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ON SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH THE QUASI-PAINLEVÉ PROPERTY II

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1. Introduction

We say that a nonlinear differential equation

$$y'' = R(x, y, y'), \qquad R(x, y, z) \in \mathbf{C}(x, y, z)$$

has the quasi-Painlevé property if, for each solution, every movable singular point is an algebraic branch point. In the previous paper [3], we have shown that

$$y'' = \frac{10}{9}y^4 + x$$

admits the quasi-Painlevé property. In this paper, we treat a more general nonlinear equation of the form

$$\text{(E)} \qquad y'' = \frac{10}{9}y^4 + P(x),$$

where $P(x)$ is a polynomial with complex coefficients satisfying $\deg P \geq 1$.

Our main results are stated as follows:

Theorem 1. *Let $y(x)$ be an arbitrary solution of* (E). *Suppose that $x = x_0$ is an algebraic branch point of $y(x)$. Then, around it,*

$$\text{(1)} \qquad y(x) = \xi^{-2/3} - \frac{9}{22}P(x_0)\xi^2 + c\xi^{8/3} + \frac{9P'(x_0)}{14}\xi^3 + \sum_{j\geq 10} c_j \xi^{j/3},$$

$\xi = x - x_0$, where c is an integration constant, c_j ($j \geq 10$) are uniquely determined polynomials of (x_0, c), and $\xi^{1/3}$ is an arbitrary branch of ϕ such that $\phi^3 = \xi$.

Theorem 2. *If $\deg P \notin 4\mathbf{N}$, then* (E) *admits no meromorphic solutions.*

Theorem 3. *If* $\deg P \in 4\mathbf{N}$, *then* (E) *admits no meromorphic solutions except at most four polynomial solutions.*

For an arbitrary polynomial $p(x)$, equation (E) with $P(x) = p''(x) - 10p(x)^4/9$ admits the polynomial solution $y = p(x)$. In particular, if $P(x) = -10(\alpha x+\beta)^4/9$, then there exist four solutions $\pm(\alpha x + \beta)$, $\pm i(\alpha x + \beta)$.

Theorem 4. *Equation* (E) *has the quasi-Painlevé property. For each solution, around every movable singularity* $x = x_0$, *it is expressible by a Puiseux series expansion of the form* (1).

2. Lemmas

We review some lemmas which will be used in the proofs of our results.

Suppose that $F(x, u, v)$ and $G(x, u, v)$ are analytic functions in a neighbourhood of $(a_0, b_0, c_0) \in \mathbf{C}^3$. For a system of differential equations

$$u' = F(x, u, v), \quad v' = G(x, u, v), \tag{2}$$

we have the following lemma due to Painlevé.

Lemma 5. *Let* Γ $(\subset \mathbf{C})$ *be a curve with finite length terminating in* $x = a_0$. *Suppose that a solution* $(u, v) = (\varphi(x), \psi(x))$ *of* (2) *has the properties below:*

(i) *for every point* $\xi \in \Gamma \setminus \{a_0\}$, $\varphi(x)$ *and* $\psi(x)$ *are analytic at* $x = \xi$;

(ii) *there exists a sequence* $\{a_n\} \subset \Gamma \setminus \{a_0\}$, $a_n \to a_0$ $(n \to \infty)$ *such that* $(\varphi(a_n), \psi(a_n)) \to (b_0, c_0) \in \mathbf{C}^2$.

Then, $\varphi(x)$ *and* $\psi(x)$ *are analytic at* $x = a_0$.

Let $f(z)$ be a meromorphic function in the whole complex plane. For $r > 0$, consider the functions defined by

$$\begin{aligned} m(r, f) &:= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})| d\phi, \\ N(r, f) &:= \int_0^r (n(t, f) - n(0, f))\frac{dt}{t} + n(0, f)\log r, \\ T(r, f) &:= m(r, f) + N(r, f), \end{aligned}$$

which are called, respectively, the proximity function, the counting function and the characteristic function ([1]). Here $\log^+ s := \max\{\log s, 0\}$ $(s > 0)$, and $n(r, f)$ denotes the number of poles in the disc $|z| \leq r$, each counted according to its multiplicity. Clearly, if $f(x)$ is entire, then $T(r, f) = m(r, f)$. It is known that $T(r, f)$ is monotone increasing with respect to r, and is a convex function of $\log r$. Furthermore, $f(z)$ is a rational function, if and only if $T(r, f) = O(\log r)$ as $r \to \infty$; and $f(z)$ is a transcendental function, if and only if $\log r/T(r, f) = o(1)$ as $r \to \infty$. The following lemma is concerning the proximity function of a meromorphic solution of a differential equation ([1, Lemma 2.4.2]).

