数理解析研究所講究録 1424 巻 2005 年 156-159

ASYMPTOTICS FOR EXTENDED CELLULAR AUTOMATA

N. JOSHI

ABSTRACT. When restricted to the integers, ultra-discrete versions of difference equations can be considered as cellular automata. In this note, we describe examples that show that asymptotic analysis of such equations are possible. We pose questions about exponential asymptotics of such equations.

1. INTRODUCTION

Difference or discrete versions of the classical Painlevé equations have stimulated a great deal of recent interest [6, 1, 2, 3, 9]. These in turn lead to so-called ultra-discrete equations, through a method that arose from min-calculus [8, 7], which are of interest because they can be interpreted as cellular automata.

In [5], we showed for the first time that ultra-discrete Painlevé equations inherit Lax pairs from the original discrete Painlevé equation. We also displayed some typical behaviours of the solutions in the limit when the independent variable approaches infinity. However, no systematic study of asymptotic properties of ultra-discrete equations appears to exist in the literature. Questions remain open on the occurrence of exponential asymptotics in ultra-discrete equations. The purpose of this paper is to stimulate this study by considering a simple example. In Section 2, we consider asymptotics of first-order examples whose solutions are related to the Airy functions.

2. A FIRST-ORDER EXAMPLE

2.1. Riccati Equation. Consider

$$y'(t) = t - y^2 \tag{2.1}$$

which is linearsiable through $y(t) = \psi'(t)/\psi(t)$. The resulting equation for $\psi(t)$ is

$$\psi''(t) = t\psi(t) \;\Rightarrow\; \psi(t) = r_0 Ai(t) + s_0 Bi(t)$$

where r_0, s_0 are constants. Therefore, we obtain the general solution $y(t)$ of the Riccati equation (2.1) in terms of Airy functions:

$$y(t) = \frac{r_0 Ai'(t) + s_0 Bi'(t)}{r_0 Ai(t) + s_0 Bi(t)}. \tag{2.2}$$

Since the asymptotic behaviours of Airy functions as $t \to +\infty$ are well known

$$Ai(t) \sim e^{-2t^{3/2}/3}/t^{1/4}, \qquad Bi(t) \sim e^{+2t^{3/2}/3}/t^{1/4}, \qquad t \to +\infty,$$

we can deduce corresponding behaviours of $y(t)$ in a straightforward fashion. In general, the behaviour of $y(t)$ is algebraic in t. If $s_0 \neq 0$, then the asymptotic

2000 *Mathematics Subject Classification.* Primary 39A13, 33E17; Secondary 37B15.
Key words and phrases. ultra-discrete Painlevé equations, Asymptotics.
Research supported by Australian Research Council Discovery Grant #DP0345505.

series expansion of $y(t)$ contains terms that are hidden beyond all orders due to the presence of the exponentially small behaviours that arise from the function $Ai(t)$ in the ratio (2.2).

2.2. Discrete and Ultra-Discrete Riccati Equations. An integrable q-discrete version (see [4]) of Equation (2.1) is

$$\overline{x} := x(n+1) = \frac{\alpha q^n + x(n)}{1 + x(n)}. \tag{2.3}$$

This is also linearisable. Transforming variables by $x(n) = F(n)/G(n)$, we obtain the linearised equations

$$\overline{\overline{F}} = (q+1)\overline{F} + q(\alpha q^n - 1)F, \quad G = q^{-n}(\overline{F} - F)/\alpha. \tag{2.4}$$

Note that to ensure the equation is well-posed, we assume $|q| \neq 1$ or 0. Without loss of generality, we assume below that q is real and that $q > 1$.

For each variable (or parameter) v in a discrete (or difference) equation, the ultra-discretisation process introduces a new variable V defined by $v = e^{\frac{V}{\epsilon}}$. (Note that this places constraints on the variables involved.) Then we take the limit $\epsilon \to 0^+$ of the equation using the identity

$$\lim_{\epsilon \to 0^+} \epsilon \log \left(\sum_{i=1}^{n} e^{\frac{A_i}{\epsilon}} \right) = \max(A_i; i = 1 \ldots n, 0). \tag{2.5}$$

Writing $x = e^{X/\epsilon}$, $\alpha = e^{A/\epsilon}$, $q = e^{Q/\epsilon}$, this ultra-discretisation process applied to Equation (2.3) yields

$$\overline{X} = \max(A + Q\,n, X) - \max(X, 0) \tag{2.6}$$

Note that by our assumption on q, we have $Q > 0$.

