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CHARACTERIZATIONS OF NORMAL COVERS ON RECTANGULAR PRODUCTS AND INFINITE PRODUCTS

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1. INTRODUCTION

An open cover $\mathcal{O}$ of a topological space X is *normal* if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that each $\mathcal{U}_n$ is a star-refinement of $\mathcal{U}_n$ for each $n \in \omega$, where $\mathcal{U}_0 = \mathcal{O}$.

We may well know the following characterization of normal covers of topological spaces. For example, it is seen in [AS, p.122], [Ho, Theorems 1.2 and 1.4] and [Mo, Theorem 1.2] etc.

Theorem 1.1 [Stone-Michael-Morita]. *Let X be a topological space and $\mathcal{O}$ an open cover of X. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero refinement which has a shrinking consisting of zero-sets.*

Now, we recall that a product space $X \times Y$ is said to be *rectangular* if every finite cozero cover of $X \times Y$ has a σ-locally finite refinement consisting of cozero rectangles. This concept was introduced by Pasynkov [Pa] in dimension theory.

The following is easily seen by the definition (see [HM, Lemma 1]).

Fact 1.2. A product space $X \times Y$ is rectangular if and only if every normal cover of $X \times Y$ has a σ-locally finite refinement consisting of cozero rectangles.

In this report, for normal covers of rectangular products, we give some characterizations analogous to Stone-Michael-Morita's Theorem above in terms of refinements consisting of cozero rectangles. Next, we can apply these characterizations to the strong rectangularity and the base-paracompactness of rectangular products as well as in [Y2]. Finally, we also give the same kind of characterization of normal covers on infinite products of metrizable spaces.

These results for normal covers of rectangular products are included in the paper [Y3] with their complete proofs. But the paper does not refer to normal covers of infinite products stated in the last section.

Throughout this paper, all spaces are *topological spaces* without any separation axiom. However, paracompact spaces are assumed to be *Hausdorff*.

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Let X be a space and $\mathcal{U}$ a cover of X. A cover $\mathcal{V}$ of X is called a *refinement* of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. A cover $\{W_U : U \in \mathcal{U}\}$ of X is called a *shrinking* of $\mathcal{U}$ if $\overline{W_U} \subset U$ for each $U \in \mathcal{U}$.

Let $X \times Y$ be a product space. A subset of the form $A \times B$ in $X \times Y$ is called a *rectangle*. For a subset R in $X \times Y$, R' and R'' denote the projections of R into X and Y, respectively. A rectangle $R = R' \times R''$ is called a *cozero rectangle (zero rectangle)* if R' and R'' are cozero-sets (zero-sets) in X and Y, respectively.

For the sake of convenience, a cover $\mathcal{G}$ of a product space $X \times Y$ is said to be *cozero rectangular (resp., zero rectangular, rectangular)* if each member of $\mathcal{G}$ is a cozero rectangle (resp., zero rectangle, rectangle) in $X \times Y$.

2. X-RECTANGULAR PRODUCTS

A product space $X \times Y$ is said to be *X-rectangular* [Oh] if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_X(\mathcal{G}) = \{G' : G \in \mathcal{G}\}$ is σ-locally finite in X.

Lemma 2.1. *A product space $X \times Y$ is X-rectangular if and only if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_X(\mathcal{G}) = \{G' : G \in \mathcal{G}\}$ is σ-discrete in X.*

Remark. X-rectangular products were originally defined for Tychonoff products in [Oh]. In the case of a Tychonoff product $X \times Y$, the proof of Lemma 3.1 can be obtained by a modification of that of [Oh, Theorem 2.2]. However, the assumption that Y is Tychonoff is necessary in his proof, because the Stone-Čech compactification βY of Y has to be used there.

Theorem 2.2. *Let $X \times Y$ be an X-rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

It is pointed out in the proof of [Ta, Theorem 1] that if X is a metric space, then the rectangularity of $X \times Y$ implies the X-rectangularity. So we have

Corollary 2.3. *Let X be a metric space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Remark. If a product space $X \times Y$ is not rectangular, there is a normal cover of $X \times Y$ which has no σ-locally finite cozero rectangular refinement (see Fact 1.2). In fact, there is a non-rectangular product with a metric factor (see [Pr], [Ta]). So we cannot exclude the assumption of rectangularity of $X \times Y$ in Corollary 2.3.

