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Some small finite groups of linear transformations and their rings of invariants and from which the automorphic forms belonging to congruence subgroups are determined

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1 Introduction

It is known that from a certain finite group of linear transformations and its invariant ring one can derive automorphic forms. However so far the construction of automorphic forms is mainly focussed on forms for subgroups of the full modular group. In the present talk we try to extend the idea to the construction of Jacobi forms. The trial is only beginning. We want to cover as far as possible cases, but here are small instances for that.

2 Finite groups of linear transformations

Let G_1 be the group generated by the linear transformations:

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}.$$

The group G_1 acts linearly on the polynomial ring $\mathbb{C}[x,y]$. Molien series $\Phi_{G_1}(t)$ of the subring $\mathbb{C}[x,y]^{G_1}$ of $\mathbb{C}[x,y]$ invariant under the action of G_1 is computed to be

$$\begin{aligned}\Phi_{G_1}(t) &= \sum_i^{\infty} \dim((\mathbb{C}[x,y]^{G_1})_i)\, t^i \\ &= \frac{1}{(1-t^4)(1-t^8)}.\end{aligned}$$

Note: The group G_1 can be regarded as the group of invariance for the weight enumerator of the class of doubly even binary self-orthogonal codes containing all one vector.

For instances, the polynomial $x^4 + y^4$ corresponds to the code with the generator matrix $(1,1,1,1)$ and the polynomial $x^8 + 14x^4y^4 + y^8$ corresponds to the Hamming $[8,4,4]$ code. The ring $\mathbb{C}[x,y]^{G_1}$ is proved to be $\mathbb{C}[x^4+y^4, x^4y^4] = \mathbb{C}[x^4+y^4, x^8+14x^4y^4+y^8]$. But we do not know such interpretation for the groups G_2 and G_3 below.

G_2 a group generated by the linear transformations:

$$\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Molien series $\Phi_{G_2}(t)$ of the invariant ring $\mathbb{C}[x,y]^{G_2}$ is computed to be

$$\Phi_{G_2}(t) = \frac{1}{(1-t^4)^2}.$$

The ring $\mathbb{C}[x,y]^{G_2}$ is proved to be $\mathbb{C}[x^4, y^4]$.

G_3 : group generated by the linear transformations represented by the matrices:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Molien series $\Phi_{G_3}(t)$ of the invariant ring $\mathbb{C}[x,y]^{G_3}$ is computed to be

$$\Phi_{G_3}(t) = \frac{1}{(1-t^2)^2}.$$

The invariant ring $\mathbb{C}[x,y]^{G_3}$ of G_3 is proved to be $\mathbb{C}[x^2, y^2]$.

One observes that

$$G_3 \subset G_2 \subset G_1,$$

and

$$\mathbb{C}[x,y]^{G_3} \supset \mathbb{C}[x,y]^{G_2} \supset \mathbb{C}[x,y]^{G_1}.$$

3 Jacobi's theta functions

3.1 Definition

Let $\mathbb{H} = \{\tau \mid \operatorname{Im}\tau > 0\}$ be the upper half plane, and τ be a variable on $\mathbb{H}$. Jacobi's theta functions are defined by

$$\begin{aligned}
\theta_0(\tau,z) &= \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2\tau + 2\pi i n z} \\
\theta_1(\tau,z) &= \frac{1}{i}\sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i (n+1/2)^2\tau + \pi i(2n+1)z} \\
\theta_2(\tau,z) &= \sum_{n\in\mathbb{Z}} e^{\pi i (n+1/2)^2\tau + \pi i(2n+1)z} \\
\theta_3(\tau,z) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}
\end{aligned}$$

Next we put

$$\begin{aligned}
\varphi_i(\tau, z) &= \theta_i(2\tau, 2z), \\
\varphi_i(\tau) &= \theta_i(2\tau) = \theta_i(2\tau, 0)
\end{aligned}$$

for $0 \leq i \leq 3$.

