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Products of weak topologies, and k-spaces

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Let X be a space, and let $\mathcal{P}$ be a cover of X. As is well-known, X has the *weak topology with respect to* $\mathcal{P}$, if $G \subset X$ is open in X iff $G \cap P$ is open in P for each $P \in \mathcal{P}$. Here, it is possible to replace "open" by "closed". For (fundamental) matters about "weak topologies", see [Du] and [T8], etc.

For a cover $\mathcal{P}$ of a space X, let us recall that X *is determined by* $\mathcal{P}$ ([GMT]) if X has the weak topology with respect to $\mathcal{P}$. Let us call such a cover $\mathcal{P}$ a *determining cover* in this paper.

For a closed cover $\mathcal{F}$ of a space X, let us recall that X *is dominated by* $\mathcal{F}$ ([M1]) (X has the *weak topology* in the sense of K. Morita [Mo1]; *Whitehead weak topology*; or *hereditarily weak topology, with respect to* $\mathcal{F}$), if any subcollection $\mathcal{P}$ of $\mathcal{F}$ is a closure-preserving cover (i.e., $\mathcal{F}$ is a closure-preserving closed cover), and a determining cover of the union of elements of $\mathcal{P}$. Let us call such a closed cover $\mathcal{F}$ a *dominating cover* in this paper.

For a closure-preserving (resp. hereditarily closure-preserving) cover $\mathcal{P}$, we say that $\mathcal{P}$ is CP (resp. HCP) in this paper.

Open cover $\Rightarrow$ *Determining cover* $\Leftarrow$ *Dominating cover* $\Leftarrow$ *HCP closed cover* $\Leftarrow$ *Locally finite closed cover.*

Remark: Let L be an infinite convergent sequence (containing its limit point). Then L has a countable increasing determining CP closed cover $\{F_n, L : n \in N\}$ ($F_n \subset F_{n+1}$) which is not a dominating cover, and L has a similar countable increasing determining closed cover which is not CP.

While, L has a countable increasing dominating cover which is not HCP. Here and after, an *increasing cover* $\mathcal{P}$ means that $\mathcal{P} = \{P_\gamma : \gamma < \delta\}$, where $P_\alpha \subset P_\beta$ if $\alpha < \beta$. When $\delta = \omega$, $\mathcal{P}$ is a *countable increasing cover.*

We assume that spaces are regular T_1, and maps are continuous and onto.

A space X is respectively a *sequential space*; *k-space*; *quasi-k-space* if X has a determining cover (consisting) of compact metric subsets; compact subsets; countably compact subsets. (We note that every space with a determining cover of sequential (sub)spaces; k-spaces; quasi-k-spaces is respectively a sequential space; k-space; quasi-k-space). Sequential spaces are k-spaces, and k-spaces are quasi-k-spaces. We recall that every sequential space; k-space; quasi-k-space is respectively characterized as a quotient space of a (locally compact) metric space; locally compact (paracompact) space; M-space.

Let X be a space, and let $\mathcal{P} = \{P_n : n \in N\}$ be an increasing determining cover of X. Then X is the inductive limit (or directed limit) of $\{P_n : n \in N\}$ (denoted by $X = \varinjlim P_n$). When all P_n are closed in X, $\mathcal{P}$ is a dominating cover of X. As is well-known, every CW-complex has a dominating cover of compact metric subsets.

Now, let us consider preservation (heredity) of weak topologies with respect to "maps", "subsets", and "products". For "maps" and "subsets", we have the following notes, for example.

Note: (1) Let $\mathcal{P}$ be a cover of a space X. As is well-known, $\mathcal{P}$ is a determining cover of X iff the obvious map $f : \Sigma\mathcal{P} \to X$ is a quotient map, where $\Sigma\mathcal{P}$ is the disjoint union of elements of $\mathcal{P}$.

(2) Let $f : X \to Y$ be a quotient (resp. closed) map. If $\mathcal{P}$ is a determining (resp. dominating) cover of X, then $\{f(P) : P \in \mathcal{P}\}$ is also a determining (resp. dominating) cover of Y.

