frac{|y|^2}{4} \right),
\end{align*}
\]

and

\[
\begin{align*}
-\Delta_{\mathbb{S}^{N-1}} Q & = \omega Q \quad \text{on} \quad \mathbb{S}^{N-1},
\end{align*}
\]

such that $0 = \omega_0 < \omega_1 = N - 1 < \omega_2 = 2N < \omega_3 < \cdots$, where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. Let $l_k$ be the dimension of the eigenspace of the eigenvalue problem (2.4) corresponding to $\omega = \omega_k$ and $\{Q_k, i\}_{i=1}^{l_k}$ the eigenfunctions of (2.4) corresponding to $\omega = \omega_k$ such that $(Q_{k,i}, Q_{k,j})_{L^2(\mathbb{S}^{N-1})} = \delta_{ij}, i, j = 1, \ldots, l_k$. In particular we may take

\[
Q_{1,i} \left( \frac{x}{|x|} \right) = c_q \frac{x_i}{|x|}, \quad i = 1, \ldots, N,
\]

for some positive constant $c_q = c_q(N) > 0$. Furthermore we have the following lemma on the eigenfunctions of (E) (see [5] and [13]).
Lemma 2.1 Let $k = 0, 1, 2, \ldots$. Let $\{\lambda_{k,i}\}_{i=0}^{\infty}$ be the eigenvalues of
\[
\begin{cases}
P_k \varphi \equiv P_0 \varphi - \frac{\omega_k}{|y|^2} \varphi = -\lambda \varphi & \text{in } \mathbb{R}^N, \\
\varphi \text{ is a radial function in } \mathbb{R}^N, \\
\varphi \in L^2(\mathbb{R}^N; \rho dy),
\end{cases}
\]
such that $\lambda_{k,0} < \lambda_{k,1} < \lambda_{k,2} < \ldots$ and $\varphi_{k,i}$ the eigenfunction corresponding to $\lambda_{k,i}$ such that $\|\varphi_{k,i}\|_{L^2(\Omega; \rho dx)} = 1$. Then
\[
\lambda_{k,i} = \frac{N + k}{2} + i, \quad \varphi_{k,0}(y) = c_k |y|^k \exp\left(-\frac{|y|^2}{4}\right)
\]
for some constants $c_k$. Furthermore $\{\lambda_{k,i}\}_{i=0}^{\infty}$ give all eigenvalue of $E$, and the eigenspace of $E$ corresponding to $\lambda$ are spanned by the eigenfunctions $\{\varphi_{k,i}\}_{k,i=0}^{\infty} \varphi_{k,i}(y)Q_{k,j}(y/|y|)$ with $\lambda = \lambda_{k,i}$.

In order to prove Theorems 2.1 and 2.2, we may assume, without loss of generality, that $\phi \in L^2(\Omega; \rho dx)$. Then, by Lemma 2.1, there exist radial functions $\{\phi_{k,j}\}_{k \in \mathbb{N}_0, j=1}^{l_k}$, such that $\phi_{k,i} \in L^2(\Omega; \rho dx)$ and
\[
\phi = \sum_{k=0}^{\infty} \sum_{j=1}^{l_k} \phi_{k,j}(|x|) Q_{k,j}\left(\frac{x}{|x|}\right) \quad \text{in } L^2(\Omega; \rho dx),
\]
Furthermore let $v_{k,j}$ be the radial solution of the Cauchy-Neumann problem
\[
\begin{cases}
\partial_t v = L_k v \equiv \Delta v - \frac{\omega_k}{|x|^2} v_k & \text{in } \Omega \times (0, \infty), \\
\partial_\nu v = 0 & \text{on } \partial \Omega \times (0, \infty), \\
v(x, 0) = \phi_{k,j}(x) & \text{in } \Omega.
\end{cases}
\]
Then the function
\[
v_{k,j}(x, t)Q_{k,j}\left(\frac{x}{|x|}\right)
\]
is a solution of (1.1) with the initial data $\phi_{k,j}(x)Q_{k,j}(x/|x|)$. Furthermore we see that
\[
u(x, t) = \sum_{k=0}^{\infty} \sum_{j=1}^{l_k} \phi_{k,j}(x, t) \quad \text{in } C^2(\Omega),
\]
for all $t > 0$. Therefore we have only to study the asymptotic behavior of the radial solution of the Cauchy-Neumann problem $(L_k^N)$ in order to study the one of the solution $u$ of (1.1).