3.2 Properties of Jacobi's theta functions

We present many properties of Jacobi's theta functions that are reproduction of well-known formulas.

Proposition 1 *It holds that*

$$\begin{aligned}
\varphi_2(-\frac{1}{\tau}, \frac{z}{\tau}) &= \sqrt{\frac{\tau}{2i}} e^{2\pi i z/\tau} (\varphi_3(\tau, z) - \varphi_2(\tau, z)) \\
\varphi_3(-\frac{1}{\tau}, \frac{z}{\tau}) &= \sqrt{\frac{\tau}{2i}} e^{2\pi i z/\tau} (\varphi_3(\tau, z) + \varphi_2(\tau, z)) \\
\varphi_2(-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{2i}}\, (\varphi_3(\tau) - \varphi_2(\tau)) \\
\varphi_3(-\frac{1}{\tau}) &= \sqrt{\frac{\tau}{2i}}\, (\varphi_3(\tau) + \varphi_2(\tau)) \\
\epsilon &= e^{-\pi i(z+\tau/4)} \\
\delta &= e^{-\pi i(2z+\tau)} \\
\theta_3(\tau, z + \frac{1}{2}) &= \theta_0(\tau, z) \\
\theta_3(\tau, z + \frac{\tau}{2}) &= \epsilon\theta_2(\tau, z) \\
\theta_3(\tau, z + \frac{\tau}{2}\frac{1}{2}+) &= i\epsilon\theta_1(\tau, z) \\
\theta_3(\tau, z + 1) &= \theta_3(\tau, z) \\
\theta_3(\tau, z + \tau) &= \delta\theta_3(\tau, z) \\
\theta_2(\tau, z + \frac{1}{2}) &= -\theta_3(\tau, z) \\
\theta_2(\tau, z + \frac{\tau}{2}) &= \epsilon\theta_3(\tau, z) \\
\theta_3(\tau, z + \frac{1}{2} + \frac{1}{2}) &= -i\epsilon\theta_0(\tau, z) \\
\theta_2(\tau, z + 1) &= -\theta_2(\tau, z) \\
\theta_2(\tau, z + \tau) &= \delta\theta_2(\tau, z) \\
\theta_2(\tau, z + 1 + \tau) &= -\delta\theta_2(\tau, z) \\
\varphi_3(\tau, z + 1) &= \varphi_3(\tau, z) \\
\varphi_3(\tau, z + \tau) &= \delta\varphi_3(\tau, z) \\
\varphi_2(\tau, z + 1) &= \varphi_2(\tau, z)
\end{aligned}$$

4 Modular group and the congruence subgroups

4.1 modular groups

Let N be a positive integer. The principal congruence subgroup $\Gamma(N)$ of level N of the modular group $SL(2,\mathbb{Z})$ is defined to be

$$\Gamma(N)=\left\{\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\pmod N\right\}$$

Another important classes of congruence subgroups are

$$\Gamma_0(N)=\left\{\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\mid c\equiv 0\pmod N\right\},$$

and

$$\Gamma_1(N)=\left\{\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\mid a\equiv d\equiv 1\pmod N, c\equiv 0\pmod N\right\},$$

We are interested in the groups $\Gamma_0(2)$, $\Gamma_0(4)$, $\Gamma(2)$, and $\Gamma_0(4)\cap\Gamma(2)$. The picture below describes the relation among some congruence subgroups. Here a downward line implies the subgroup relation with the relative index attached.

$$\begin{tikzcd}
 & \Gamma(1)=SL(2,\mathbb{Z}) \arrow[d, dash, "\mathbf{3}\text{ index}"] & \\
 & \Gamma_0(2) \arrow[dl, dash, "\mathbf{2}"'] \arrow[dr, dash, "\mathbf{2}"] & \\
\Gamma(2) \arrow[d, dash, "\mathbf{2}"'] & & \Gamma_0(4) \arrow[dll, dash, "\mathbf{2}"'] \arrow[d, dash, "\mathbf{2}"] \\
\Gamma(2)_0 \arrow[dr, dash, "\mathbf{2}"'] & & \Gamma_1(4) \arrow[dl, dash, "\mathbf{2}"] \\
 & \Gamma_2(4)=\Gamma_1(4)\cap\Gamma(2) \arrow[d, dash, "\mathbf{2}"] & \\
 & \Gamma(4) &
\end{tikzcd}$$