(3) Let $f : X \to Y$ be a closed map. If $\mathcal{P}$ is a determining (resp. dominating) cover of Y, then $\{f^{-1}(P) : P \in \mathcal{P}\}$ is also a determining (resp. dominating) cover of X (cf. [T1]).

Note: Let $\mathcal{P}$ be a determining cover of a space X. For a subset S of X, let $\mathcal{P}|S = \{P \cap S : P \in \mathcal{P}\}$. Every $\mathcal{P}|S$ need not be a determining cover of S even if $\mathcal{P}$ is a countable increasing dominating cover of X by compact metric subsets.

Let $\mathcal{P}$ be a determining (resp. dominating) cover of a space X, and let S be a subset X. Then $\mathcal{P}|S$ is a determining (resp. dominating) cover of S if S is open or closed in X, or S is a k-space. Also, the following theorem holds. (Here, the "determining" of the cover $\mathcal{P}$ is unessential in the proof).

(Theorem): Let $\mathcal{P}$ be a (determining) cover of a space X. Then the following are equivalent.

(a) For any subset S of X, $\mathcal{P}|S$ is a determining cover of S.

(b) For any subset S of X and any $a \in clS$, there exists $P \in \mathcal{P}$ such that $a \in P \cap cl(S \cap P)$.

When $\mathcal{P}$ is a dominating (or closed) cover, the following are equivalent.

(a') For any subset S of X, $\mathcal{P}|S$ is a dominating cover of S.

(b') For any subset S of X, $\mathcal{P}|S$ is CP in X.

(Corollary): For a space X, the following are equivalent.

(a) X is Fréchet.

(b) There is a determining cover $\mathcal{P}$ of X by compact metric subsets such that $\mathcal{P}|S$ is a determining cover of S for every (countable) subset S of X.

(c) For some (or any) determining cover $\mathcal{P}$ of X by Fréchet spaces, $\mathcal{P}|S$ is a determining cover of S for every (countable) subset S of X.

(d) X is sequential, and for any determining cover $\mathcal{P}$ of X and any subset S of X, $\mathcal{P}|S$ is always a determining cover of S.

As a generalization of Fréchet spaces or locally compact spaces, let us recall k'-spaces. A space X is a *k'-space*, if whenever $a \in clA$, then there exists a compact subset K of X such that $a \in cl(A \cap K)$ (when the compact set K is metric, such a space X is precisely Fréchet). Analogously, a space X is a k'-space iff for some (or any) dominating cover $\mathcal{F}$ of X by k'-spaces, $\mathcal{F}|S$ is a dominating (or determining) cover of S for any subset S of X.

Concerning (the surround of) k-spaces, the author has been studying products of these spaces, in [T3, T5, T6, T7, T9] and so on. As for weak topologies, let us recall the following (classic) problems on products of weak topologies. The question (**W**) is considered in [T10]. In this paper, we will give some answers to the problems in §1 and §2.

Problems: (**W**) For each i = 1, 2, let $\mathcal{P}_i$ be a determining cover of a space X_i. Is $\mathcal{P}_1 \times \mathcal{P}_2$ a determining cover of $X_1 \times X_2$?

(**HW**) Same as (**W**), but replace "determining" by "dominating".

1. Negative answers

Let $L = \{x_n : n \in N\} \cup \{x_0\}$ be a sequence converging to x_0 with $x_0 \neq x_n$. For an infinite cardinal number α, let $\Sigma(\alpha)$ be the disjoint union of α copies of L. Let S_α be the space obtained from $\Sigma(\alpha)$ by identifying all the limit points. S_ω is called the *sequential fan*, in particular. S_α is a Fréchet space with the obvious HCP closed cover (hence, dominating cover) $\mathcal{F}_\alpha$ by α copies of L. Let T_ω be the space obtained from the disjoint union $\Sigma(\omega) + L$ by identifying each limit point $p_n \in \Sigma(\omega)$ with $x_n \in L$. T_ω is called the *Arens' space* (denoted by S_2, usually). The space T_c, $c = 2^\omega$, is similarly defined as T_ω, but replace "L" by the closed interval "[0, 1]" (identify each limit point $p_\alpha \in \Sigma(c)$ with $\alpha \in (0,1]$). T_ω has the obvious point-finite determining cover $\mathcal{P}_\omega$ by ω copies of L, and it is the prefect pre-image of S_ω. These properties also hold on T_c by replacing "ω" by "c". The following examples give negative answers to **Problems**: (**W**) & (**HW**).