Lemma 6. *Suppose that the differential equation $w^{p+1} = \Phi(z,w)$, $p \in \mathbf{N}$ admits a meromorphic solution $w = f(z)$, where $\Phi(z,w)$ is a polynomial of $(z, w, w', ..., w^{(q)})$. If the total degree of $\Phi(z,w)$ with respect to w and its derivatives does not exceed p, then*

$$m(r,f) = O(\log T(r,f) + \log r)$$

as $r \to \infty$, $r \notin E$, where E is an exceptional interval with finite length.

3. Proof of Theorem 1

For an algebraic branch point x_0 of $y(x)$, we have $|y(x_0)| = +\infty$. Indeed, if $|y(x_0)| < +\infty$, then, for some $x_1 \in \mathbf{C}$,

$$|y'(x)| \leq |y'(x_1)| + \left| \int_{x_1}^{x} \Big(\frac{10}{9} y(t)^4 + P(t)\Big) dt \right| < +\infty$$

along a curve tending to x_0. By Lemma 5, $y(x)$ is analytic at $x = x_0$, which is a contradiction. Putting

$$y(x) = c_0 \xi^{\alpha}(1 + o(1)), \quad \xi = x - x_0, \quad \alpha \in \mathbf{Q}, \quad \alpha < 0, \quad c_0 \neq 0,$$

and substituting into (E), we have $(10/9)c_0^3 \xi^{3\alpha} = \alpha(\alpha - 1)\xi^{-2}$. Hence

$$y(x) = \xi^{-2/3} + \sum_{j=l}^{\infty} c_j \xi^{j/(3p)}, \qquad p \in \mathbf{N}, \quad l \in \mathbf{Z}, \quad l \geq -2p + 1,$$

where $\xi^{1/3}$ is an arbitrary branch of ϕ such that $\phi^3 = \xi$. Substituting this into (E) again, we have

$$\begin{aligned} &\frac{10}{9}\xi^{-8/3} + \frac{l(l-3p)}{9p^2} c_l \xi^{l/(3p)-2} + \cdots \\ =&\frac{10}{9}\big(\xi^{-8/3} + 4c_l \xi^{l/(3p)-2} + \cdots + P(x_0) + P'(x_0)\xi + \cdots\big). \end{aligned}$$

From this it follows that $l = 6p$ and that $c_{6p} = -9P(x_0)/22$. In general, the coefficients c_j are determined by

$$\text{(3)} \qquad \left(\frac{j}{p} + 5\right)\left(\frac{j}{p} - 8\right) c_j = Q_j(x_0, c_k; k \leq j - 1), \quad c_{-2p} = 1,$$

where Q_j are polynomials of x_0 and c_k. Suppose that $c_j \neq 0$ for some j satisfying $j/p \notin \mathbf{Z}$, and let j_0 be the minimal one among such numbers. Then $Q_{j_0} = 0$, and hence $c_{j_0} = 0$, which is a contradiction. This implies $p = 1$. Using (3), we have

$$c_6 = -\frac{9}{22}P(x_0), \quad c_7 = 0, \quad c_8 = c, \quad c_9 = \frac{9}{14}P'(x_0),$$

where c is an arbitrary constant. Thus we obtain the desired series expansion.

4. Proofs of Theorems 2 and 3

Suppose that $\deg P \notin 4\mathbf{N}$, and that (E) admits a meromorphic solution $Y(x)$. Then $Y(x)$ is entire, because $Y(x)$ has no poles. If $Y(x)$ is a polynomial, then $Y(x) = C_0 x^m + O(x^{m-1})$, $m \in \mathbf{N} \cup \{0\}$, $C_0 \neq 0$ around $x = \infty$. Substitution of this into (E) yields that $\deg P = 4m$, which is a contradiction. Hence $Y(x)$ is transcendental and entire. Applying Lemma 6 to the equality $10Y(x)^4/9 = Y''(x) - P(x)$, we have $T(r, Y) = m(r, Y) = O(\log T(r, Y) + \log r)$ as $r \to \infty$, $r \notin E_0$, $\mu(E_0) < \infty$. Hence $T(r, Y) \leq K_0 \log r$ for $r \notin E_0$, where K_0 is some positive number. For each $r > 0$, there exists a number $r'(r) \geq r$ such that $r'(r) - r \leq 2\mu(E_0)$ and that $r'(r) \notin E_0$. Since $T(r, Y)$ is monotone increasing,

$$T(r, Y) \leq T(r'(r), Y) \leq K_0 \log r'(r) \leq K_0 \log(r + 2\mu(E_0)) = O(\log r),$$

which contradicts the transcendency of $Y(x)$. Thus Theorem 2 has been proved. Next suppose that $\deg P = 4m$, $m \in \mathbf{N}$, namely $P(x) = P_0 x^{4m} + \cdots + P_{4m}$, and that $Y_0(x) = C_0 x^d + C_1 x^{d-1} + \cdots + C_0$ is a polynomial solution of (E). Then $4d = 4m$ and $10C_0^4/9 + P_0 = 0$. Hence the number of polynomial solutions does not exceed four. Thus we obtain Theorem 3.