Let $v_k$ be the solution of the Cauchy-Neumann problem $(L_k^N)$ with the initial data $\varphi$, where $\varphi$ is a radial function belonging to $L^2(\Omega, \rho dx)$. In order to study the asymptotic behavior of the solution $v_k$, we define a rescaled function $w_k$ of the solution $v_k$ as follows:

$$w_k(y, s) = (1+t)^{\frac{N+k}{2}} v_k(x, t), \quad y = (1+t)^{-\frac{1}{2}} x, \quad s = \log(1+t).$$

Then the function $w_k$ satisfies

$$\left\{ \begin{array}{ll}
\partial_s w_k = P_k w_k + \frac{N+k}{2} w_k & \text{in } W, \\
\partial_{\nu} w_k = 0 & \text{on } \partial W, \\
w_k(y, 0) = \varphi(y) & \text{in } \Omega,
\end{array} \right.$$ 

where

$$\Omega(s) = e^{-s/2} \Omega, \quad W = \bigcup_{0<s<\infty} (\Omega(s) \times \{s\}), \quad \partial W = \bigcup_{0<s<\infty} (\partial \Omega(s) \times \{s\}).$$

We study the asymptotic behavior of the first eigenvalue and the first eigenfunction of the operator $P_k$, and obtain the asymptotic behavior of the solution $w_k$ in the space $L^2$ with weight $\rho$. Furthermore, for $k = 0, 1, 2$, by using the radially symmetry of $v_k$, the equations $(L_k^N)$ and $(P_k^N)$, and the Ascoli-Arzera theorem, we study the asymptotic behavior of $v_k$, $\partial_r v_k$, and $\partial_r^2 v_k$ as $t \to \infty$.

For the case $k = 0$, we extend the domain of $w_0$ to $\mathbb{R}^N$, and apply the Ascoli-Arzera theorem to $w_0$. Then, by using the results on the asymptotic behavior of $w_0$ in the space $L^2$ with weight $\rho$, we obtain a result on the asymptotic behavior of $v_0$ and $\partial_r v_0$, where $r = |x|$. Furthermore we obtain a result on the asymptotic behavior of $\partial_r^2 v_0$ as $t \to \infty$ by using the ones of $v_0$ and $\partial_r v_0$.

**Proposition 2.1** Let $\varphi$ be a radial function in $\Omega$ satisfying (2.1). Let $v_0$ be a radial solution of $(L_0^N)$ with the initial data $\varphi$. Then

$$\lim_{t \to \infty} t^{\frac{N}{2}} v_0(x, t) = (4\pi)^{-\frac{N}{2}} \int_{\Omega} \varphi(x) dx$$
uniformly on any compact set in $Ω$. Furthermore, for any positive constants $ε$, there exist positive constants $C$, $R$, and $T$ such that

$$\partial_r v_0(x, t) \leq -Ct^{-\frac{N+1}{2}} \int_Ω \varphi(x) dx$$

for all $x \in Ω$ with $ε(1 + t)^{1/2} \leq |x| \leq R(1 + t)^{1/2}$ and all $t \geq T$.