Examples: The following (a) is well-known, and (b) is essentially given in [D] (see [Du]). Q is the space of rational numbers.

(a) $\{Q\} \times \mathcal{F}_\omega$ is not a determining cover of $Q \times S_\omega$.

(b) $\mathcal{F}_\omega \times \mathcal{F}_c$ is not a determining cover of $S_\omega \times S_c$.

In (a) or (b), it is possible to replace "S_ω" by "T_ω"; or "S_c" by "T_c" (changing "$\mathcal{F}_\omega$" to "$\mathcal{P}_\omega$"; or "$\mathcal{F}_c$" to "$\mathcal{P}_c$" respectively).

Remark: (1) For countable determining closed covers $\mathcal{F}_i$ $(i = 1, 2)$ of spaces X_i by locally compact subsets, $\mathcal{F}_1 \times \mathcal{F}_2$ is a determining cover of $X_1 \times X_2$ (thus, $X_1 \times X_2$ is a k-space). It is possible to replace "closed covers $\mathcal{F}_i$" by "increasing covers $\mathcal{F}_i$ (or, covers $\mathcal{F}_i$ such that, for any A, $B \in \mathcal{F}_i$, $C \supset A$ and $C \supset B$ for some $C \in \mathcal{F}_i$")".

(2) For CP covers $\mathcal{P}_i$ $(i = 1, 2)$ of spaces X_i, $\mathcal{P}_1 \times \mathcal{P}_2$ is also a CP cover of $X_1 \times X_2$.

(3) Let X_i $(i = 1, 2)$ be quasi-k-spaces which are not discrete. Then, for HCP covers $\mathcal{F}_i$ of X_i, $\mathcal{P}_1 \times \mathcal{P}_2$ is a HCP cover of $X_1 \times X_2$ iff $\mathcal{F}_i$ are locally finite in X_i.

2. Positive answers

We give some positive answers to **Problems**: (**W**) & (**HW**).

(I) *When X_1 is locally compact,*

(**W**) is positive if one of the following (a) $\sim$ (e) holds.

(a) $\mathcal{P}_1$ is an open cover.

(b) $\mathcal{P}_1$ is a countable increasing cover.

(c) $\mathcal{P}_1$ is a point-countable closed cover.

(d) $\mathcal{P}_1$ is a dominating cover.

(e) Elements of $\mathcal{P}_2$ are k-spaces.

(**HW**) is positive if $\mathcal{P}_1$ is a HCP closed cover, or an increasing closed cover.

(II) *When $X_1 \times X_2$ is a quasi-k-space,*

(**W**) is positive if the following (a) or (b) holds ([T10]).

(a) X_1 is sequential (in particular, $X_1 \times X_2$ is sequential).

(b) $X_1 \times X_2$ is a k-space, and elements of $\mathcal{P}_1$ are k-spaces.

(**HW**) is positive if $\mathcal{P}_1$ is a HCP closed cover, or an increasing closed cover.

The author has the following question in view of the above positive answers. The question (**W**) for $X_1 \times X_2$ being a k-space is given in [T10].

Question: Let X_1 be a locally compact space, or $X_1 \times X_2$ be a k-space. Is (**W**) or (**HW**) positive ?

3. Applications

As applications of the positive answers in §2, we have the following.

Theorem 3.1: Let X_1 be a locally compact space with a dominating cover (resp. HCP closed cover) $\mathcal{P}_1$. Let $\mathcal{P}_2$ be a determining (resp. dominating) cover of X_2. Then $\mathcal{P}_1 \times \mathcal{P}_2$ is a determining (resp. dominating) cover of $X_1 \times X_2$.