5. Proof of Theorem 4

Let $y(x)$ be an arbitrary solution of (E).

5.1. Equivalent system of equations. Suppose that $x = x_0$ is an algebraic branch point of $y(x)$. Since $P(x_0) = P(x) - P'(x_0)\xi + O(\xi^2)$, $\xi = x - x_0$, series (1) is written in the form

$$y(x) = \xi^{-2/3}\left(1 - \frac{9}{22}P(x)\xi^{8/3} + c\xi^{10/3} + \frac{81}{77}P'(x)\xi^{11/3} + \cdots\right).$$

Putting $y = u^{-2}$, we have

$$\xi^{1/3} = \pm u\left(1 - \frac{9}{44}P(x)u^8 + \frac{c}{2}u^{10} \pm \frac{81}{154}P'(x)u^{11} + \cdots\right),$$

and

$$\begin{aligned} y'(x) &= -\frac{2}{3}\xi^{-5/3} - \frac{9}{11}P(x)\xi + \frac{8}{3}c\xi^{5/3} + \frac{9 \cdot 47}{154}P'(x)\xi^2 + \cdots \\ &= \pm\, u^{-5}\left(-\frac{2}{3} - \frac{3}{2}P(x)u^8 + \frac{13}{3}cu^{10} + \cdots\right) + \frac{9}{2}P'(x)u^6. \end{aligned}$$

Observing these facts, we define new unknown variables u, v by

$$\begin{aligned} y &= u^{-2}, \\ y' &= \mp\frac{2}{3}u^{-5}\left(1 + \frac{9}{4}P(x)u^8 + u^{10}v\right) + \frac{9}{2}P'(x)u^6. \end{aligned} \tag{4}$$

Then, we have a system of equations with respect to u, v. Now regarding x, v as unknown functions of u, we get

$$(5)\qquad \frac{dx}{du} = \pm 3u^2 U(x,u,v)^{-1}, \quad \frac{dv}{du} = 3u^3 V(x,u,v)U(x,u,v)^{-1},$$

with

$$\begin{aligned} U(x,u,v) =& 1 + \frac{9}{4}P(x)u^8 + u^{10}v \mp \frac{27}{4}P'(x)u^{11}, \\ V(x,u,v) =& -\frac{5}{3}u^6v^2 + \frac{3}{4}u^4(-8P(x) \pm 33P'(x)u^3)v \\ &- \frac{81}{16}u^2(P(x) \mp 3P'(x)u^3)(P(x) \mp 6P'(x)u^3) + \frac{27}{4}P''(x). \end{aligned}$$

Note that (5) is equivalent to (E), and that $(x(u), v(u))$ is a solution of (5) analytic at $u = 0$ and satisfying $x(0) = x_0$, $v(0) = -13c/2$.

5.2. Lyapunov function. The second equation of (4) is written in the form

$$\left(y' - \frac{9}{2}P'(x)y^{-3}\right)^2 = \frac{4}{9}u^{-10}\left(1 + \frac{9}{4}P(x)y^{-4} + y^{-5}v\right)^2,$$

which implies that

$$(6)\qquad V = (y')^2 - 9P'(x)y^{-3}y' - \frac{4}{9}y^5 - 2P(x)y$$

satisfies

$$(7)\qquad V = -\frac{81}{4}P'(x)^2y^{-6} + \frac{9}{4}P(x)^2y^{-3} + \left(\frac{8}{9} + 2P(x)y^{-4}\right)v + \frac{4}{9}y^{-5}v^2.$$

Furthermore, $V(x)$ satisfies the first order equation

$$V' - 27P'(x)y^{-4}V = 243P'(x)^2y^{-7}y' - 9P''(x)y^{-3}y' + 45P(x)P'(x)y^{-3}.$$

Using this equation, we have

Lemma 7. *If $y(x)^{-1}$ is bounded along a path Γ with finite length, then $V(x)$ is also bounded along Γ.*

5.3. Derivation of Theorem 4. Suppose that $x = a$ is a singular point of $y(x)$. Let Γ be a segment terminating in a such that each $\xi \in \Gamma \setminus \{a\}$ is at most an algebraic branch point of $y(x)$. Modifying Γ if necessary, we may suppose that Γ is a curve terminating in a, and that $y(x)$ is analytic along $\Gamma \setminus \{a\}$. Put

$$A = \liminf_{x \to a, x \in \Gamma} |y(x)|.$$

We divide into three cases (i) $0 < A < +\infty$, (ii) $A = +\infty$, (iii) $A = 0$.