The generators of some of these groups are computed to be

$$\begin{aligned}
\Gamma(1) &= \left\langle\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\right\rangle\\
\Gamma_0(2) &= \left\langle\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},\begin{pmatrix} 1 & -1\\ 2 & -1\end{pmatrix},\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}\right\rangle\\
&= \left\langle\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},\begin{pmatrix} 1 & 0\\ 2 & 1\end{pmatrix},\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}\right\rangle
\end{aligned}$$

$$\begin{aligned}
\Gamma_0(4) &= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \\
\Gamma(2) &= \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \\
\Gamma_1(4) \cap \Gamma(2) &= \left\langle \begin{pmatrix} 3 & -2 \\ 8 & -5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} -7 & 2 \\ -4 & 1 \end{pmatrix} \right\rangle
\end{aligned}$$

4.2 A definition of modular forms

A holomorphic function $f(\tau)$ on $\mathbb{H}$ is called a modular form of weight k for a group $\Gamma \subset SL(2,\mathbb{Z})$ if it satisfies the conditions:
(1) $f\left(\frac{atau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,
and
(2) holomorphic at each cusp of Γ.
We denote by $M_k(\Gamma)$ the linear space of modular forms of weight k belonging to Γ.

4.3 A summary of known results

Proposition 2 *We have the following isomorphisms between the rings*

$$\begin{array}{ccc}
\mathbb{C}[x,y]^{G_1} = \mathbb{C}[x^4+y^4, x^4y^4] & \xrightarrow{iso} & \bigoplus_{k\in\mathbb{Z}_{>0}} M_k(\Gamma_0(2)) \\
\cap & & \cap \\
\mathbb{C}[x,y]^{G_2} = \mathbb{C}[x^4, y^4] & \xrightarrow{iso} & \bigoplus_{k\in\mathbb{Z}_{>0}} M_k(\Gamma_0(4)) \\
\cap & & \cap \\
\mathbb{C}[x,y]^{G_3} = \mathbb{C}[x^2, y^2] & \xrightarrow{iso} & \bigoplus_{k\in\mathbb{Z}_{>0}} M_k(\Gamma_0(4)\cap\Gamma(2)),
\end{array}$$

where the isomorphism is realized by sending x to $\varphi_3(\tau)$ and y to $\varphi_2(\tau)$ respectively.

These are not new results. Actually the first and second lines are due to Maher[9], and the third line is due to Hiramatsu [8].

4.4 Yet another group

Let G_4 be the group generated by a linear transformation $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Then it can be easily proved that the invariant ring $\mathbb{C}[x,y]^{G_4}$ equals $\mathbb{C}[x,y^4]$. We set a map from $\mathbb{C}[x,y^4]$ to

$\oplus_{k\in\mathbb{Z}} M_k(\Gamma_0(4), \mathbf{1}_k)$ by substituting $x = \varphi_3(\tau)$ and $y = \varphi_2(\tau)$, where $M_k(\Gamma_0(4), \mathbf{1}_k)$ is a linear space of modular forms discussed in H. Cohen [5]. Cohen showed that

$$M_k(\Gamma_0(4), \mathbf{1}_k) \cong \mathbb{C}[\varphi_3(\tau), \mathcal{F}_2(\tau)],$$

where $\mathcal{F}_2(\tau) = \dfrac{\eta\ 8(4\tau)}{\eta^4(2\tau)}$ with Dedekind's eta function $\eta(\tau)$. We show