Theorem 3.2: (1) Let X_1 be a k-space with a determining cover $\mathcal{P}_1$. Let X_2 be a space with a determining cover $\mathcal{P}_2$ of locally compact subsets. Then $X_1 \times X_2$ is a k-space iff $\mathcal{P}_1 \times \mathcal{P}_2$ is a determining cover of $X_1 \times X_2$.

(2) Let X_i $(i = 1, 2)$ have a determining cover $\mathcal{P}_i$ of first countable subsets. Then the following (a) $\sim$ (d) are equivalent.

(a) $X_1 \times X_2$ is a sequential space.

(b) $X_1 \times X_2$ is a k-space.

(c) $X_1 \times X_2$ is a quasi-k-space.

(d) $\mathcal{P}_1 \times \mathcal{P}_2$ is a determining cover of $X_1 \times X_2$.

(3) In (1) and (2), it is possible to replace "determining" by "dominating" if the cover $\mathcal{P}_1$ is a HCP closed cover, or an increasing closed cover.

Corollary 3.3: Let X_i $(i = 1, 2)$ be a space with a dominating cover $\mathcal{F}_i$ of locally compact subsets. Suppose that X_1 and X_2 satisfy (a), (b), or (c) below. Then $X_1 \times X_2$ is a k-space with a determining cover $\mathcal{F}_1 \times \mathcal{F}_2$.

(a) Locally separable.

(b) Locally ω_1-compact (i.e., each point has a nbd whose closure is ω_1-compact; that is, the closure has no uncountable discrete closed subsets).

(c) Character $\leq \omega_1$ (i.e., each point has a local base of cardinality $\leq \omega_1$).

4. Countable products of weak topologies

For a determining cover $\mathcal{P}$ of a space X, let $\mathcal{P}^* = \{P \cup F : P \in \mathcal{P}, F$ is a finite subset of $X\}$, and let $[\mathcal{P}]$ be the collection of all finite unions of elements of $\mathcal{P}$. Then $\mathcal{P}^*$ and $[\mathcal{P}]$ are also a determining cover of X.

Theorem 4.1: (1) Let $\mathcal{P}$ be a determining cover of X. If X^ω is a sequential space, then $\mathcal{P}^{*\omega}$ is a determining cover of X^ω (see [T1]).

(2) Suppose that $\mathcal{P}$ is a dominating cover or a point-countable determining cover. If X^ω is a sequential space, then $[\mathcal{P}]^\circ (= \{intP : P \in [\mathcal{P}]\})$ is an open cover of X (by refering to [T3] and [GMT]).

Remark 4.2: (1) In Theorem 4.1(1), it is impossible to replace "$\mathcal{P}^{*\omega}$ " by "$\mathcal{P}^\omega$ ". (Indeed, for a discrete space $D = \{0, 1\}$ with a cover $\mathcal{F} = \{\{0\}, \{1\}\}$, $\mathcal{F}^\omega$ is not a determining cover of a compact metric space D^ω). While, for a determining cover $\mathcal{P}$ of a space X, for each $n \in N$, $\mathcal{P}^n$ is a determining cover of X^n if X^n is sequential (see §3).

(2) Let $\mathcal{P}$ be a dominating cover or a point-countable determining cover of a space X. Then $[\mathcal{P}]^\omega$ is a determining cover of X^ω if X^ω is a quasi-k-space. Also, when the elements of $\mathcal{P}$ are locally compact closed subsets, $[\mathcal{P}]^\omega$ is a determining cover of X^ω iff X^ω is a k-space (or quasi-k-space). In these results, it is possible to replace "$[\mathcal{P}]^\omega$" by "$\mathcal{P}^\omega$" if $\mathcal{P}$ is an increasing dominating cover, or a countable increasing determining cover (without the closedness of the locally compact subsets).

Corollary 4.3: (1) For a space X with a dominating cover $\mathcal{F}$ of first countable spaces (resp. metric spaces), the following are equivalent.

(a) X^ω is a sequential space.

(b) X^ω is a quasi-k-space.

(c) X is a first countable space (resp. metric space).

(d) $[\mathcal{P}]^\circ$ is an open cover of X.

(e) $\mathcal{F}^{*\omega}$ is a determining cover of X^ω.

