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A square principle in the context of $\mathcal{P}_\kappa\lambda$

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Abstract

We introduce a combinatorial principle for $\mathcal{P}_\kappa\lambda$ based upon $\square_\kappa$. Although we cannot transfer one of the clauses of $\square_\kappa$ to this context, we can replicate some of the desired consequences of that clause. We discuss this situation and its implications along with proving the relative consistency of some $\mathcal{P}_\kappa\lambda$ versions of $\square_\kappa$.

1 Introduction

In this paper, we discuss the problem of generalising the square principle to the context of $\mathcal{P}_\kappa\lambda$. The research presented below is discussed in the author's thesis, [9]. (In fact, the principles presented there are slight variations on those defined below.) This combinatorial research follows a well-established tradition and is guided by the idea of transferring interesting notions from the theory of the combinatorics of ordinal numbers. For example, Jensen's diamond principle (see [5]) has been usefully generalised to this context (originally by Jech in [4], but also by Matet in [8] and by Džamonja in [3]).

The square principle cannot be directly transferred to the context of $\mathcal{P}_\kappa\lambda$ for various reasons, as discussed below. The general approach that we follow is to establish a basic nontrivial square principle for $\mathcal{P}_\kappa\lambda$ then explicitly add

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further properties of square in more complex forcings. In this paper, following the basic principle, we define a second principle in which the square principle's non-reflection property is added.

Throughout this paper, κ is a regular infinite cardinal and λ is a cardinal with $\kappa \leq \lambda$. We now give some basic definitions and clarify the notation used in this paper.

Let $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$. $\mathcal{P}_\kappa\lambda$ is typically ordered by $\subseteq$ and will be throughout this paper. Combinatorial ideas such as clubs and stationarity can be defined in this context, as described in [6]. Note that κ and λ are arguments and may be replaced by specified cardinals or sets respectively. In this paper, we will frequently consider $\mathcal{P}_{|x|}(x)$ where x is a set. Note that $\mathcal{P}_\kappa\lambda$ is also commonly written as $[\lambda]^{<\kappa}$.

The notation used in this paper is mostly standard. By $x \subset y$ we mean ($x \subseteq y$ and $x \neq y$). We write $\lim(\alpha)$ as an abbreviation for "α is a limit ordinal"; $\mathrm{otp}(X)$ denotes the ordertype of a wellordered set X and for an ordinal α, $\mathrm{cf}(\alpha)$ denotes the cofinality. For a function f, $\mathrm{dom}(f)$ denotes the domain of f and $\mathrm{im}(f)$ denotes image of f, while $f\lceil X$ denotes the restriction of f to X where $X \subseteq \mathrm{dom}(f)$. In forcing proofs, we follow the convention that for two conditions p, q, $p \leq q$ means p is a *weaker* condition than q. Lastly, we use $\mathrm{reg}(X)$ to denote the set of elements of X of regular cardinality.

Before we develop a version of the square principle in the context of $\mathcal{P}_\kappa\lambda$, we introduce the standard principle. This principle, denoted $\square_\kappa$, was developed by Jensen and has proved a useful tool in various areas of mathematical logic. It is defined as follows (although it should be noted that other equivalent formulations exist).

Definition 1.1 *$\square_\kappa$ is the statement that there is a sequence $\langle C_\alpha : \alpha \in \kappa^+, \lim(\alpha)\rangle$ with the following properties:*

(i) C_α is a club subset of α

(ii) if $\mathrm{cf}(\alpha) < \kappa$ then $\mathrm{otp}(C_\alpha) < \kappa$

(iii) (Coherence:) if $\beta \in C_\alpha$ and $\lim(\beta)$ then $C_\beta = C_\alpha \cap \beta$.

Forcing can be used to produce a model of set theory in which $\square_\kappa$ holds. This approach uses a partial order whose elements are initial segments of potential square sequences. It is also known that $\square_\kappa$ holds in L, the universe of constructible sets. The best-known proof uses fine structure theory and is due to Jensen; a good account of this proof is given in [2].

The square principle encapsulates various interesting properties. Coherence and anticoherence are discussed in the next section. Another property, non-reflection, is discussed further in the final section of this paper.

2 A square principle in the context of $\mathcal{P}_\kappa\lambda$

We will define a square-like principle that asserts the existence of a coherent set of subsets of $\mathcal{P}_\kappa\lambda$ indexed by the elements of $\mathcal{P}_\kappa\lambda$. Note that in considering $C_x \subseteq \mathcal{P}_{|x|}(x)$ for $x \in \mathcal{P}_\kappa\lambda$ we require $|x|$ to be regular and hence no club of $\mathcal{P}_{|x|}(x)$ will have cardinality $< |x|$. Thus, the cardinalities of the clubs cannot be limited as they are for those corresponding to singular ordinals in $\square_\kappa$. It is necessary, therefore, to introduce alternative non-triviality conditions that add some of the basic properties of $\square_\kappa$. Also, note that if κ is a successor cardinal, coherence is trivial for a club of $\mathcal{P}_\kappa\lambda$, that is for the elements of $[\lambda]^\kappa$. Thus, while $\square_\kappa$ actually asserts a property of κ^+, the principle defined below does not "look ahead" at $\mathcal{P}_{\kappa^+}\lambda$.

For the reasons mentioned above, we must assume that κ is a regular limit cardinal. In fact, since we require stationary-many regular cardinals below κ, for the remainder of this paper we assume that κ is a Mahlo cardinal.

Definition 2.1 *Suppose κ is a Mahlo cardinal and λ is an infinite cardinal with $\kappa \leq \lambda$. Suppose also that S is a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\square_{\mathcal{P}_\kappa\lambda}(S)$ is the statement that there is a family of sets $\{C_x : x \in S\}$ with the following properties:*

(i) C_x is a club subset of $\mathcal{P}_{|x|}(x)$ for all $x \in S$

(ii) (Coherence:) if $x \in S$ and $y \in C_x \cap S$ then $C_y = C_x \cap \mathcal{P}_{|y|}(y)$

(iii) (Anticoherence:) the set $\{x \in S :$ there is a cofinal set of $y \in S \cap \mathcal{P}_{|x|}(x)$ such that $C_y \neq C_x \cap \mathcal{P}_{|y|}(y)\}$ is stationary in $\mathcal{P}_\kappa\lambda$.