Proposition 3 *It holds that*

$$\varphi_2^4(\tau) = 2^4\mathcal{F}_2(\tau).$$

A proof is done by using the infinite product expansions of both $\varphi_2(\tau)$ and $\eta(\tau)$:

$$\begin{aligned} \eta(\tau) &= q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m) \quad q = e^{2\pi i\tau} \\ \varphi_2(\tau) &= 2q^{\frac{1}{4}} \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m})^2. \end{aligned}$$

5 Diagonalized groups

Let $G_1 \oplus G_1$ be the group generated by the linear transformations:

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$\begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \end{pmatrix}$$

The ring of invariants for $G_1 \oplus G_1$ is denoted by

$$\mathcal{R}_1 = \mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}.$$

Molien series for $G_1 \oplus G_1$ is computed to be

$$\Phi_{G_1 \oplus G_1}(t) = \frac{1 + 3t^4 + 13t^8 + 9t^{12} + 6t^{16}}{(1 - t^4)^2(1 - t^8)^2}.$$

Primary invariants (i.e. the polynomials implied by the numerator of $\Phi_{G_1 \oplus G_1}(t)$) of $\mathcal{R}_1$ are given by

$$\begin{gathered} x^4 + y^4 \\ u^4 + v^4 \\ x^8 + y^8, \\ u^8 + v^8 \end{gathered}$$

Secondary invariants (i.e. the polynomials implied by the denominator of $\Phi_{G_1 \oplus G_1}(t)$) are given by

$$
\begin{aligned}
a_0 &= 1,\\
a_{4,2} &= x^2u^2 + y^2v^2,\\
a_{4,1} &= x^3u + y^3v,\\
a_{4,3} &= xu^3 + yv^3,\\
a_{8,4} &= x^4u^4 + 2x^2y^2u^2v^2 + y^4v^4,\\
b_{8,3} &= x^5u^3 + x^3y^2uv^2 + x^2y^3u^2v + y^5v^3,\\
b_{8,5} &= x^3u^5 + x^2yu^2v^3 + xy^2u^3v^2 + y^3v^5,\\
a_{8,2} &= x^6u^2 + 2x^3y^3uv + y^6v^2,\\
a_{8,4} &= x^4u^4 + x^3yuv^3 + xy^3u^3v + y^4v^4,\\
a_{8,6} &= x^2u^6 + 2xyu^3v^3 + y^2v^6,\\
a_{8,4} &= x^4u^4 + y^4v^4,\\
b_{8,2} &= x^4y^2v^2 + x^2y^4u^2,\\
b_{8,6} &= x^2u^2v^4 + y^2u^4v^2,\\
a_{8,1} &= x^4y^3v + x^3y^4u,\\
a_{8,5} &= x^3uv^4 + y^3u^4v,\\
a_{8,3} &= x^4yv^3 + xy^4u^3,\\
a_{8,7} &= xu^3v^4 + yu^4v^3,\\
a_{12,6} &= x^6u^6 + x^4y^2u^4v^2 + x^2y^4u^2v^4 + y^6v^6,\\
a_{12,3} &= x^6y^3u^2v + x^5y^4u^3 + x^4y^5v^3 + x^3y^6uv^2,\\
a_{12,7} &= x^5u^3v^4 + x^3y^2uv^6 + x^2y^3u^6v + y^5u^4v^3,\\
a_{12,5} &= x^6yu^2v^3 + x^4y^3v^5 + x^3y^4u^5 + xy^6u^3v^2,\\
b_{12,5} &= x^3u^5v^4 + x^2yu^6v^3 + xy^2u^3v^6 + y^3u^4v^5,\\
a_{12,5} &= x^7u^5 + x^4y^3u^4v + x^3y^4uv^4 + y^7v^5,\\
b_{12,7} &= x^5u^3v^4 + x^3y^2u^5v^2 + x^2y^3u^2v^5 + y^5u^4v^3\\
b_{12,4} &= x^7yuv^3 + x^4y^4u^4 + x^4y^4v^4 + xy^7u^3v,\\
a_{12,8} &= x^4u^4v^4 + x^3yu^5v^3 + xy^3u^3v^5 + y^4u^4v^4,\\
a_{16,8} &= x^8u^8 + 2x^6y^2u^6v^2 + 2x^4y^4u^4v^4 +\\
&\quad 2x^2y^6u^2v^6 + y^8v^8,\\
a_{16,9} &= x^7u^5v^4 + 2x^5y^2u^3v^6 + x^4y^3u^8v +\\
&\quad x^3y^4uv^8 + 2x^2y^5u^6v^3 + y^7u^4v^5,\\
a_{16,7} &= x^8yu^4v^3 + 2x^6y^3u^2v^5 + x^5y^4u^7 + x^4y^5v^7\\
&\quad +2x^3y^6u^5v^2 + xy^8u^3v^4,\\
a_{16,6} &= x^9yu^3v^3 + x^7y^3uv^5 + x^6y^4u^6 + x^6y^4u^2v^4\\
&\quad +x^4y^6u^4v^2 + x^4y^6v^6 + x^3y^7u^5v + xy^9u^3v^3,\\
a_{16,10} &= x^6u^6v^4 + x^5yu^7v^3 + x^4y^2u^4v^6 +\\
&\quad 2x^3y^3u^5v^5 + x^2y^4u^6v^4 + xy^5u^3v^7 + y^6u^4v^6,\\
a_{16,8} &= x^8u^4v^4 + x^6y^2u^6v^2 + 2x^5y^3u^3v^5 +
\end{aligned}
$$