**Proposition 2.2** Let $φ$ be a radial function in $Ω$ satisfying (2.1). Let $v_0$ be a radial solution of $(L^N)$ with the initial data $φ$. Then there exist positive constants $R$ and $T$ such that

$$\partial_r v_0(x, t) \leq -\frac{1}{4}(4\pi)^{-\frac{N}{2}} t^{-\frac{N+2}{2}} \int_Ω \varphi(x) dx$$

for all $x \in Ω$ with $|x| \leq L + R(1 + t)^{1/2}$ and $t \geq T$, where $r = |x|$. Furthermore, for any $R > L$,

$$\partial_r v_0(x, t)$$

$$= -\frac{1}{2}(4\pi)^{-\frac{N}{2}} (1 + o(1)) |x|(1 - LN|x|^{-N}) t^{-\frac{N+2}{2}} \int_Ω \varphi(x) dx,$$

$$\partial^2_r v_0(x, t)$$

$$= -\frac{1}{2}(4\pi)^{-\frac{N}{2}} (1 + o(1))(1 + (N - 1)L^N r^{-N}) t^{-\frac{N+2}{2}} \int_Ω \varphi(x) dx$$

as $t \to \infty$, uniformly on $Ω \cap B(0, R)$.

On the other hand, for the case $k = 1$, the inequality

$$\sup_{s > 1} \|\nabla^2 w_1(\cdot, s)\|_{C(Ω(s))} < \infty$$

does not necessarily holds, and $w(y, s)$ tends to 0 uniformly for all $y$ with $|y| \leq Re^{-s/2}$ with any $R > L$. So it is not useful to apply the Ascoli-Arzera theorem to $w_1$ for the aim at studying the asymptotic behavior of $w_1$ and $\partial_r w_1$ in the domain \{\(y \in Ω(s) : |y| \leq Re^{-s/2}\}\}, as $s \to \infty$. To overcome this difficulty, we may apply the Ascoli-Arzera theorem $w_1$ in the any annulus $D(ε, R) = \{y \in R^N : ε \leq |y| \leq R\}$ with $0 < ε < R$, and obtain the asymptotic behavior of $w_1$ in the annulus $D(ε, R)$. Furthermore, by using the equation $(L_1)$ effectively, we study the asymptotic behavior of $v_1$, $\partial_r v_1$ and $\partial^2_r v_1$ as $t \to \infty$, and obtain the following proposition.
Proposition 2.3 Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_1 \) be a radial solution of \((L_N^N)\) with the initial data \( \varphi \). Put
\[
U_L^N(r) = c_1 r \left( 1 + \frac{L^N}{N-1} r^{-N} \right), \quad a_{\varphi}^N = \int_{\Omega} \varphi(x) U_L^N(|x|) dx.
\]
Then there exists a positive constant \( C \) such that
\[
||\nabla v_1(x, t)||_{L^\infty(\Omega)} \leq C (|a_{\varphi}^N| + o(1)) t^{-\frac{N+2}{2}}
\]
for sufficiently large \( t \). Furthermore, for any \( R > L \),
\[
v_1(x, t) = (a_{\varphi}^N + o(1)) U_L^N t^{-\frac{N+2}{2}}, \\
\partial_r v_1(x, t) = c_1 (a_{\varphi}^N + o(1)) (1 - L^N r^{-N}) t^{-\frac{N+2}{2}}, \\
\partial_r^2 v_1(x, t) = c_1 (a_{\varphi}^N + o(1)) NL^N r^{-(N+1)} t^{-\frac{N+2}{2}}
\]
as \( t \to \infty \), uniformly on \( \Omega \cap B(0, R) \).

Similarly we study the asymptotic behavior of \( v_2, \partial_r v_2 \) and \( \partial_r^2 v_2 \) as \( t \to \infty \), and obtain the following proposition.

Proposition 2.4 Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_2 \) be a radial solution of \((L_2^N)\) with the initial data \( \varphi \). Then there exists a positive constant \( C_1 \) such that
\[
||v_2(\cdot, t)||_{L^\infty(\Omega)} \leq C_1 t^{-\frac{N+2}{2}}, \\
||\partial_t v_2(\cdot, t)||_{L^\infty(\Omega)} \leq C_1 t^{-\frac{N+3}{2}}
\]
for sufficiently large \( t \). Furthermore, for any \( R > L \), there exists a constant \( C_2 \) such that
\[
|\partial_r^2 v_2(x, t)| \leq C_2 t^{-\frac{N+3}{2}}
\]
for all \( x \in \Omega \) with \( |x| \leq R \) and all sufficiently large \( t \).