3. Products with a σ-space factor

Recall that a regular T_1-space X is a *σ-space* if there is a σ-discrete (closed) net of X.

Theorem 3.1. *Let X be a paracompact σ-space and Y a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(b) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Theorem 3.1 immediately yields the following extension of Corollary 2.3.

Corollary 3.2. *Let X be a paracompact σ-space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Theorem 3.3. *If X is a paracompact Σ-space and Y is a paracompact P-space, then every open cover of $X \times Y$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Comparing Theorems 3.1 and 3.3, it is natural to raise the following question.

Question. Can "σ-space" be replaced by "Σ-space" in Theorem 3.1?

4. Products with a factor defined by topological games

Telgársky [Te] introduced the topological game $G(\mathbf{DC}, X)$, where $\mathbf{DC}$ denotes the class of all spaces which have a discrete cover consisting of compact sets.

According to [GT], a function s from the family of all closed sets in X to itself is called a *winning strategy for Player I* in $G(\mathbf{DC}, X)$ if it satisfies

(a) $s(F) \in \mathbf{DC}$ and $s(F) \subset F$ for each closed set F in X,
(b) if $\{F_n\}$ is a decreasing sequence of closed sets in X such that $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\bigcap_{n\in\omega} F_n = \emptyset$.

A space X is said to be **DC**-*like* if there is a winning strategy for Player I in $G(\mathbf{DC}, X)$.

Theorem 4.1. *Let X be a paracompact* **DC**-*like space and Y a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(b) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

If a Hausdorff space X is subparacompact and C-scattered or it has a σ-closure-preserving cover consisting of compact sets, then Player I has a winning strategy for $G(\mathbf{DC}, X)$, that is, X is **DC**-like (see [Te, Theorems 9.7 and 14.7]). So the following is an immediate consequences of Theorem 4.2.

Corollary 4.2. *Suppose that a paracompact space X is C-scattered or has a σ-closure-preserving cover consisting of compact sets and Y is a space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

5. Applications to rectangular products

A product space $X \times Y$ is called a *strongly rectangular* [Y1] if every finite cozero (or normal) cover of $X \times Y$ has a locally finite cozero rectangular refinement.

Theorem 2.2 immediately yields

Corollary 5.1. *If a product space $X \times Y$ is X-rectangular, then it is strongly rectangular.*

Theorem 4.1 immediately yields

Corollary 5.2. *Let X be a paracompact σ-space. Then $X \times Y$ is rectangular if and only if it is strongly rectangular for any space Y.*

Moreover, Corollary 4.2 immediately yields

Corollary 5.3. *Let X be a paracompact space which is C-scattered or has a σ-closure-preserving cover consisting of compact sets. Then $X \times Y$ is rectangular if and only if it is strongly rectangular for any space Y.*

A Hausdorff space X is said to be *base-paracompact* [Po] if there is a base $\mathcal{B}$ of X such that $|\mathcal{B}| = w(X)$ and every open cover of X has a locally finite refinement consisting of members of $\mathcal{B}$.

Proposition 5.4. *Let X and Y be base-paracompact spaces. Assume that every normal cover of $X \times Y$ has a locally finite cozero rectangular refinement which has a zero rectangular shrinking. If $X \times Y$ is paracompact, then it is base-paracompact.*

Theorem 3.1 and Proposition 5.4 immediately yield

Corollary 5.5. *Let X be a base-paracompact σ-space and Y a base-paracompact space. If $X \times Y$ is paracompact and rectangular, then it is base-paracompact.*

Zhong [Zh] actually proved that the product $X \times Y$ of a stratifiable space X and a paracompact space Y is rectangular if it is (countably) paracompact. So our Corollaries 5.2 and 5.5 are extensions of [Y2, Corollaries 4.2 and 4.4], respectively.

6. Infinite products of metrizable spaces

Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product space. A subset of the form $\prod_{\lambda \in \Lambda} Y_\lambda$ in X is called a *rectangle* if there is a finite subset θ of Λ such that $Y_\lambda = X_\lambda$ for each $\lambda \in \Lambda \setminus \theta$. Recall that a subset U of X is *R-distinguished* if $\pi_R^{-1}\pi_R(U) = U$ holds, where $R \subset \Lambda$ and π_R denotes the projection from X onto $\prod_{\lambda \in R} X_\lambda$. A subset V of X is called a *cylinder* (resp., *ω-cylinder*) if V is R-distinguished for some finite (resp., countable) $R \subset \Lambda$.

$$\text{rectangle} \implies \text{cylinder} \implies \omega\text{-cylinder}$$

A cover $\mathcal{G}$ of X is said to be *rectangular* (resp., *cylindrical, ω-cylindrical*) if each member of $\mathcal{G}$ is a rectangle (resp., cylinder, ω-cylinder) in X.

