Geometry of finite-dimensional maps (Pasynkovの定理の精密化)

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Abstract. In [2 and 3], Pasynkov proved the following theorem: If \( f : X \to Y \) is a map of compacta such that \( f \) is a \( k \)-dimensional map and \( \dim Y = p < \infty \), then the set of maps \( g \) in the space \( C(X, \mathbb{I}^{p+2k+1}) \) such that the diagonal product \( f \times g : X \to Y \times \mathbb{I}^{p+2k+1} \) is an embedding is a \( G_\delta \)-dense subset of \( C(X, \mathbb{I}^{p+2k+1}) \). In this paper, furthermore we investigate the geometric properties of finite-dimensional maps and finite-to-one maps. We prove that if \( f : X \to Y \) is a map as above, then for each \( 0 \leq i \leq p + k \), the set of maps \( g \) in the space \( C(X, \mathbb{I}^{p+2k+1-i}) \) such that the diagonal product \( f \times g : X \to Y \times \mathbb{I}^{p+2k+1-i} \) is an \((i+1)\)-to-1 map is a \( G_\delta \)-dense subset of \( C(X, \mathbb{I}^{p+2k+1-i}) \). The case \( i = 0 \) implies the result of Pasynkov. Also, if \( Y \) is a one point set, our result implies the following Hurewicz's theorem: If \( \dim X = n < \infty \) and \( 0 \leq i \leq n \), then the set of maps \( g \) in the space \( C(X, \mathbb{I}^{2n+1-i}) \) such that \( g : X \to \mathbb{I}^{2n+1-i} \) is an \((i+1)\)-to-1 map is a \( G_\delta \)-dense subset of \( C(X, \mathbb{I}^{2n+1-i}) \). As a corollary, we have the following representation theorem of finite-dimensional maps: For a map \( f : X \to Y \) of compacta such that \( 0 \leq k < \infty \) and \( \dim Y = p < \infty \), \( f \) is a \( k \)-dimensional map if and only if \( f \) can be represented as the composition \( f = g_{p+2k+1} \circ \ldots \circ g_{p+k+2} \circ g_{p+k+1} \circ g_{p+k} \circ \ldots \circ g_1 \) of maps \( g_i (i = 1, 2, \ldots, p + k + 1) \) parallel to the unit interval \( I \) such that \( g_i \) is an \((i+1)\)-to-1 map for each \( i = 1, 2, \ldots, p + k \) and \( g_{p+k+1} \) is a zero-dimensional map.

\[
\begin{array}{cccccccc}
X = X_0 & \xrightarrow{g_1} & X_1 & \ldots & \xrightarrow{g_{p+k}} & X_{p+k} & \xrightarrow{g_{p+k+1}} & X_{p+k+1} \\
& \xrightarrow{g_{p+2k+2}} & X_{p+k+2} & \ldots & \xrightarrow{g_{p+2k+1}} & X_{p+2k} & \xrightarrow{g_{p+2k+1}} & X_{p+2k+1} = Y
\end{array}
\]

1 Introduction.

All spaces considered in this paper are assumed to be separable metric spaces. Maps are continuous functions. Let \( I = [0, 1] \) be the unit interval. By a compactum we mean a nonempty compact metric space. Let \( X \) and \( Y \) be compacta. Then \( C(X, Y) \) denotes the space of all maps \( g : X \to Y \) with the usual sup-metric. Note that \( C(X, Y) \) is a complete metric space.

A map \( f : X \to Y \) is a \( k \)-dimensional map \((0 \leq k < \infty)\) if for each \( y \in Y \) \( \dim f^{-1}(y) \leq k \), where \( \dim Z \) denotes the topological dimension of a space \( Z \). If a map \( f : X \to Y \) is a \( k \)-dimensional map, we write \( \dim f \leq k \). A map \( f : X \to Y \) is a \( k \)-to-1 map if for each \( y \in Y \), the cardinal number \( |f^{-1}(y)| \) of \( f^{-1}(y) \) is equal to or less than \( k \).
In [2 and 3], Pasynkov proved that if $f : X \to Y$ is a $k$-dimensional map from a compactum $X$ to a finite dimensional compactum $Y$, then there is a map $g : X \to I^k$ such that $\dim (f \times g) = 0$. Also, he proved that if $f : X \to Y$ is a map of compacta such that $f$ is a $k$-dimensional map and $\dim Y = p < \infty$, then the set of maps $g$ in the space $C(X, I^{p+2k+1})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1}$ is an embedding is a $G_\delta$-dense subset of $C(X, I^{p+2k+1})$.