We write $\square_{\mathcal{P}_\kappa\lambda}$ to mean that there is a stationary $S \subseteq \mathcal{P}_\kappa\lambda$ such that $\square_{\mathcal{P}_\kappa\lambda}(S)$ holds.

Note that the restriction to a stationary subset is not as serious a restriction as it may appear since in this context we can at best have C_x defined for all $x \in \mathrm{reg}(\mathcal{P}_\kappa\lambda)$, which is stationary if Mahlo is κ, but cannot be club. Note that stronger forms of stationarity could be substituted with appropriate adjustments to the forcing below.

The anticoherence property is implicit in the definition of $\Box_\kappa$ but must be explicitly required for $\Box_{\mathcal{P}_\kappa\lambda}$. This ensures that the principle cannot be satisfied trivially, e.g. by setting $C_x = \mathcal{P}_{|x|}(x)$ for all $x \in \mathrm{reg}(\mathcal{P}_\kappa\lambda)$.

The $\Box_{\mathcal{P}_\kappa\lambda}$ principle is consistent with ZFC+ "κ is Mahlo", as we assert in the following theorem. Important questions remain unanswered, however. Noteably, it is not known whether the principle holds in ZFC + "V=L"or even in ZFC.

Before we proceed with the consistency proof, note that we could also develop a principle based on clubs of $\mathcal{P}_{\kappa_x}(x)$ for each $x \in S$. Recall that $\kappa_x = x \cap \kappa$ if this is an ordinal and is undefined otherwise. Here, we would insist that S contains only elements x for which κ_x is a regular cardinal. Assuming that κ is Mahlo, the consistency of such a principle can be proved with a forcing analogous to the one for $\Box\mathcal{P}_\kappa\lambda$.

Theorem 2.2 *Suppose M is a countable model of a sufficiently rich fragment of ZFC in which κ is Mahlo and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which κ is Mahlo and $\Box_{\mathcal{P}_\kappa\lambda}$ holds.*

This theorem is proved by forcing with the partial order defined below. Essentially, the partial order is composed of fragments of possible witnesses to $\Box_{\mathcal{P}_\kappa\lambda}$.

Definition 2.3 *Let P be a set whose elements p are characterised as follows:*

(i) p is a function with $dom(p) \in \mathcal{P}_\kappa(reg(\mathcal{P}_\kappa\lambda))$

(ii) for all $x \in dom(p)$, $p(x)$ is either club in $\mathcal{P}_{|x|}(x)$ or the empty set

(iii) if $x \in dom(p)$ and $y \in p(x) \cap reg(\mathcal{P}_\kappa\lambda)$ then $y \in dom(p)$ and either $p(y) = p(x) \cap \mathcal{P}_{|y|}(y)$ or $p(y) = \emptyset$

(iv) if $dom(p) \cap \mathcal{P}_{|y|}(y)$ is stationary in $\mathcal{P}_{|y|}(y)$ then $y \in dom(p)$.

For $p, q \in P$, $p \leq q$ (meaning q is stronger than p) iff $p \subseteq q$. We will also use the symbols $<, \geq$ and $>$ in the natural way.

Note that if we let $\emptyset$ be the function with empty domain then $\emptyset$ is the unique minimal element of P. Clearly, P is non-empty. We must now establish various properties of $(P, \leq)$ to show that a suitable generic object exists and that the resulting forcing preserves cofinalities and cardinalities.

Lemma 2.4 *$(P, \leq)$ is separative.*

Proof. Let $p \in P$ and let $x \in \mathrm{reg}(\mathcal{P}_\kappa\lambda)$ be such that $\bigcup \mathrm{dom}(p) \in \mathcal{P}_{|x|}(x)$, which is possible because $|\bigcup \mathrm{dom}(p)| < \kappa$. We now define $q \in P$ such that $p \leq q$ and $x \in \mathrm{dom}(q)$ and $q(x) \neq \emptyset$. For $y \in \mathrm{dom}(p)$, let $q(y) = p(y)$. Let $q(x)$ be any club of $\mathcal{P}_{|x|}(x)$ that does not intersect $\mathrm{dom}(p)$. Such a club exists because p satisfies (iv) of Definition 2.3. It is straightforward to check that $q \in P$, by checking against conditions (i)-(iv).

Now let $r \geq p$ be defined as follows. Let $r(y) = q(y)$ if $y \neq x$, let $r(x) = \emptyset$ and let $r(y)$ be undefined otherwise. Then $r \in P$ and q, r are clearly incompatible extensions of p.

$\dashv$.

Since P is separative, there is a generic object G in $M[G]$ that is not in the ground model, M. We will see that this generic provides an example of a $\square_{\mathcal{P}_\kappa\lambda}$ set. First, however, we must show that the forcing preserves cofinalities and cardinalities.

Lemma 2.5 *P satisfies the κ^+-chain condition.*

Proof. Suppose $X \subseteq P$ and $|X| = \kappa^+$. We show that X is not an antichain. Let $\mathcal{A} = \{\mathrm{dom}(p) : p \in X\}$. By a Δ-system argument, using the fact that κ is strongly inaccessible, we can find $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \kappa^+$ and $\mathcal{B}$ is a Δ-system with root R. That is, for all $X, Y \in \mathcal{B}$, $X \cap Y = R$.

Consider the numbers of functions with domain R such that for each function f and each $x \in R$, $f(x) \subseteq \mathcal{P}_{|x|}(x)$. Clearly, if we impose no further conditions on the value of $f(x)$, the number of distinct functions is equal to $\Pi_{x\in R}|\mathcal{P}(\mathcal{P}_{|x|}(x))|$. Now for all $x \in R$, $|\mathcal{P}_{|x|}(x)| < \kappa$ and since $\kappa^{<\kappa} = \kappa$, it follows that $\mathcal{P}(\mathcal{P}_{|x|}(x))| \leq \kappa$. Furthermore, since $|R| < \kappa$, it follows that $\Pi_{x\in R}|\mathcal{P}(\mathcal{P}_{|x|}(x))| \leq \kappa$. In other words there are only κ-many suitable functions defined on R. But $\mathcal{B} = \kappa^+$ so by the pigeonhole principle there must be some function f defined on R such that $p\lceil R = f$ for κ^+ many $p \in X$ with $\text{dom}(p) \in \mathcal{B}$.

