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Absolute embeddings in Hausdorff spaces

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Arhangel'skiĭ and Tartir [3] characterized compactness by some relative separation property and posed the following problem; *characterize Tychonoff spaces X, for which there is a Tychonoff space Y containing disjoint closed copies X_1 and X_2 of X such that these copies cannot be separated in Y by open subsets.* Answering this question, Bella and Yaschenko [4] proved the following theorem. We note that this theorem also follows from Matveev, Pavlov and Tartir [6, Theorem 2.3].

Theorem 1 (Bella-Yaschenko [4]; see also [6]). *For a Tychonoff space X, the following conditions are equivalent.*

(*a*) *X is Lindelöf.*

(*b*) *If a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be separated in Y by open subsets.*

As another type of absolute embeddings, Bella and Yaschenko [4] also obtained the following characterization of absolute weak C-embeddings; recall that a subspace Y of a space X is *weakly C-embedded* in X if every continuous real-valued function f on Y has an extension over X which is continuous at every point of Y ([1]). A Tychonoff space X is *almost compact* if $|\beta X\setminus X|\leq 1$, where βX denotes the Stone-Čech compactification of X.

Theorem 2 (Bella-Yaschenko [4]). *A Tychonoff space X is weakly C-embedded in every larger Tychonoff space if and only if X is almost compact or Lindelöf.*

Concerning Theorem 2, Arhangel'skiĭ [2] posed the following problem; *when is a Hausdorff (Tychonoff) space Y weakly C-embedded in every larger Hausdorff space X?* Yamazaki [9] answered this problem as follows.

Theorem 3 (Yamazaki [9]). *A Hausdorff space X is weakly C-embedded in every larger Hausdorff space if and only if either X is compact or every continuous real-valued function on X is constant.*

In view of these results, it is natural to consider a characterization of spaces X satisfying the condition (*b*) of Theorem 1 in the realm of Hausdorff spaces. We give a characterization of such spaces as follows.

Theorem 4. *For a Hausdorff space X, the following conditions are equivalent.*

(*a*) *X is compact.*

(*b*) *If a Hausdorff space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be separated in Y by open subsets.*

For the detail of the proof, see [5].

Remark 5. Using [6, Theroem 2.3], we obtain the regular case of Theorem 1 as follows; *for a regular space X, the following conditions are equivalent.*

(*a*) *X is Lindelöf.*

(*b*) *If a regular space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be separated in Y by open subsets.*

Remark 6. Yajima [7] proved that the following condition (*c*) is equivalent to the conditions (*a*) and (*b*) in Theorem 1; (*c*) *For every compactification αX of X, any two disjoint closed copies of X in $(X \times \alpha X) \cup (\alpha X \times X)$ are completely separated in it.*

Remark 7. It was proved in [8]; *for a Tychonoff space X, the following conditions are equivalent.*

(*a*) *X is compact.*

(*b*) *If a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be completely separated in Y.*

How about the corresponding case of regular (Hausdorff) spaces? Indeed, for a non-empty regular (respectively, Hausdorff) space X, we can construct a regular (respectively, Hausdorff) space Y contains two disjoint closed copies X_1 and X_2 of X such that these copies cannot be completely separated in Y ([5]).

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