In this paper, furthermore we investigate the geometric properties of finite-dimensional maps and finite-to-one maps. We prove that if $f : X \to Y$ is a map of compacta such that $f$ is a $k$-dimensional map and $\dim Y = p < \infty$, then for each $0 \leq i \leq p + k$, the set of maps $g$ in the space $C(X, I^{p+2k+1-i})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1-i}$ is an $(i+1)$-to-1 map is a $G_\delta$-dense subset of $C(X, I^{p+2k+1-i})$. Note that the restriction $g|f^{-1}(y) : f^{-1}(y) \to I^{p+2k+1-i}$ is an $(i+1)$-to-1 map for each $y \in Y$. Also, note that the case $i = 0$ implies the result of Pasynkov, and our proof in this paper is different from the proof of Pasynkov (see [3]). Also, if $Y$ is a one point set, our result implies that if $\dim X = n < \infty$ and $0 \leq i \leq n$, then the set of maps $g$ in the space $C(X, I^{2n+1-i})$ such that $g : X \to I^{2n+1-i}$ is an $(i+1)$-to-1 map is a $G_\delta$-dense subset of $C(X, I^{2n+1-i})$. As a corollary, we have the following representation theorem of finite-dimensional maps: For a map $f : X \to Y$ of compacta such that $0 \leq k < \infty$ and $\dim Y = p < \infty$, if $f$ is a $k$-dimensional map and if only if $f$ can be represented as the composition $f = g_{p+2k+1} \circ \ldots \circ g_{p+k+2} \circ g_{p+k+1} \circ g_{p+k} \circ \ldots g_1$ of maps $g_i$ ($i = 1, 2, \ldots, p+2k+1$) parallel to the unit interval $I$ (for the definition, see section 3) such that $g_i$ is an $(i+1)$-to-1 map for each $i = 1, 2, \ldots, p + k$ and $g_{p+k+1}$ is a zero-dimensional map.

$$X = X_0 \xrightarrow{g_1} X_1 \rightarrow \ldots \rightarrow X_{p+k} \xrightarrow{g_{p+k+1}} X_{p+k+1}$$

Note that the maps $g_i$ ($p + k + 2 \leq i \leq p + 2k + 1$) are 1-dimensional maps.

2 Main theorem.

A map $h : X \to Y$ is a $(p, \epsilon)$-map ($\epsilon > 0$) if for each $y \in Y$, there are subsets $A_1, A_2, \ldots, A_p$ of $h^{-1}(y)$ such that $h^{-1}(y) = \bigcup_{i=1}^{p} A_i$ and $\text{diam } A_i < \epsilon$ for each $i$. Let $f : X \to Y$ be a map and $A \subset X$. Then $f|A : A \to Y$ is a strict embedding for $f$ if $f|A$ is an embedding and $f^{-1}(f(A)) = A$. Note that $f|A : A \to Y$ is a strict embedding for $f$ if and only if $A \subset \{ x \in X | f^{-1}(f(x)) = \{x\} \}$.

In this paper, we need the following key lemma of Toruńczyk [4, Lemma 2].

**Lemma 2.1.** Let $\epsilon > 0$. Suppose that $f : X \to Y$ is a map of compacta with $\dim f = 0$ and $\dim Y = p < \infty$. For each $i = 1, 2, \ldots, l$, let $K_i$ and $L_i$ be closed
disjoint subsets of $X$. Then there are open subsets $E_i$ of $X$ separating $X$ between $K_i$ and $L_i$ such that $f|(\text{Cl}(E_1) \cup ... \cup \text{Cl}(E_l))$ is a $(p, \epsilon)$-map.

The next proposition was proved by Pasynkov in [2] (see also [4, Corollary 1] and [1, p. 48]).

**Proposition 2.2.** If $f : X \to Y$ is a $k$-dimensional map from a compactum $X$ to a finite dimensional compactum $Y$, then the set of maps $g$ in $C(X, I^k)$ such that $\dim (f \times g) = 0$ is a $G_δ$-dense subset of $C(X, I^k)$.

The following lemma is easily proved.

**Lemma 2.3.** Let $X$ and $Y$ be compacta and $A$ a closed subset of $X$. Let $C(X, Y; A, p)$ be the set of all maps $g : X \to Y$ such that $g|A$ is a $p$-to-1 map. Then $C(X, Y; A, p)$ is $G_δ$ in $C(X, Y)$.

**Theorem 2.4.** If $f : X \to Y$ is a map of compacta such that $f$ is a $k$-dimensional map and $\dim Y = p < \infty$, then for each $0 \leq i \leq p + k$, the set of maps $g$ in the space $C(X, I^{p+2k+1-i})$ such that the diagonal product $f \times g : X \to Y \times I^{p+2k+1-i}$ is an $(i + 1)$-to-1 map is a $G_δ$-dense subset of $C(X, I^{p+2k+1-i})$. Hence the restriction $g|f^{-1}(y) : f^{-1}(y) \to I^{p+2k+1-i}$ is an $(i + 1)$-to-1 map for each $y \in Y$.

## 3 Finite-dimensional maps and compositions of maps parallel to the unit interval.

A map $f : X \to Y$ is said to be embedded in a map $f_0 : X_0 \to Y_0$ (see [2 and 3]) if there exists embeddings $g : X \to X_0$ and $h : Y \to Y_0$ such that $h \circ f = f_0 \circ g$. A map $f : X \to Y$ is parallel to the unit interval $I$ (see [2 and 3]) if $f$ can be embedded in the natural projection $p : Y \times I \to Y$. In [2 and 3], Pasynkov proved the following theorem: If $f : X \to Y$ is a map such that $\dim f = k$ and $\dim Y < \infty$, then $f$ can be represented as the composition $f = h_k \circ ... \circ h_1 \circ g$ of a zero-dimensional map $g$ and maps $h_i$ ($i = 1, 2, ..., k$) parallel to the unit interval $I$ (see Proposition 2.2).