Now let $Y = \{p \in X : \text{dom}(p) \in \mathcal{B}$ and $p\lceil R = f\}$. For $p, q \in P$, if $p(x) = q(x)$ for all $x \in \text{dom}(p) \cap \text{dom}(q)$, it is easily proved that $p \cup q$ is a common extension of p, q and hence that p, q are compatible. Thus, the elements of Y are pairwise compatible because they agree on R, which is the intersection of their domains, by the definition of $\mathcal{B}$. Hence, X is not an antichain.

$\dashv$.

We can now conclude that the forcing preserves cofinalities and cardinalities $> \kappa$. We now prove that P is $< \kappa$-directed closed. It will then follow that the forcing preserves cofinalities and cardinalities $\leq \kappa$.

Lemma 2.6 *P is $< \kappa$-directed closed.*

Proof. Suppose $\mu < \kappa$ and $\{p_\alpha : \alpha < \mu\}$ is a set of pairwise compatible conditions from P. We define $p^*_\mu = \bigcup_{\alpha<\mu} p_\alpha$. This is a function since the conditions are pairwise compatible. It is easily checked that p^*_μ satisfies (i)-(iii) of Definition 2.3. However, there may be $x \in \text{reg}(\mathcal{P}_\kappa\lambda) \backslash \text{dom}(p^*_\mu)$ such that $\text{dom}(p^*_\mu)$ is stationary in $\mathcal{P}_{|x|}(x)$ so condition (iv) may not hold. We now make a small adjustment to p^*_μ to obtain p_μ still satisfying (i)-(iii) but also satisfying (iv). Let $p_\mu(x) = p^*_\mu(x)$ for all $x \in \text{dom}(p^*_\alpha)$. For $x \in (\mathcal{P}(\bigcup \text{dom}(p^*_\mu)) \backslash \text{dom}(p^*_\mu)$, let $p_\mu(x) = \emptyset$. Then p_μ is as required since (i)-(iv) hold and for all $\alpha < \mu$, $p_\alpha < p_\mu$.

$\dashv$.

Note that since the forcing is $< \kappa$-closed, no new sets of ordinals of size $< \kappa$ are introduced. Hence, $(\mathcal{P}_\kappa\lambda)^{M[G]} = (\mathcal{P}_\kappa\lambda)^M$ and we can write $\mathcal{P}_\kappa\lambda$ for the name $\mathcal{P}_{\underset{\sim}{\kappa}}\lambda$.

We must now ensure that for any generic G of P, the set $\{x \in \mathcal{P}_\kappa\lambda : (\exists p \in G)(x \in \text{dom}(p)$ and $p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa\lambda$ in the generic extension. Before we do this we give a lemma that will be needed several times in the proof.

Lemma 2.7 *Suppose $p \in P$ and $p \Vdash (\underset{\sim}{C}$ is a club of $\mathcal{P}_\kappa\lambda)$ and suppose $y \in \mathcal{P}_\kappa\lambda$. Then there is $x \in \mathcal{P}_\kappa\lambda$ and $q \in P$ such that $q \geq p$ and $q \Vdash (y \subset x$ and $x \in \underset{\sim}{C})$.*

Proof. Let $\underset{\sim}{x}$ be a name such that $p \Vdash (y \subseteq \underset{\sim}{x}$ and $\underset{\sim}{x} \in \underset{\sim}{C})$. This is possible because $p \Vdash (\underset{\sim}{C}$ is club). Also, $p \Vdash ((\exists \mu < \kappa)(|\underset{\sim}{x}| < \mu))$ because κ is a limit cardinal.

Let $p_0 \geq p$ and ν a cardinal such that $p_0 \Vdash (\mu = \nu)$. So $p_0 \Vdash (|\underset{\sim}{x}| < \nu$ and $\nu < \kappa)$. Thus we can find a condition $p_1 \geq p_0$ and a name for an enumeration of $\underset{\sim}{x}$ in an ordertype $< \alpha$ so that: $p_1 \Vdash (\underset{\sim}{i}^* < \nu$ and $\underset{\sim}{x} = \{\underset{\sim}{\gamma}_i : i < i^*\})$ and we can extend again to obtain β and p_2 such that $p_2 \Vdash (\underset{\sim}{x} = \{\underset{\sim}{\gamma}_i : i < \beta\})$.

Now let $q_0 \geq p_2$ be such that $q_0 \Vdash (\underset{\sim}{\gamma}_0 = \delta_0)$. That is, q_0 identifies the value of the name $\underset{\sim}{\gamma}_i$. By induction on $i < \beta$ we construct an increasing sequence $\langle q_\alpha : \alpha < \beta \rangle$ and a sequence $\langle \delta_\alpha : \alpha < \beta \rangle$ such that $q_\alpha \Vdash ((\forall \xi < \alpha)(\underset{\sim}{\gamma}_\xi = \delta_x i))$. This is possible because P is $< \kappa$-closed.

Again, by the $< \kappa$-closure of P, it is possible to find $q \in P$ that identifies all the elements of $\underset{\sim}{x}$. That is, there is $z \in \mathcal{P}_\kappa\lambda$ and $q \geq p$ such that $q \Vdash (y \subseteq z$ and $z \in \underset{\sim}{C})$, as required.

$\dashv$.

Lemma 2.8 *Let G be a generic of P. Then $M[G] \models \{x \in \mathcal{P}_\kappa\lambda : (\exists p \in G)(x \in dom(p)$ and $p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa\lambda$.*

Proof. Let $\underset{\sim}{S}$ be a name of the set $\{x \in \mathcal{P}_\kappa\lambda : (\exists p \in G)(x \in \mathrm{dom}(p)$ and $p(x) \neq \emptyset)\}$.

Suppose $p_0 \in G$ is such that $p_0 \Vdash (\underset{\sim}{C}$ is club in $\mathcal{P}_\kappa\lambda$ and $\underset{\sim}{C} \cap \underset{\sim}{S} = \emptyset$ and $x_0 \in \underset{\sim}{C} \cap \mathcal{P}_\kappa\lambda)$. Note that we use the previous lemma to obtain $p_0 \Vdash (x_0 \in \underset{\sim}{C} \cap \mathcal{P}_\kappa\lambda)$. We derive a contradiction by finding $p \geq p_0$ such that $p \Vdash (\underset{\sim}{C} \cap \underset{\sim}{S} \neq \emptyset)$. The strategy is to fix a chain of elements of $\underset{\sim}{C}$ and $\underset{\sim}{S}$ up to a regular limit where the two chains intersect.