By Propositions 2.1–2.4, we may obtain the asymptotic behavior of the solutions \( u_{k,j} \), \( k = 0, 1, 2 \), \( j = 1, \ldots, l_k \). Finally, by (2.6), we put
\[
\phi_3 = \phi - \sum_{k=0}^{2} \sum_{j=1}^{l_k} \phi_{k,j}(|x|) Q_{k,j} \left( \frac{x}{|x|} \right), \tag{2.7}
\]
and study the solution of (1.1) with the initial data \( \phi_3 \). Then we have
Proposition 2.5 Assume (2.1). Let $\phi_3$ be a function defined by (2.6) and (2.7). Let $u_3$ be a function of (1.1) with the initial data $\phi_3$. Then there exists a constant $C$ such that
\[ \|\nabla^k_x u_3(\cdot, t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N+3}{2}}, \quad k = 0, 1, 2, \]
for all sufficiently large $t$.

By Propositions 2.1–2.5, we obtain the asymptotic behavior of $u$, $\nabla_x u$, and $\nabla^2_x u$ as $t \to \infty$, and may obtain Theorems 2.1 and 2.2.

3 On the Cauchy-Neumann Problem (1.2)

In this section we assume that

(3.1) \[ \phi \in L^2(\Omega, \rho dx), \quad m_\phi > 0, \]

where $\rho(x) = \exp(|x|^2/4)$ and

\[
m_\phi = \begin{cases} 
\int_\Omega \phi(x) \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right) dx & \text{if } N \geq 3, \\
\int_\Omega \phi(x) \log \frac{|x|}{L} dx & \text{if } N \geq 2.
\end{cases}
\]

We first give the following results on the asymptotic behavior of the solution $u$ of (1.2), which imply that the hot spots $H(t)$ run away from the boundary $\partial\Omega$ as $t \to \infty$.

Theorem 3.1 (See Theorem 1.1 in [9].) Let $u$ be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1) and $N \geq 3$. Then

(3.2) \[ \lim_{t \to \infty} \int_\Omega u(x, t) dx = m_\phi > 0 \]

and

(3.3) \[ \lim_{t \to \infty} t^\frac{N}{2} u(x, t) = (4\pi)^{-\frac{N}{2}} m_\phi \left(1 - \frac{L^{N-2}}{|x|^{N-2}}\right) \]

uniformly for all $x$ on any compact set in $\overline{\Omega}$. 

**Theorem 3.2** (See Theorem 1.2 in [9].) Let $u$ be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1) and $N = 2$. Then there exists a constant $C$ such that

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq C (\log t)^{-1} \|\phi\|_{L^2(\Omega, \rho dx)}$$

for all $t \geq 1$. Furthermore

$$\lim_{t \to \infty} (\log t) \int_\Omega u(x, t) dx = 2m_\phi$$

and

$$\lim_{t \to \infty} (\log t)^2 u(x, t) = \frac{1}{\pi} m_\phi \log \frac{|x|}{L}$$

uniformly for all $x$ on any compact set in $\overline{\Omega}$.

**Remark 3.1** Collet, Martínes, and Martín [4] used the probability method to prove the asymptotic behavior of the Dirichlet heat kernel $G = G(x, y, t)$ on the exterior domain of a compact set as $t \to \infty$. In particular, for the exterior domain $\mathbb{R}^N \setminus B(0, L)$, they obtained that

$$\lim_{t \to \infty} t^{\frac{N}{2}} G(x, y, t) = (4\pi)^{-\frac{N}{2}} \left( 1 - \frac{L^{N-2}}{|x|^{N-2}} \right) \left( 1 - \frac{L^{N-2}}{|y|^{N-2}} \right)$$

if $N \geq 3$,

$$\lim_{t \to \infty} t (\log t)^2 G(x, y, t) = \frac{1}{\pi} \log \frac{|x|}{L} \log \frac{|y|}{L}$$

if $N = 2$,

for all $x, y \in \Omega$ (see also [6]). By (3.3) and (3.6), we may obtain (3.7) and (3.8), and the proof of this paper is complete different from the one of [4]. Furthermore we remark that Herruz [7] applied the comparison method to the Cauchy-Dirichlet problem (1.2) in the exterior domain of a compact set, and obtained the similar results to Theorems 3.1 and 3.2 for nonnegative initial data $\phi$.