$$2x^3y^5u^5v^3 + x^2y^6u^2v^6 + y^8u^4v^4$$

Let $G_2 \oplus G_2$ be the group generated by the linear transformations:

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

The ring of invariants for $G_2 \oplus G_2$ is denoted by

$$\mathcal{R}_2 = \mathbb{C}[x, y, u, v]^{G_2 \oplus G_2}.$$

Molien series for $G_2 \oplus G_2$ is

$$\Phi_{G_2 \oplus G_2}(t) = \frac{1 + 6t^4 + 9t^8}{(1 - t^4)^4}.$$

PrimaryInvariants of the group $G_2 \oplus G_2$ are

$$\begin{aligned} c_{4,0} &= x^4, \\ d_{4,0} &= y^4, \\ c_{4,4} &= u^4, \\ d_{4,4} &= v^4 \end{aligned}$$

and secondary invariants of $G_2 \oplus G_2$ are

$$\begin{aligned} c_{0,0} &= 1, \\ c_{4,2} &= x^2u^2, \\ d_{4,2} &= y^2v^2, \\ c_{4,1} &= x^3u, \\ c_{4,3} &= xu^3, \\ d_{4,1} &= y^3v, \\ d_{4,3} &= yv^3, \\ c_{8,4} &= x^2y^2u^2v^2, \\ c_{8,3} &= x^2y^3u^2v, \\ c_{8,5} &= x^2yu^2v^3, \\ d_{8,3} &= x^3y^2uv^2, \\ d_{8,5} &= xy^2u^3v^2, \\ c_{8,2} &= x^3y^3uv, \\ d_{8,4} &= x^3yuv^3, \\ e_{8,4} &= xy^3u^3v, \\ c_{8,6} &= xyu^3v^3. \end{aligned}$$

Let $G_3 \oplus G_3$ be the group generated by the linear transformations:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The ring of invariants for $G_3 \oplus G_3$ is denoted by

$$\mathcal{R}_3 = \mathbb{C}[x, y, u, v]^{G_3 \oplus G_3}.$$

Molien series for $G_3 \oplus G_3$ is computed to be

$$\Phi_{G_3 \oplus G_3}(t) = \frac{1 + 2t^2 + t^4}{(1 - t^2)^4}.$$

Primary invariants of the group are given by;

$$\begin{aligned} f_{2,0} &= x^2, \\ g_{2,0} &= y^2, \\ f_{2,2} &= u^2, \\ g_{2,2} &= v^2 \end{aligned}$$

and secondary invariants are

$$\begin{aligned} f_{0,0} &= 1, \\ f_{2,1} &= xu, \\ g_{2,1} &= yv, \\ f_{4,2} &= xyuv \end{aligned}$$