Let $y_0 \in \mathrm{reg}(\mathcal{P}_\kappa\lambda)$ be such that $(\bigcup \mathrm{dom}(p_0) \cup x_0) \in \mathcal{P}_{|y_0|}(y_0)$. We now identify $p_0^* \geq p_0$ such that $y_0 \in \mathrm{dom}(p_0^*)$.

Let D_0 be a linearly ordered club of $\mathcal{P}_{|y_0|}(y_0)$ that does not intersect $\mathrm{dom}(p_0)$. Such a club exists by definition of P (in particular, clause (iv) of Definition 2.3). Note that having D_0 linearly ordered is convenient but not strictly necessary; it is possible because $|y_0|$ is regular.

$$\text{Let } p_0^*(u) = \begin{cases} p_0(u) & \text{if } u \in \mathrm{dom}(p_0) \\ D_0 & \text{if } u = y_0 \\ \emptyset & \text{if } u \in \mathrm{reg}(\mathcal{P}(y_0) \backslash (\mathrm{dom}(p_0) \cup \{y_0\}) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then by checking against Definition 2.3, it is apparent that $p_0^* \in P$. Note also that $p_0^* \geq p_0$.

Now using the preceding lemma, let $p_1 \geq p_0^*$ be such that for some $x_1 \in \mathcal{P}_\kappa\lambda$, $p_1 \Vdash (x_1 \in \underset{\sim}{C} \cap \mathcal{P}_\kappa\lambda$ and $y_0 \subseteq x_1)$.

We now proceed inductively to define $p_\alpha, x_\alpha, y_\alpha, p_\alpha^*$ so that for all $\beta < \alpha$, $y_\beta \in p_\alpha(y_\alpha)$ and $p_\beta \leq p_\alpha \leq p_\alpha^*$. In the case when α is a limit ordinal, we describe the condition under which the induction will stop. We will then observe that this condition will be met at some stage $\alpha < \kappa$.

Case 1: $\alpha = \beta + 1$

By the inductive definition, p_α and x_α are already defined. We now define p_α^* and y_α then also define $p_{\alpha+1}$ and $x_{\alpha+1}$. Let $y_\alpha \in \mathrm{reg}(\mathcal{P}_\kappa\lambda)$ be such that $\bigcup\mathrm{dom}(p_\beta) \cup x_\alpha \in \mathcal{P}_{|y_\alpha|}(y_\alpha)$. We now identify $p_\alpha^* \geq p_\alpha$ such that $y_\alpha \in \mathrm{dom}(p_\alpha^*)$. Unlike in the case $\alpha = 0$, we will define $p_\alpha^*(y_\alpha)$ so that it has non-trivial coherence. In particular, for all $\beta < \alpha$, we will have $y_\beta \in p_\alpha^*(y_\alpha)$.

The inductive hypothesis implies that $y_\beta \in \mathrm{dom}(p_\alpha)$ so we can find a linearly ordered club D_α of $\mathcal{P}_{|y_\alpha|}(y_\alpha)$ that does not intersect $\mathrm{dom}(p_\alpha)$ and satisfies $u \in D_\alpha \Rightarrow y_\beta \subseteq u$. Such a club exists by (iv) of Definition 2.3 and by intersecting with the club $\{u \in \mathcal{P}_\kappa\lambda : y_\beta \subseteq u\}$. Now let $D_\alpha^* = p_\alpha(y_\beta) \cup \{y_\beta\} \cup D_\alpha$.

$$\text{Let } p_\alpha^*(u) = \begin{cases} p_\alpha(u) & \text{if } u \in \mathrm{dom}(p_\alpha) \\ D_\alpha^* & \text{if } u = y_\alpha \\ \emptyset & \text{if } u \in \mathrm{reg}(\mathcal{P}(y_\alpha) \backslash (\mathrm{dom}(p_\alpha) \cup \{y_\alpha\}) \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is easily checked that p_α^* satisfies (i) to (iv) of Definition 2.3 and that $p_\alpha^* \geq p_\alpha$. Note also that $y_\beta \in p_\alpha^*(y_\alpha)$.

Now using the previous lemma, let $p_{\alpha+1} \geq p_\alpha^*$ be such that for some $x_{\alpha+1} \in \mathcal{P}_\kappa\lambda$, $p_\alpha \Vdash (x_\alpha \in \underset{\sim}{C} \cap \mathcal{P}_\kappa\lambda$ and $y_\alpha \subseteq x_\alpha)$.

Case 2: α is a limit ordinal $< \kappa$

Note that x_α and p_α are not yet defined . Let $p_\alpha \in P$ be such that $p_\alpha \geq p_\beta$ for all $\beta < \alpha$. This is possible because P is $< \kappa$-closed. Let $s_\alpha = \bigcup\{y_\beta : \beta < \alpha\}$.

If $|s_\alpha|$ is regular then this will be the final stage of the induction. We then proceed to define y and p as described below. So suppose now that $|s_\alpha|$ is singular. Note in particular that $s_\alpha \notin \mathrm{reg}(\mathcal{P}_\kappa\lambda)$ so $s_\alpha \notin \mathrm{dom}(p^*_\alpha)$.

By the inductive definitions of y_β, $s_\alpha = \bigcup\bigcup\{\mathrm{dom}(p_\beta) : \beta < \alpha\}$, that is s_α is the set of ordinals that are in at least one element of the domain of at least one p_β. Let $y_\alpha \in \mathrm{reg}(\mathcal{P}_\kappa\lambda)$ be such that $s_\alpha \in \mathcal{P}_{|y_\alpha|}(y_\alpha)$. Thus, for any $\beta < \alpha$, if $u \in \mathrm{dom}(p_\beta)$ then $u \in \mathcal{P}_{|y_\alpha|}(y_\alpha)$.