(2) For a space X with a point-countable determining closed cover $\mathcal{P}$ of first countable spaces (resp. metric spaces), the same equivalence in (1) remains true, but the equivalence for the parenthetic part holds if X is a paracompact space.

Remark 4.4: For a space X with a point-countable determining cover of metric spaces (resp. locally separable, metric spaces), (a), (b), and (c) in Corollary 4.3(1) are equivalent (by refering to [T3], and [L] (resp. [T4])).

5. Applications to products of paracompact spaces

As special applications of "Products of weak topologies" in §2, in terms of determining or dominating covers, let us consider "Products of paracompact spaces". First, let us recall the following note on CP covers by certain spaces.

Note: (1) Every space with a CP cover of compact subsets is meta-compact. While, there exists a normal space X with a CP cover of finite sets, but X is not paracompact (for these, see [Y], for example).

(2) Every space X is a P-space if X has a σ-HCP closed cover of P-spaces (in view of [Mo4]); or a σ-CP closed cover of countably compact subsets ([Y]). Also, every space X with a dominating cover of perfect spaces (i.e., every closed set is a G_δ-set) is a perfect space, hence a P-space.

(3) Every space with a countable closed cover of Σ-spaces is a Σ-space ([N]). While, there exists a space X with a HCP cover of compact subsets, but X is not a Σ-space ([M2]) (hence, X can not be expressed as a countable closed cover of locally compact subsets).

(4) Every separable space X with a CP closed cover $\mathcal{F}$ of σ-spaces; P-spaces; Σ-space is respectively a σ-space; P-space; Σ-space.

Theorem 5.1: Every space with a dominating cover of paracompact spaces; normal spaces; σ-spaces is respectively a paracompact space; normal space ([M1] or [Mo2]); σ-space ([T2], etc.).

Remark 5.2: There exists a locally compact, separable, σ-space X with a determining CP closed cover of metric spaces, but X is not meta-compact, nor normal. Thus, the "dominating" cover in Theorem 5.1 is essential.

Question 5.3: (1) Is any space with a determining CP closed cover of σ-spaces (or metric spaces) a σ-space ?

(2) Is any space with a dominating cover of (paracompact) P-spaces a P-space ?

Theorem 5.4: Let X be a paracompact space with a σ-CP cover of compact subsets. Let Y be a paracompact space. Then $X \times Y$ is a paracompact space ([Y]).

Corollary 5.5: (1) Let Y be a paracompact space. Then $X \times Y$ is a paracompact space if one of the following holds.

(a) X is a paracompact space with a countable closed cover of locally compact subsets ([Mo3]).

(b) X is a paracompact space with a CP closed cover of locally compact subsets, and X is a locally separable space.

(c) X has a dominating cover of compact spaces.

(d) X has a dominating cover of locally compact, paracompact σ-spaces.

(e) X has a dominating cover of locally compact, paracompact spaces, and X is a locally ω_1-compact space or X has character $\leq \omega_1$.

(f) X is a paracompact space with a point-countable determining closed cover of locally compact subsets, and X is a locally ω_1-compact space or X has character $\leq \omega_1$.

Question 5.6: (1) Is it possible to omit "σ-spaces" in (d) ?

(2) When X is paracompact, is it possible to replace "dominating cover" to "CH closed cover" in (d) or (e) ?

Theorem 5.7: Let X be a paracompact P-spaces X, and Y be a paracompact Σ-space. Then $X \times X$ is paracompact ([N]).

Question 5.8: Let X be a space with a dominating cover of paracompact P-spaces (resp. paracompact Σ-space), and let Y be a paracompact Σ-space (resp. paracompact P-space). Is $X \times Y$ a paracompact space ?

Theorem 5.9: Let X have a HCP closed cover or an increasing dominating cover $\mathcal{P}$, and let Y have a dominating cover $\mathcal{F}$. Suppose that $X \times Y$ is a quasi-k-space. Then $X \times Y$ is paracompact (resp. normal) iff all elements of $\mathcal{P} \times \mathcal{F}$ are paracompact (resp. normal).

Corollary 5.10: Questions 5.6(1) and 5.8 are positive if $X \times Y$ is a quasi-k-space.

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