In this section, furthermore we study the properties of finite-dimensional maps and compositions of maps parallel to the unit interval. In fact, we show that the zero-dimensional map $g$ as in the above theorem of Pasynkov can be represented as a composition of some special maps parallel to $I$.

First, we prove the following proposition (Proposition 3.2) which is related to results of Uspenskij [6], Tuncali and Valov [5]. Our proof is similar to the proof of Theorem 2.4. We give the proof which is different from the proofs of Uspenskij, Tuncali and Valov (see [6] and [5]).

**Lemma 3.1.** Let $X, Y$ and $Z$ be compacta and $0 \leq k < \infty$. Let $T$ be the set of maps $g = u \times v : X \to Y \times Z$ in $C(X, Y \times Z)$ such that $\dim v(u^{-1}(y)) \leq k$ for each $y \in Y$. Then $T$ is a $G_δ$-set of $C(X, Y \times Z)$.
Proposition 3.2. Let $f : X \rightarrow Y$ be a map of compacta such that $f$ is a $k$-dimensional map and $\dim Y = p < \infty$. Let $T$ be the set of all maps $h = g \times u : X \rightarrow I^k \times I$ in $C(X, I^{k+1})$ such that $\dim h(f^{-1}(y)) \leq k$, $\dim u((f \times g)^{-1}(y, t)) = 0$ for each $y \in Y$, $t \in I^k$, and $\dim (f \times x) = 0$ and $f \times h$ is a $(p + k + 1)$-to-1 map. Then $T$ is a $G_{\delta}$-dense subset of $C(X, I^{k+1})$.

Corollary 3.3. Let $f : X \rightarrow Y$ be a map of compacta such that $f$ is a $k$-dimensional map and $\dim Y = p < \infty$. Let $\tilde{E}(X, I^{p+2k+1})$ be the set of maps in the space $C(X, I^{p+2k+1})$ such that (1) $f \times g$ is an embedding, (2) for each $1 \leq i \leq p + k$, $f \times (p_i \circ g) : X \rightarrow Y \times I^{p+2k+1-i}$ is an $(i + 1)$-to-1 map, and (3) for $h = p_{p+k} \circ g = g' \times u : X \rightarrow I^k \times I$, $\dim h(f^{-1}(y)) \leq k$, $\dim u((f \times g')^{-1}(y, t)) = 0$ for each $y \in Y$ and $t \in I^k$, and $\dim (f \times g') = 0$, where $p_i : I^{p+2k+1} \rightarrow I^{p+2k+1-i}$ is the natural projection. Then $\tilde{E}(X, I^{p+2k+1})$ is a $G_{\delta}$-dense subset of $C(X, I^{p+2k+1})$.

\[
\begin{array}{ccc}
Y \times I^{p+2k+1} & \xrightarrow{f \times g} & X \\
\downarrow Pr & & \\
Y \times I^{p+2k+1-i} & \xrightarrow{Pr} & Y \\
\downarrow Pr & & \\
Y \times I^k & \xrightarrow{Pr} & Y
\end{array}
\]

Now, we have the following representation theorem of finite-dimensional maps.

Theorem 3.4. Let $f : X \rightarrow Y$ be a map of compacta such that $0 \leq k < \infty$ and $\dim Y = p < \infty$. Then $f$ is a $k$-dimensional map if and only if $f$ can be represented as the composition

\[f = g_{p+2k+1} \circ \ldots \circ g_{p+k+2} \circ g_{p+k+1} \circ g_{p+k} \circ \ldots \circ g_1\]

of maps $g_i (i = 1, 2, \ldots, p + 2k + 1)$ parallel to $I$ such that $g_i$ is an $(i + 1)$-to-1 map for each $i = 1, 2, \ldots, p + k$ and $g_{p+k+1}$ is a zero-dimensional map.

\[
\begin{array}{cccccccc}
X = X_0 \xrightarrow{g_1} X_1 & \ldots & \xrightarrow{g_{p+k}} X_{p+k} & \xrightarrow{g_{p+k+1}} X_{p+k+1} \\
\downarrow g_{p+k+2} & \ldots & \downarrow g_{p+2k} & \downarrow g_{p+2k+1} & \\
X_{p+k+2} & \ldots & X_{p+2k} & X_{p+2k+1} = Y
\end{array}
\]

Remark. In the proof of Theorem 3.4, the maps $g_i (i = 1, 2, \ldots, p + k)$ satisfy the condition that $g_i \circ \ldots \circ g_1 (i \leq p + k)$ is an $(i + 1)$-to-1 map. In particular, $g_i (i \leq p + k)$ is an $(i + 1)$-to-1 map.
References