Let D_α be a linearly ordered club of $\mathcal{P}_{|y_\alpha|}(y_\alpha)$ that does not intersect $\mathrm{dom}(p_\alpha)$ and such that if $u \in D_\alpha$ then $s_\alpha \subseteq u$. Let $D^*_\alpha = \bigcup\{p_\beta(y_\beta) : \beta < \alpha\} \cup \{s_\alpha\} \cup D_\alpha$.

$$\text{Let } p^*_\alpha(u) = \begin{cases} p_\alpha(u) & \text{if } u \in \mathrm{dom}(p_\alpha) \\ D^*_\alpha & \text{if } u = y_\alpha \\ \emptyset & \text{if } u \in \mathrm{reg}(\mathcal{P}(y_\alpha) \backslash (\mathrm{dom}(p_\alpha) \cup \{y_\alpha\}) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then $p^*_\alpha \in P$ and $(\forall\beta < \alpha)(p^*_\alpha \geq p_\beta)$.

As before, let $p_{\alpha+1} \geq p^*_\alpha$ be such that for some $x_{\alpha+1} \in \mathcal{P}_\kappa\lambda$, $p_\alpha \Vdash (x_\alpha \in \underset{\sim}{C} \cap \mathcal{P}_\kappa\lambda$ and $y_\alpha \subseteq x_\alpha)$.

We repeat this procedure until we reach a limit ordinal $\alpha = \mu < \kappa$ such that s_α (as defined in Case 2) has inaccessible cardinality. There must be such a μ because κ is Mahlo. Otherwise the set $\{s_\alpha : \alpha < \kappa \text{ and } \lim(\alpha)\}$ would be a club subset of κ that does not intersect the set of regular cardinals, contradicting the fact that κ is Mahlo. So suppose $|s_\alpha|$ is regular. Then $|s_\alpha|$ is inaccessible because the sequence $\langle |y_\beta| : \beta < \alpha\rangle$ is strictly increasing by the inductive definitions of y_β for $\beta < \alpha$.

Let $y = s_\alpha$ and let $E = \bigcup\{\mathrm{dom}(p_\beta) : \beta < \alpha\}$. Now define p as follows.

$$\text{Let } p(u) = \begin{cases} p_\beta(u) & \text{if } (\exists\beta < \mu)(u \in \mathrm{dom}(p_\beta)) \\ \bigcup\{p_\beta(y_\beta) : \beta < \mu\} & \text{if } u = y \\ \emptyset & \text{if } u \in \mathrm{reg}(\mathcal{P}(y) \backslash (E \cup \{y\}) \\ \text{undefined} & \text{otherwise} \end{cases}$$

As before, by checking against (i)-(iv) of Definition 2.3, we see that $p \in P$. We now show that $p \Vdash \underset{\sim}{C} \cap \underset{\sim}{S} \neq \emptyset$.

Note that $\bigcup_{\beta<\mu} y_\beta = y = \bigcup_{\beta<\mu} x_\beta$ because for any $\beta < \mu$, $x_\beta \subset y_\beta \subseteq x_{\beta+1} \subset y_{\beta+1}$. By the definition of p, it is clear that $p(y) \neq \emptyset$ and hence that $p \Vdash y \in \underset{\sim}{S}$. Also, since $p \Vdash (\underset{\sim}{C}$ is club in $\mathcal{P}_\kappa\lambda$ and $(\forall\beta < \mu)(x_\beta \in \underset{\sim}{C}))$ it follows that $p \Vdash y \in \underset{\sim}{C}$. Hence, $p \Vdash y \in \underset{\sim}{C} \cap \underset{\sim}{S}$, which is a contradiction because $p \geq p_0$ and $p_0 \Vdash \underset{\sim}{C} \cap \underset{\sim}{S} = \emptyset$.

$\dashv$.

We now establish that the proposed witness to $\square_{\mathcal{P}_\kappa \lambda}$ satisfies the anticoherence condition.

Lemma 2.9 *Let G be a generic of P. Then let* $\underset{\sim}{S} = \{x \in \mathcal{P}_\kappa\lambda : (\exists p \in G)(x \in dom(p)$ *and* $p(x) \neq \emptyset)\}$ *and let* $\underset{\sim}{T} = \{x \in \underset{\sim}{S}$: *there is a cofinal set of* $y \in \underset{\sim}{S} \cap \mathcal{P}_{|x|}(x)$ *such that* $(\exists p \in G)(\{x, y\} \subseteq dom(p)$ *and* $p(y) \neq p(x) \cap \mathcal{P}_{|y|}(y))\}$.

Then $M[G] \models T$ is stationary in $\mathcal{P}_\kappa\lambda$.

Proof (outline). We proceed as in Lemma 2.8, forming the sequence of forcing conditions as before but at each stage, we interrupt the induction after setting p^*_α but before setting $x_{\alpha+1}$. We set $z_\alpha \supset y_\alpha$ and define $q \geq p^*_\alpha$ such that $z_\alpha \in \mathrm{dom}(q)$ but $q(z_\alpha) \cap q(y_\alpha) = \emptyset$. Now continue as before but defining $x_{\alpha+1}$ so that $z_\alpha \subset x_{\alpha+1}$ and with $q \leq p$.

$\dashv$.

Finally, we need to verify that κ is Mahlo in the generic extension $M[G]$.

Lemma 2.10 *If G is a generic of P then $M[G] \models \kappa$ is Mahlo.*

Proof. Working in $M[G]$, suppose C is a club in κ. Then if $C^* = \{x \in \mathcal{P}_\kappa\lambda : |x| \in C\}$, it follows that C^* is club in $\mathcal{P}_\kappa\lambda$. By Lemma 2.8, we can find y in $C^* \cap \{x \in \mathcal{P}_\kappa\lambda : (\exists p \in G)(x \in \mathrm{dom}(p)$ and $p(x) \neq \emptyset)\}$. Then $|y|$ is a regular cardinal in both M and $M[G]$, by the preservation of cofinalities and cardinalities. Furthermore, $|y| \in C$. Hence the set of regular cardinals is stationary in κ. To see that κ remains a strong limit, note that for all $\mu < \kappa$, $(2^\mu)^{M[G]} = (2^\mu)^M$ by $< \kappa$-closure so κ remains a strong limit in the generic extension. Hence κ is Mahlo in $M[G]$ as required.

$\dashv$.

Given generic G of P, let $S = \{x \in \mathrm{reg}(\mathcal{P}_\kappa\lambda) : (\exists p \in G)(p(x) \neq \emptyset)$ and for $x \in S$, let $C_x = p(x)$ where p is an element of G with $x \in \mathrm{dom}(p)$. The preceding series of lemmas together prove that this S and $\{C_x : x \in S\}$ provides a witness to $\Box_{\mathcal{P}_\kappa\lambda}$ in $M[G]$. Thus, Theorem 2.2 is proved.