By Theorem 4.3, the following is easily seen by induction.

Lemma 6.1. *Let $X = \prod_{i \leq n} X_i$ be a finite product of paracompact Σ-spaces. Then every open cover of X has a locally finite, σ-discrete, cozero rectangular refinement which has a rectangular shrinking.*

For the countable product case, we have

Lemma 6.2. *Let $X = \prod_{n \in \omega} X_n$ be a countable product of metrizable spaces. Then every binary open cover of X has a locally finite, σ-discrete, open cylindrical refinement which has a cylindrical shrinking.*

For the uncountable product case, we can prove

Lemma 6.3. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of metrizable spaces. Then every binary cover of X by open F_σ-sets has a locally finite, σ-discrete, open ω-cylindrical refinement which has a ω-cylindrical shrinking.*

A basic idea of the proof of Lemma 6.3 is due to Yamazaki's which is found in that of [Ya, Theorem 1.3].

Remark. Lemma 6.3 is an extension of [Kl, Theorem 1].

Finally, by Lemmas 6.1, 6.2 and 6.3, we can obtain the following result.

Theorem 6.4. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of metrizable spaces and $\mathcal{G}$ an open cover of X. Then the following are equivalent.*

(a) *$\mathcal{G}$ is normal.*
(b) *$\mathcal{G}$ has a σ-locally finite open rectangular refinement.*
(c) *$\mathcal{G}$ has a σ-discrete open rectangular refinement.*
(d) *$\mathcal{G}$ has a locally finite open rectangular refinement.*
(e) *$\mathcal{G}$ has a locally finite, σ-discrete, open rectangular refinement which has a rectangular shrinking.*

References

[AS] R. A. Alo and H. L. Shapiro, *Normal topological spaces*, (Cambridge University Press, London, 1974).

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, NJ, 1960.

[GT] F. Galvin and R. Telgársky, *Stationary strategies in topological games*, Topology and Appl. **22** (1986), 51–69.

[Ho] T. Hoshina, *Extensions of mappings II*, in: K. Morita and J. Nagata, Eds., Topics in General Topology, (North-Holland, 1989), pp. 41–80.

[HM] T. Hoshina and K. Morita, *On rectangular products of topological spaces*, Topology Appl. **11** (1980), 47–57.

[Kl] B. S. Klebanov, *On subspaces of products of metric spaces*, Soviet Math. Dokl. **21** (1980), 303–306.

[Mo] K. Morita, *Paracompactness and product spaces*, Fund. Math. **50** (1962), 223–236.

[Na] K. Nagami, *Σ-spaces*, Fund. Math. **65** (1969), 169–192.

[Oh] H. Ohta, *Extensions of zero-sets in the product of topological spaces*, Topology and Appl. **35** (1990), 21–39.

[Pa] B. A. Pasynkov, *On the dimension of rectangular products*, Soviet Math. Dokl. **16** (1975), 344–347.

[Po] J. E. Porter, *Base-paracompact spaces*, Topology and Appl. **128** (2003), 145–156.

[Pr] J. C. Przymusiński, *A solution to a problem of E. Michael*, Pacific J. Math. **114** (1984), 235–242.

[Ta] K. Tamano, *A note on E. Michael's example and rectangular products*, J. Math. Soc. Japan **34** (1982), 187–190.

[Te] R. Telgársky, *Spaces defined by topological games*, Fund. Math. **88** (1975), 193–223.

[Y1] Y. Yajima, *Topological games and products I*, Fund. Math. **113** (1981), 141–153.

[Y2] Y. Yajima, *A base property in paracompact products and its applications*, Topology and Appl. **142** (2004), 19–30.

[Y3] Y. Yajima, *Characterizations of normal covers of rectangular products*, Topology and Appl. (to appear).

[Ya] K. Yamazaki, *A base-normality and product spaces*, Topology and Appl. (to appear).

[Zh] N. Zhong, *Products with an M_3-factor*, Topology and Appl. **45** (1992), 131–144.

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