Next we give a result on the rate for the hot spots $H(t)$ to run away from the boundary $\Omega$ as $t \to \infty$.

**Theorem 3.3** (See Theorem 1.3 in [9].) Let $u$ be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1). Put

$$\zeta(t) = 2(N - 2)L^{N-2} t \quad \text{if } N \geq 3, \quad \zeta(t) = 2t (\log t)^{-1} \quad \text{if } N = 2.$$
Then
\[ \lim_{t \to \infty} \sup_{x \in H(t)} |\zeta(t)^{-1}|x|^{N} - 1| = 0. \]  

Furthermore there exists a positive constant \( T \) such that, if \( x \in H(t) \) and \( t \geq T \), then
\[ H(t) \cap l_{x} = \{x\}, \]
where \( l_{x} = \{kx/|x| : k \geq 0\} \).

Next we give a sufficient condition for the hot spots \( H(t) \) to consist of one point \( x(t) \) after a finite time. Furthermore we give the limit of \( x(t)/|x(t)| \) as \( t \to \infty \).

**Theorem 3.4** (See Theorem 1.4 in [9].)
Let \( u \) be a solution of the Cauchy-Dirichlet problem (1.2) under the condition (3.1). Assume that
\[ A_{\phi}^{D} = \int_{\Omega} x\phi(x) \left(1 - \frac{L^{N}}{|x|^{N}}\right) dx \neq 0. \]

Then there exist a positive constant \( T \) and a smooth curve \( x = x(t) \in C^{\infty}([T, \infty) : \Omega) \) such that \( H(t) = \{x(t)\} \) for all \( t \geq T \) and
\[ \lim_{t \to \infty} \frac{x(t)}{|x(t)|} = A_{\phi}^{D}/|A_{\phi}^{D}|. \]

Therefore, by Theorems 3.3 and 3.4, we see that, under the assumptions (3.1) and \( A_{\phi}^{D} \neq 0 \), the set of hot spots \( H(t) \) consists of one points \( x(t) \) after a finite time, and
\[ \lim_{t \to \infty} \zeta(t)^{-1/N}|x(t)| = 1, \quad \lim_{t \to \infty} x(t)/|x(t)| = A_{\phi}^{D}/|A_{\phi}^{D}|. \]

Next we explain the outline of the proofs of Theorems 3.1–3.3. In the similar way to the Cauchy-Neumann problem (1.1), we have only to study the asymptotic behavior of the radial solutions \( v_{k} \) of the Cauchy-Dirichlet problem
\[ (L_{k}^{D}) \begin{cases} \partial_{t}v = L_{k}v \equiv \Delta v - \frac{\omega_{k}}{|x|^{N}}v_{k} & \text{in } \Omega \times (0, \infty), \\ v = 0 & \text{on } \partial\Omega \times (0, \infty), \\ v(x, 0) = \varphi(x) & \text{in } \Omega, \end{cases} \]
where \( \varphi \) is a radial function belonging to \( L^2(\Omega, \rho dx) \) and \( k = 0, 1, 2 \ldots \). Furthermore, by the same argument with in the Cauchy-Neumann problem (1.1), we introduce a rescaled function \( w_k \) of \( v_k \), and study the asymptotic behavior of the rescaled functions \( w_k \) as \( s \to \infty \). For the case \( N \geq 3 \), we study the asymptotic behavior of \( \varphi = \varphi(y, s) \) in the space \( L^2 \) with weight \( \rho \), and obtain the one of \( \varphi = \varphi(x, t) \) for all \( x \in \Omega \) with \( |x| \sim t^{1/2} \) as \( t \to \infty \). Furthermore, by using the radially symmetry of \( \varphi \) and \( (L_0^D) \), we obtain the asymptotic behavior of \( \varphi \), \( \partial_r \varphi \), \( \partial_r^2 \varphi \), and \( \partial_t \varphi \) for all \( x \in \Omega \) with \( |x| = O(t^{1/2}) \) as \( t \to \infty \).