We proved in Lemma 2.10 that this forcing preserves the fact that κ is Mahlo. In fact, we can do more than this and preserve supercompactness. Since forcing with P is κ-directed closed, if κ is supercompact in the ground model and we first force with a Laver preparation, then the supercompactness of κ is preserved when we force with P.

Theorem 2.11 *Suppose M is a countable model of a sufficiently rich fragment of ZFC in which κ is supercompact and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which κ is supercompact and $\Box_{\mathcal{P}_\kappa\lambda}$ holds.*

Proof. This follows by forcing with a Laver preparation followed by forcing with P. We use the fact that P is κ-directed closed.

$\dashv$.

3 A $\mathcal{P}_\kappa\lambda$ version of square with a non-reflection property

One of the useful properties encapsulated by the square sequence is that of stationary non-reflection. This is demonstrated in the theorem presented below, which makes use of Fodor's Lemma, which we present here without proof.

Lemma 3.1 (Fodor's Lemma) *Suppose that S is a stationary subset of a regular cardinal μ. Suppose also that $f : S \longrightarrow \mu$ is such that $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is a stationary subset $T \subseteq S$ such that f is constant on T.*

The following well-known theorem is presented here with proof to motivate the work towards a $\mathcal{P}_\kappa\lambda$ version of the theorem discussed in the remainder of this section.

Theorem 3.2 *If $\Box_\kappa$ holds then κ^+ has a non-reflecting stationary subset.*

Proof. Suppose $\langle C_\alpha : \alpha < \kappa^+ \text{ and } \lim(\alpha)\rangle$ is as specified in the definition of $\square_\kappa$. Let $T = \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) < \kappa < \alpha\}$. To see that this is stationary, let C be an arbitrary club of κ^+ and let $C^* = C \backslash \kappa$. Then the ωth element of C^* is an element of T.

Now define $F : T \to \kappa$ by $F(\alpha) = \mathrm{otp}(C_\alpha)$. By part (ii) of Definition 1.1 and the definition of T, $F(\alpha) < \kappa < \mathrm{otp}(\alpha)$ for all $\alpha \in T$. Hence, by Fodor's Lemma, we can select a stationary subset $R \subseteq T$ such that F is constant on R.

Now suppose R reflects in α for some $\alpha \in R$. Let $\beta, \gamma \in R \cap C_\alpha$ with $\beta < \gamma$. Then $C_\beta \cup \{\beta\} \subseteq C_\gamma$ as $\beta = \sup(C_\beta)$. Thus $F(\gamma) = \mathrm{otp}(C_\gamma) \geq \mathrm{otp}(C_\beta) + 1 > F(\beta)$. But this is a contradiction because F is constant on R.

$\dashv$.

We now extend $\square_{\mathcal{P}_\kappa \lambda}$ to produce a square principle that has a non-reflection property explicitly built into the definition. We then give a non-reflection theorem using this new principle.

Definition 3.3 *$\square_{\mathcal{P}_\kappa \lambda}(S, f)$ holds if $f : S \to \kappa$ and S is stationary and there is a witness $\{C_x : x \in S\}$ to $\square_{\mathcal{P}_\kappa \lambda}(S)$ such that in addition to (i)-(iii) from Definition 2.1 we have:*

(iv) $f(x) \in x$

(v) if $y \in C_x$ then $f(x) \neq f(y)$.

We now prove the relative consistency of this principle by extending the partial order P used in the proof of Theorem 2.2.

Theorem 3.4 *Suppose M is a countable model of a sufficiently rich fragment of ZFC in which κ is Mahlo and $\lambda \geq \kappa$. Then there is a generic extension of this model which preserves cofinalities and cardinalities and in which κ is Mahlo and for some f, S, $\square_{\mathcal{P}_\kappa \lambda}(S, f)$ holds.*

We force with the poset Q defined below.

Definition 3.5 *$p, q \in Q$ iff $p \in P$ and q is as follows:*

(i) q is a function with domain $\{x \in dom(p) : p(x) \neq \emptyset\}$

(ii) $q(x) \in x$ for all $x \in dom(q)$

(iii) if $x \in dom(p)$ and $y \in p(x) \cap dom(p)$ and $p(y) \neq \emptyset$ then $q(y) \neq q(x)$.

If $(p, q), (p', q') \in Q$ then $(p, q) \leq (p', q')$ iff $p \subseteq p'$ and $q \subseteq q'$.

We do not present all of the details of the forcing proof. Instead we describe how to upgrade the proof of Theorem 2.2 to include the new property.

Note that $(\emptyset, \emptyset) \in Q$ so Q is non-empty and has a minimal element. We must now establish various properties of $(Q, \leq)$ to show that a suitable generic object exists and that the resulting forcing preserves cofinalities and cardinalities.

Lemma 3.6 *$(Q, \leq)$ is separative.*

Proof. Let $(p, q) \in Q$ and let $x \in \text{reg}(\mathcal{P}_\kappa\lambda) \backslash \text{dom}(p)$ such that there is $\gamma \in x \backslash \text{im}(q)$. Let $(p_0, q_0) \geq (p, q)$ be such that $p_0(x)$ is a club in $\mathcal{P}_{|x|}(x)$ that does not intersect $\text{dom}(p)$ and let $q_0(x) = \gamma$. Such a p_0 can be found by Definition 2.3 (iv) and because $|\text{dom}(p)| < \kappa \leq |\text{reg}(\mathcal{P}_\kappa\lambda)|$ so there must be some $x \in \text{reg}(\mathcal{P}_\kappa\lambda) \backslash \text{dom}(p)$. Now let $(p_1, q_1) \geq (p, q)$ be such that $x \in \text{dom}(p_1)$ and $p_1(x) = \emptyset$ and hence $x \notin \text{dom}(q_1)$. Clearly (p_0, q_0) and (p_1, q_1) are incompatible extensions of (p, q). Hence, Q is separable.

$\dashv$.

We now prove that forcing with Q preserves cofinalities and cardinalities by showing that Q has the κ^+-chain condition and is $< \kappa$-directed closed.

We now use the Δ-System Lemma to show that Q has the κ^+-chain condition.