2.3. Asymptotic Analysis. The asymptotic analysis of the discrete equation (2.3) as $n \to \infty$ can be deduced by investigating balances (i.e., limiting forms of the equation in which the largest terms remain). The only consistent balance yields

$$x(n) \sim \alpha^{1/2} q^{-1/4} q^{n/2}, \text{ as } n \to \infty.$$

A similar asymptotic analysis of Equation (2.6) yields

$$X(n) \sim \frac{Q}{2} n, \text{ as } n \to \infty.$$

However, these leading order results do not admit possible terms that are exponentially small in the asymptotic expansion of X. To show this, we take

$$X(n) = \frac{Q}{2} n + \xi(n), \qquad \xi(n) \ll \frac{Q}{2} n, \quad n \to \infty.$$

Substitution into Equation (2.6) shows that

$$\overline{\xi} = \max(A + Q\,n, Q\,n/2 + \xi) - Q\,n/2 - Q/2 - \max(Q\,n/2 + \xi, 0).$$

If ξ is bounded as $n \to \infty$, this implies

$$\overline{\xi} + \xi = A - \frac{Q}{2},$$

where we have used the fact that $Q > 0$ to eliminate the maxima in the equation. This has the explicit solution

$$\xi(n) = \xi_0(-1)^n + \frac{A}{2} - \frac{Q}{4},$$

where ξ_0 is an arbitrary number. If ξ is not bounded, but is such that $\xi \gg Q\,n/2$, then we get a contradiction. Therefore, we have captured the asymptotic behaviour of the general solution and there appears to be no hidden behaviours.

Asymptotic analysis through the linearized equation gives the same result. Writing the ultra-discrete representation of $F = e^{\Phi/\epsilon}$, the first of the equations (2.4) yields

$$\overline{\overline{\Phi}} = \max(C + \overline{\Phi}, D(n) + \Phi) \tag{2.7}$$

where $q + 1 = e^{C/\epsilon}$ and $q(\alpha q^n - 1) = e^{D(n)/\epsilon}$. When $n \gg 1$, we have $D(n) \sim (n+1)Q + A$. In other words, the leading asymptotic behaviour is given by

$$\Phi \sim \frac{Q}{4} n(n-1), \qquad \text{as } n \to \infty.$$

We get the above leading asymptotic behaviour for X after using this in the second of the linearized equations (2.4).

In other words, there appears to be no exponentially small asymptotic term in the behaviour of the ultra-discrete Riccati equation.

2.4. **Concluding Remarks.** Despite the fact that equations such as (2.6) can be regarded as cellular automata, asymptotic analysis is possible because the automata are embedded in more general equations in which analysis is possible in limits. We considered a simple example to obtain an explicit comparison with well-known special functions that exhibit exponential asymptotics. However, as we have shown, the ultra-discrete equation (2.6) does not appear to allow asymptotic expansions with exponentially small terms hidden beyond all orders. Such terms appear to be lost in the limiting procedure (see (2.5)) that leads from an ODE to an ultra-discrete equation. This procedure is not invertible. If we consider the asymptotic expansion of a function as a formal sum, and apply this procedure, the exponentially small terms do not appear to remain after the limit $\epsilon \to 0+$. Nevertheless, it is intriguing to ask whether there is an interpretation of such terms which will lead to the appearance of exponential asymptotics in the ultra-discrete setting.

References

[1] B. Grammaticos, A. Ramani, and V. Papageorgiou. Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.*, 67(14):1825–1828, 1991.

[2] Basile Grammaticos, Frank W. Nijhoff, and Alfred Ramani. Discrete Painlevé equations. In *The Painlevé property*, CRM Ser. Math. Phys., pages 413–516. Springer, New York, 1999.

[3] M. Jimbo and H. Sakai. A q-analogue of the sixth Painlevé equation *Letts Math. Phys*, 38(2):145–154, 1996.

[4] N. Joshi. Singularity analysis and integrability for discrete dynamical systems. *J. Math. Anal. Appl.*, 184(3):573–584, 1994.

[5] N. Joshi and F. Nijhoff and C. Ormerod. Lax pairs for ultra-discrete Painlevé cellular automata *J. Phys A: Math. Gen.*, 2004 (to appear).

[6] A. Ramani, B. Grammaticos, and J. Hietarinta. Discrete versions of the Painlevé equations. *Phys. Rev. Lett.*, 67(14):1829–1832, 1991.

[7] D. Takahashi, T. Tokihiro, B. Grammaticos, Y. Ohta, and A. Ramani. Constructing solutions to the ultradiscrete Painlevé equations. *J. Phys. A*, 30(22):7953–7966, 1997.

[8] J. Matsukidaira J. Satsuma T.Tokihiro, D. Takahashi. From soliton equations to integrable cellular automata through a limiting procedure. *Phys. Rev. Lett.*, 76:3247, 1996.

[9] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé equations *Comm. Math. Phys.*, 220(1):165–229, 2001.

School of Mathematics and Statistics F07, University of Sydney, NSW2006 Sydney, Australia

E-mail address: nalini@maths.usyd.edu.au