**Proposition 3.1** Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_0 \) be a radial solution of \( (L_0^D) \) with the initial data \( \varphi \) and \( N \geq 3 \). Put

\[
U_L^{D,0}(r) = c_0 \left( 1 - \frac{L^{N-2}}{r^{N-2}} \right), \quad a_{\varphi}^{D,0} = \int_{\Omega} \varphi(x) U_L^{D,0}(|x|) dx.
\]

Then there hold that

\[
v_0^*(r, t) = t^{-\frac{N}{2}}(a_{\varphi}^{D,0} + o(1))U_L^{0}(r) + \frac{N}{2}t^{-\frac{N+2}{2}}(a_0 + o(1))O(r^2)
+ O(t^{-\frac{N+4}{2}})O(r^4),
\]

\[
(\partial_r v_0^*)(r, t) = t^{-\frac{N}{2}}(a_{\varphi}^{D,0} + o(1))\partial_r U_L^{0}(r)
- \frac{Nc_0}{4}rt^{-\frac{N+2}{2}}(a_{\varphi}^{D,0} + o(1))(1 + O(r^{-1}))+ O(t^{-\frac{N+4}{2}})O(r^3),
\]

\[
(\partial_r^2 v_0^*)(r, t) = t^{-\frac{N}{2}}(a_{\varphi}^{D,0} + o(1))\partial_r^2 U_L^{0}(r) - U_L^{0}(r)\frac{N}{2}t^{-\frac{N+2}{2}}(a_{\varphi}^{D,0} + o(1))
+ O(t^{-\frac{N+4}{2}})O(r^2),
\]

\[
(\partial_t v_0^*)(r, t) = -\frac{N}{2}t^{-\frac{N+2}{2}}(a_{\varphi}^{D,0} + o(1))U_L^{0}(r) + O(t^{-\frac{N+4}{2}})O(r^2)
\]

for all \( r \geq L \) and \( t \geq 1 \).

For the case \( N = 2 \), the behavior of \( v_0 \) is different from the one for the case \( N \geq 3 \). By the similar way to in the case \( N \geq 3 \), we first obtain \( \max_{x \in \Omega} |\partial_r v_0(x, t)| = O(t^{-1}(\log t)^{-1}) \) as \( t \to \infty \). This gives that \( ||v_0(\cdot, t)||_{L^1(\Omega)} = O((\log t)^{-1}) \) as \( t \to \infty \). By using the similar argument to in the case \( N \geq 3 \) again, we have \( \max_{x \in \Omega} |\partial_r v_0(x, t)| = O(t^{-1}(\log t)^{-2}) \) as \( t \to \infty \), and obtain the following proposition.
Proposition 3.2 Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_0 \) be a radial solution of \( (L_0^D) \) with the initial data \( \varphi \) and \( N = 2 \). Put

\[
\tilde{a}_{\varphi}^{D,0} = 4c_0^2 \int_{\Omega} \varphi(x) \log \frac{|x|}{L} \ dx.
\]

Then there exists a function \( \zeta_1 = \zeta_1(t) \) and \( \zeta_2(t) \) with

\[
\lim_{t \to \infty} t(\log t)^2 \zeta_1(t) = \tilde{a}_{\varphi}^{D,0}, \quad \lim_{t \to \infty} t^2(\log t)^2 \zeta_2(t) = \tilde{a}_{\varphi}^{D,0},
\]

such that

\[
v_0(r, t) = \zeta_1(t) \log \frac{r}{L} + O(r^2 \log r) \zeta_1(t) + O(r^4) O(t^{-3}(\log t)^{-1}),
\]