Lemma 3.7 *Q satisfies the κ^+-chain condition.*

Proof. Let A be a subset of Q of size κ^+. Now let $\mathcal{A} = \{\text{dom}(p) : \exists q(p, q) \in A\}$. By the Δ-System Lemma, using the fact that κ is a strong limit, we can find $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \kappa^+$ and $\mathcal{B}$ is a Δ-system with root R.

Consider the number of pairs of functions (p, q) definable on R such that for each function (p, q) and each $x \in R$, $p(x) \in \mathcal{P}(\mathcal{P}_{|x|}(x))$ and $q(x) \in x$. By the argument in the proof of Lemma 2.5, the number of possible values that $p(x)$ can take is $< \kappa$. The number of possible values that $q(x)$ can take is clearly $|x|$. Since $|x| < \kappa$, the number of possible pairs $(p(x), q(x))$ is $< \kappa$. But $|\mathcal{B}| = \kappa^+$ so by the pigeonhole principle there must be some pair of functions (g, h) defined on R such that $p\lceil R = g$ and $q\lceil R = h$ for κ^+ many $(p, q) \in X$ with $\mathrm{dom}(p) \in \mathcal{B}$.

Now let $Y = \{(p, q) \in X : p\lceil R = g \text{ and } q\lceil R = h\}$. For any $(p_0, q_0), (p_1, q_1) \in Y$, using the fact that p_0, p_1 and q_0, q_1 agree R, it is straightforward to verify that $(p_0 \cup p_1, q_0 \cup q_1) \in Q$. Thus, $(p_0, q_0), (p_1, q_1)$ have a common extension in Q and hence are compatible. Hence, A is not an antichain.

$\dashv$.

Lemma 3.8 *Q is $< \kappa$-directed closed.*

Proof. Suppose $\mu < \kappa$ and $\{(p_\alpha, q_\alpha) : \alpha < \mu\}$ is a set of pairwise compatible conditions from Q. We define $p^*_\mu = \bigcup_{\alpha<\mu} p_\alpha$ and $q^*_\mu = \bigcup_{\alpha<\mu} q_\alpha$. Now extend p^*_μ to p_μ as in the proof of the $< \kappa$-directed closure of P. Note that we need not add new elements to the domain of q^*_μ since $x \in \mathrm{dom}(p_\mu)\backslash\mathrm{dom}(p^*_\mu) \Rightarrow p_\mu(x) = \emptyset$. That is, we may set $q_\mu = q^*_\mu$. Now for any $x, y \in \mathrm{dom}(q_\mu)$, there is some $\alpha < \mu$ such that $x, y \in \mathrm{dom}(q_\alpha)$. Since $(p_\gamma, q_\gamma) \in Q$ it follows that $x \in p_\mu(x) \Rightarrow q_\mu(x) \neq q_\mu(y)$ and vice versa as required. It follows that $(p_\alpha, q_\alpha) \in Q$ and for all $\beta < \mu$, $(p_\alpha, q_\alpha) \leq (p_\mu, q_\mu)$.

$\dashv$.

It follows from the preceding lemmas that forcing with Q preserves cofinalities and cardinalities. As with P, this forcing is $< \kappa$-closed so for a generic G of Q, $(\mathcal{P}_\kappa\lambda)^{M[G]} = (\mathcal{P}_\kappa\lambda)^M$ and we can write $\mathcal{P}_\kappa\lambda$ for the name $\mathcal{P}_{\underset{\sim}{\kappa}}\lambda$ in the following. We must now ensure that for any generic G of Q, the set $\{x \in \mathcal{P}_\kappa\lambda : (\exists(p, q) \in G)(x \in \mathrm{dom}(p) \text{ and } p(x) \neq \emptyset\}$ is stationary in $\mathcal{P}_\kappa\lambda$. Note that the following variation on Lemma 2.7 holds. The proof is almost identical to the proof of Lemma 2.7.

Lemma 3.9 *Suppose $(p, q) \in Q$ and $(p, q) \Vdash (\underset{\sim}{C}$ is a club of $\mathcal{P}_\kappa\lambda)$. Then there is $x \in \mathcal{P}_\kappa\lambda$ and $(p', q') \in Q$ such that $(p', q') \geq (p, q)$ and $(p', q') \Vdash x \in \underset{\sim}{C}$.*

Lemma 3.10 *Let G be a generic of Q. Then $M[G] \models \{x \in \mathcal{P}_\kappa\lambda : (\exists p \in G)(x \in dom(p)$ and $p(x) \neq \emptyset)\}$ is stationary in $\mathcal{P}_\kappa\lambda$.*

Proof. We proceed as in the proof of Lemma 2.8 but define (p_α, q_α) and (p^*_α, q^*_α) at each stage. We now describe how to set q_α. Let $\gamma \in y_0 \backslash \{q(y_0)\}$. We insist, without loss of generality, that for all α, γ is not in the image of q_α or q^*_α. For all $\alpha < \mu$ we set $q(y_\alpha) = \gamma_\alpha \in y_\alpha \backslash \bigcup_{\beta<\alpha} y_\beta$. By definition of y_α, such a γ_α wil always exist. At the final stage, when defining (p, q), we define p as before and set $q(y) = \gamma$.

$\dashv$.

The last two lemmas that we need follow by arguments exactly analogous to the corresponding lemmas for P.

Lemma 3.11 *Let G be a generic of Q. Then let* $\underset{\sim}{S} = \{x \in \mathcal{P}_\kappa\lambda : (\exists(p,q) \in G)(x \in dom(p)$ *and* $p(x) \neq \emptyset)\}$ *and let* $\underset{\sim}{T} = \{x \in \underset{\sim}{S}$: *there is a cofinal set of* $y \in \underset{\sim}{S} \cap \mathcal{P}_{|x|}(x)$ *such that* $(\exists(p,q) \in G)(\{x,y\} \subseteq dom(p)$ *and* $p(y) \neq p(x) \cap \mathcal{P}_{|y|}(y))\}$.

Then $M[G] \models T$ *is stationary in* $\mathcal{P}_\kappa\lambda$.