6 From finite groups to Jacobi forms

6.1 Definition of Jacobi form

Let Γ be a subgroup of $SL(2, \mathbb{Z})$. A holomorphic function $f(\tau, z)$ defined on $\mathbb{H} \times \mathbb{C}$ is called a Jacobi form of weight k and index m for a group $\Gamma \times \mathbb{Z}^2$ if it satisfies

(3) $f\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau + d)^k e^{2\pi i m c z/(c\tau+d)} f(\tau, z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

and

(4) $f(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2 + 2\lambda z)} f(\tau, z)$ for any $(\lambda, \mu) \in \mathbb{Z}^2$.

For a subgroup Γ of the modular group we let $Jac(k, m, \Gamma \times \mathbb{Z}^2)$ to denote Jacobi forms of weight k and index m associated with the group $\Gamma \times \mathbb{Z}^2$.

6.2 A result

Proposition 4 *We have the following injective isomorphisms between the rings*

$$\begin{array}{ccc}
\mathbb{C}[x,y,u,v]^{G_1\oplus G_1} & \hookrightarrow & \bigoplus_{k\in\mathbb{Z}_{>0},m\geq 0} Jac(k,m,\Gamma_0(2)\times\mathbb{Z}^2) \\
\cap & & \cap \\
\mathbb{C}[x,y,u,v]^{G_2\oplus G_2} & \hookrightarrow & \bigoplus_{k\in\mathbb{Z}_{>0},m\geq 0} Jac(k,m,\Gamma_0(4)\times\mathbb{Z}^2) \\
\cap & & \cap \\
\mathbb{C}[x,y,u,v]^{G_3\oplus G_3} & \hookrightarrow & \bigoplus_{k\in\mathbb{Z}_{>0}} Jac(k,m,(\Gamma_0(4)\cap\Gamma(2))\times\mathbb{Z}^2),
\end{array}$$

where the isomorphism is realized by sending x to $\varphi_3(\tau)$,y to $\varphi_2(\tau)$, u to $\varphi_3(\tau,z)$ and v to $\varphi_2(\tau,z)$ respectively.

7 Jacobi forms associated with $\Gamma_0(4)$

Let $G_4\oplus G_4$ be the group generated by the linear transformation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

then we know that

$$\mathbb{C}[x,y,u,v]^{G_4\oplus G_4} = \mathcal{R}_2 \oplus y^2v^2\mathcal{R}_2 \oplus vy^3\mathcal{R}_2 \oplus yv^3\mathcal{R}_2,$$

where $\mathcal{R}_2 = \mathbb{C}[x,y^4,u,v^4]$. To u we associate $\varphi_3(\tau,z)$, and it gives Jacobi forms of weight $\frac{1}{2}$ and index 1 for $\Gamma_0(4)$ and character $\mathbf{1}_k$. In the same way we have

$$\begin{array}{rcl}
v^4 & \mapsto & \varphi_2(\tau,z)^4 \text{ Jacobi forms of weight 2 and index 4} \\
y^2v^2 & \mapsto & \varphi_2(\tau)^2\varphi_2(\tau,z)^2 \text{ Jacobi forms of weight 2 and index 2} \\
y^3v & \mapsto & \varphi_2(\tau)^3\varphi_2(\tau,z) \text{ Jacobi forms of weight 2 and index 1} \\
yv^3 & \mapsto & \varphi_2(\tau)\varphi_2(\tau,z)^3 \text{ Jacobi forms of weight 2 and index 3}
\end{array}$$

For the proof of these statements we need algebraic properties of the automorphic factor of the Jacobi forms.

References

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