\[
(\partial_r v_0)(r, t) = \frac{\zeta_1(t)}{r} - \zeta_1(t) r \log r (1 + o(1)) + O(r^3) O(t^{-3}(\log t)^{-1}),
\]

\[
(\partial_r^2 v_0)(r, t) = -\frac{\zeta_1(t)}{r^2} - U_L^0(r) \zeta_1(t) + O(r^2) O(t^{-3}(\log t)^{-1}),
\]

\[
(\partial_t v_0)(r, t) = -\left( \log \frac{r}{L} \right) \zeta_2(t) + O(r^2) O(t^{-3}(\log t)^{-1})
\]

for all \( r \geq L \) and \( t \geq 2 \).

Furthermore, by the similar argument to the problem (1.1), we obtain the asymptotic behavior of the solutions \( v_1 \) and \( v_2 \).

Proposition 3.3 Let \( \varphi \) be a radial function in \( \Omega \) satisfying (2.1). Let \( v_1 \) be a radial solution of \( (L_1^D) \) with the initial data \( \varphi \) and \( N \geq 2 \). Put

\[
U_L^{D,1}(r) = c_1 r \left( 1 - \frac{L^N}{r^N} \right), \quad a_{\varphi}^{D,1} = \int_{\Omega} \varphi(x) U_L^{D,1}(|x|) \ dx.
\]

Then there hold that

\[
v_1^*(r, t) = t^{-\frac{N+2}{2}} (a_{\varphi}^{D,1} + o(1)) U_L^1(r) + O(r^2) O(t^{-\frac{N+3}{2}}),
\]

\[
(\partial_r v_1^*)(r, t) = t^{-\frac{N+2}{2}} (a_{\varphi}^{D,1} + o(1)) \partial_r U_L^1(r) + O(r) O(t^{-\frac{N+3}{2}}),
\]

\[
(\partial_r^2 v_1^*)(r, t) = t^{-\frac{N+2}{2}} (a_{\varphi}^{D,1} + o(1)) \partial_r^2 U_L^1(r) + O(t^{-\frac{N+3}{2}})
\]

for all \( r \geq L \) and \( t > 1 \).
Proposition 3.4 Let $\varphi$ be a radial function in $\Omega$ satisfying (2.1). Let $v_2$ be a radial solution of $(L^D_2)$ with the initial data $\varphi$ and $N \geq 2$. Then there hold that

\begin{align*}
v_2^*(r,t) &= O(t^{-\frac{N+4}{2}} \log t)U_{L}^{D,2}(r) + O(t^{-\frac{N+4}{2}})O(r^2 \log r), \\
\partial_r v_2^*(r,t) &= O(t^{-\frac{N+4}{2}} \log t)\partial_r U_{L}^{D,2}(r) + O(t^{-\frac{N+4}{2}})r \log \frac{r}{L}, \\
\partial_r^2 v_2^*(r,t) &= O(t^{-\frac{N+4}{2}} \log t)\partial_r^2 U_{L}^{D,2}(r) + O(t^{-\frac{N+4}{2}})\log \frac{r}{L}
\end{align*}

for all $r \geq L$ and $t > 1$, where

$$U_{L}^{D,2}(r) = c_2 r^2 \left(1 - \frac{L^{N+2}}{r^{N+2}}\right).$$

Therefore, by the similar argument to the problem (1.1) and Propositions 3.1-3.4, we may prove Theorems 3.1-3.3. In order to prove Theorem 3.4, we study the asymptotic behavior of $x/|x|$ for all $x \in H(t)$ and all sufficiently large $t$, by using the asymptotic behavior of $v_0$ and $v_1$. Furthermore we compare the hot spots $H(t)$ with the radial solution of (1.2) with the initial data $\varphi \in L^2(\Omega, \rho dx)$ with $m_{\varphi} = m_\phi$. Then we may prove that, if $t$ is sufficiently large, then the matrix $\{-\partial_{x_i}\partial_{x_j}u(x,t)\}_{i,j=1}^{N}$ is positive definite for all points near the hot spots $H(t)$, and complete the proof Theorem 3.4.

References