Lemma 3.12 *If G is a generic of Q then $M[G] \models \kappa$ is Mahlo.*

By forcing with the partial order $(Q, \leq)$, Theorem 3.4 is proved. We set $S = \{x \in \mathcal{P}_\kappa\lambda : (\exists(p,q) \in G)(x \in \mathrm{dom}(p)$ and $p(x) \neq \emptyset)\}$ and set $f = \bigcup\{q : \exists p((p,q) \in G)\}$. Then f and $\{C_x : (\exists(p,q) \in G)(C_x = p(x) \neq \emptyset)\}$, together witness that $\Box_{\mathcal{P}_\kappa\lambda}(S,f)$ holds, as required.

We now show how $\Box_{\mathcal{P}_\kappa\lambda}(S,f)$ gives non-reflection in $\mathcal{P}_{|x|}(x)$ for stationary many $x \in \mathcal{P}_\kappa\lambda$. We then state without proof some related results proved by Abe in [1] and by Koszmider in [7].

The following is proved by forcing and draws on Gitik's method of shooting clubs in $\mathcal{P}_\kappa\lambda$.

Theorem 3.13 (Abe) *Let $V \subset W$ be two models of ZFC with the same ordinals, $(\kappa^+)^V = (\kappa^+)^W$; let C be a club subset of κ of V-inaccessibles; let κ be an inaccessible cardinal in W and let $T = \{x \in \mathcal{P}_\kappa\kappa^+ : V \models |x|$ is not inaccessible$\}$. Then there is a forcing notion in W that preserves cofinalities and cardinalities and such that there is a stationary $S \subset \mathcal{P}_\kappa\kappa^+$ such that $S \cap \mathcal{P}_{\kappa_x}(x)$ is non-stationary for any $x \in T$.*

Koszmider in [7] gives a different kind of non-reflection result, considering reflection in $\mathcal{P}_\kappa(X)$ where $X \subset \lambda$.

Theorem 3.14 (Koszmider) *It is consistent that there is a stationary set $S \subset \mathcal{P}_\kappa\lambda$ such that $S \cap \mathcal{P}_\kappa X$ is non-stationary in $\mathcal{P}_\kappa X$ for any $X \subset \lambda$ with $|X| \geq \kappa$ in the generic extension.*

Finally we consider the following theorem of Abe which gives a form of non-reflection when κ is supercompact.

Theorem 3.15 (Abe) *If it is consistent that there is a supercompact cardinal then it is consistent that there is a supercompact κ, a cardinal $\lambda \geq \kappa$ and a stationary set $X \subset \mathcal{P}_\kappa\lambda$ such that $X \cap \mathcal{P}_\kappa\alpha$ is non-stationary in $\mathcal{P}_\kappa\alpha$ for any $\alpha < \lambda$.*

The following definition presents the form of non-reflection that we examine with $\square_{\mathcal{P}_\kappa\lambda}(S, f)$.

Definition 3.16 *A stationary set $S \subseteq \mathcal{P}_\kappa\lambda$ reflects in $\mathcal{P}_{|x|}(x)$ if $S \cap \mathcal{P}_{|x|}(x)$ is stationary in $\mathcal{P}_{|x|}(x)$.*

The non-reflection theorem follows easily from the $\square_{\mathcal{P}_\kappa\lambda}(S, f)$ principle. Note that the proof is closely analogously to the proof of non-reflection from $\square_\kappa$ in the theory of cardinals. This theorem draws on the variation on Fodor's Lemma presented below. Lacking a suitable reference, we present a proof.

Lemma 3.17 *Suppose that S is a stationary subset of $\mathcal{P}_\kappa\lambda$. Suppose also that $f : S \to \lambda$ is such that $f(x) \in x$ for all $x \in S$. Then there is a stationary subset $T \subseteq S$ such that f is constant on T.*

Proof. Suppose $f : S \to \lambda$ is a counterexample. For each $\alpha < \lambda$ choose C_α club in $\mathcal{P}_\kappa\lambda$ with $(f^{-1}(\alpha)) \cap C_\alpha = \emptyset$. Now let D be the diagonal intersection of the C_α, $D = \Delta\langle C_\alpha : \alpha < \lambda\rangle$ and take $y \in S \cap D$, guaranteed to exist because D is club. Then $f(y) \in y$ so since $y \in D$ we have $y \in C_{f(y)}$. Hence, $y \in f^{-1}(f(y)) \cap C_{f(y)}$, contradicting the choice of $C_{f(y)}$.

$\dashv$.

Theorem 3.18 *Suppose κ is Mahlo and $\lambda \geq \kappa$. Then if $\Box_{\mathcal{P}_\kappa\lambda}(S,f)$ holds then there is a stationary set $T \subseteq S$ such that T does not reflect in $\mathcal{P}_{|x|}(x)$ for any $x \in S$.*

Proof. Let $\{C_x : x \in S\}$ witness $\Box_{\mathcal{P}_\kappa\lambda}(S,f)$. Note that since $f(x) \in x$, by the preceding lemma it follows that there is a stationary set $T \subseteq S$ such that $f(x)$ is constant on T. Now suppose T reflects in $\mathcal{P}_{|x|}(x)$ for some $x \in S$. Let $y \in T \cap C_x$. The set $\{u \in \mathcal{P}_{|x|}(x) : y \subseteq u$ and $|y| < |u|\}$ is club in $\mathcal{P}_{|x|}(x)$ so we can find $z \in T \cap C_x$ such that $y \in \mathcal{P}_{|z|}(z)$. By the definition of $\Box_{\mathcal{P}_\kappa\lambda}(S,f)$, we have that $C_z = C_x \cap \mathcal{P}_{|z|}(z)$ so $y \in C_z$. But then $f(y) \neq f(z)$, contradicting the definition of T. Thus T cannot reflect in $\mathcal{P}_{|x|}(x)$.

$\dashv$.

It should be noted that for some κ, for example the first Mahlo cardinal, the conclusion of this theorem holds in ZFC. (Simply let $S = T = reg(\mathcal{P}_\kappa\lambda)$.) The theorem becomes more relevant for cardinals higher in the Mahlo hierarchy (i.e. those that are $\alpha - Mahlo$ for $\alpha > 0$).

As with $\Box_{\mathcal{P}_\kappa\lambda}(S)$ we may use a Laver preparation to prove that $\Box_{\mathcal{P}_\kappa\lambda}(S,f)$ is consistent even for supercompact κ. Thus, supercompactness of κ does not prevent this principle or the corresponding non-reflection theorem.

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