Absolute embeddings in Hausdorff spaces

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Arhangel’skii and Tartir [3] characterized compactness by some relative separation property and posed the following problem; characterize Tychonoff spaces $X$, for which there is a Tychonoff space $Y$ containing disjoint closed copies $X_1$ and $X_2$ of $X$ such that these copies cannot be separated in $Y$ by open subsets. Answering this question, Bella and Yaschenko [4] proved the following theorem. We note that this theorem also follows from Matveev, Pavlov and Tartir [6, Theorem 2.3].

Theorem 1 (Bella-Yaschenko [4]; see also [6]). For a Tychonoff space $X$, the following conditions are equivalent.

(a) $X$ is Lindelöf.

(b) If a Tychonoff space $Y$ contains two disjoint closed copies $X_1$ and $X_2$ of $X$, then these copies can be separated in $Y$ by open subsets.

As another type of absolute embeddings, Bella and Yaschenko [4] also obtained the following characterization of absolute weak $C$-embeddings; recall that a subspace $Y$ of a space $X$ is weakly $C$-embedded in $X$ if every continuous real-valued function $f$ on $Y$ has an extension over $X$ which is continuous at every point of $Y$ ([1]). A Tychonoff space $X$ is almost compact if $|\beta X \setminus X| \leq 1$, where $\beta X$ denotes the Stone-Čech compactification of $X$.

Theorem 2 (Bella-Yaschenko [4]). A Tychonoff space $X$ is weakly $C$-embedded in every larger Tychonoff space if and only if $X$ is almost compact or Lindelöf.

Concerning Theorem 2, Arhangel’skii [2] posed the following problem; when is a Hausdorff (Tychonoff) space $Y$ weakly $C$-embedded in every larger Hausdorff space $X$? Yamazaki [9] answered this problem as follows.

Theorem 3 (Yamazaki [9]). A Hausdorff space $X$ is weakly $C$-embedded in every larger Hausdorff space if and only if either $X$ is compact or every continuous real-valued function on $X$ is constant.

In view of these results, it is natural to consider a characterization of spaces $X$ satisfying the condition (b) of Theorem 1 in the realm of Hausdorff spaces. We give a characterization of such spaces as follows.
Theorem 4. For a Hausdorff space $X$, the following conditions are equivalent.

(a) $X$ is compact.

(b) If a Hausdorff space $Y$ contains two disjoint closed copies $X_1$ and $X_2$ of $X$, then these copies can be separated in $Y$ by open subsets.

For the detail of the proof, see [5].

Remark 5. Using [6, Theorem 2.3], we obtain the regular case of Theorem 1 as follows; for a regular space $X$, the following conditions are equivalent.

(a) $X$ is Lindelöf.

(b) If a regular space $Y$ contains two disjoint closed copies $X_1$ and $X_2$ of $X$, then these copies can be separated in $Y$ by open subsets.

Remark 6. Yajima [7] proved that the following condition (c) is equivalent to the conditions (a) and (b) in Theorem 1; (c) For every compactification $\alpha X$ of $X$, any two disjoint closed copies of $X$ in $(X \times \alpha X) \cup (\alpha X \times X)$ are completely separated in it.

Remark 7. It was proved in [8]; for a Tychonoff space $X$, the following conditions are equivalent.

(a) $X$ is compact.

(b) If a Tychonoff space $Y$ contains two disjoint closed copies $X_1$ and $X_2$ of $X$, then these copies can be completely separated in $Y$.

How about the corresponding case of regular (Hausdorff) spaces? Indeed, for a non-empty regular (respectively, Hausdorff) space $X$, we can construct a regular (respectively, Hausdorff) space $Y$ contains two disjoint closed copies $X_1$ and $X_2$ of $X$ such that these copies cannot be completely separated in $Y$ ([5]).

References


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