Case (i) $0 < A < +\infty$: By Lemma 7, $V(x)$ is bounded on Γ near a. Then, by (6), there exists a sequence $\{a_n\} \subset \Gamma$ such that $y(a_n) \to y_0$ ($\neq 0, \infty$), $y'(a_n) \to y_1$ ($\neq \infty$), $a_n \to a$. This fact together with Lemma 5 implies that $y(x)$ is analytic at $x = a$.

Case (ii) $A = +\infty$: By supposition, $y(x) \to \infty$ as $x \to a$ along Γ, and $V(x)$ is bounded along Γ near a. Then, regarding (7) as a quadratic equation with respect to v, we can choose a branch $v_-(x)$ of $v(x)$ which is bounded along Γ. Let $u_-(x)$ be the corresponding branch of $u(x)$ such that $u_-(x)^{-2} = y(x)$ (cf. (4)). Denote by $x = x(u)$ the inverse function of $u = u_-(x)$. Then, $x = x(u)$ and $v = v_-(x(u))$ are analytic functions of u along $\Gamma^* = u_-(\Gamma \setminus \{a\})$ satisfying

(a) $x(u) \to a$ as $u \to u_-(a) = 0$ along Γ^*;

(b) $v_-(x(u))$ is bounded along Γ^*.

Take a sequence $\{b_n\} \subset \Gamma^*$ satisfying $b_n \to u_-(a) = 0$, $x(b_n) \to a$, $v_-(x(b_n)) \to v_0 \neq \infty$. Observe that $(x(u), v_-(x(u)))$ is a solution of (5). By Lemma 5, $x(u)$ is analytic at $u = 0$, implying that $x = a$ is at most an algebraic branch point of $y(x) = u(x)^{-2}$.

Case (iii) $A = 0$: For $y(x)$, we note the following lemma, which is obtained from [2, Lemma 2.2] with $R_0 = \Delta = 1/2$, $K = 1 + |a|$.

Lemma 8. *Set* $\theta_0 = (1 + |a|)^{-1}/42$. *Let* c *be a point such that* $|c - a| < 1/4$, *and suppose that* $y(x)$ *is analytic at* $x = c$. *If the inequalities* $|y(c)| \leq \theta_0/6$, $|y'(c)| \geq 2$ *hold, then* $y(x)$ *is analytic for* $|y'(c)||x - c| < \theta_0$ *and satisfies* $|y(x)| \geq \theta_0/4$ *on the circle* $|y'(c)||x - c| = \theta_0/2$.

Let us consider the set $\Gamma_0 = \{x \in \Gamma \mid |y(x)| \leq \theta_0/6\}$. By the supposition $A = 0$, we have $\Gamma_0 \cap \{x \mid |x - a| < \varepsilon\} \neq \emptyset$ for every $\varepsilon > 0$. We may suppose that $|y'(x)| \geq 2$ for $x \in \Gamma_0$. Indeed, if this is not the case, then there exists a sequence $\{a_n\} \subset \Gamma_0$, $a_n \to a$ such that $y(a_n)$ and $y'(a_n)$ are bounded, and hence $y(x)$ is analytic at $x = a$. Now we proceed along Γ toward $x = a$. Suppose that we meet the first point $c_1 \in \Gamma_0$. By Lemma 8, there exists a disc $D_1 : |x - c_1| \leq |y'(c_1)|^{-1}\theta_0/2$ such that $|y(x)| \geq \theta_0/4$ on the boundary ∂D_1. Note that $a \notin D_1$. Restarting from a point in $\Gamma \cap \partial D_1$, we proceed along Γ until we meet the next point $c_2 \in \Gamma_0$. Take the disc $D_2 : |x - c_2| \leq |y'(c_2)|^{-1}\theta_0/2$, and repeat the procedure above. In this way, we get a sequence of discs $\{D_j\}$ such that $|y(x)| \geq \theta_0/4$ on the boundary ∂D_j. Then, $|y(x)| \geq \theta_0/6$ on the boundary of the set $\Gamma \cup \left(\bigcup_{j=1}^{\infty} D_j\right)$, which contains a curve γ with the properties: (i) γ terminates in a; (ii) $|y(x)| \geq \theta_0/6$; (iii) $y(x)$ is analytic along $\gamma \setminus \{a\}$. Hence this case is reduced to either (i) or (ii), which completes the proof of Theorem 4.

6. A remark

As was shown in [3], the equation

$$y'' = \frac{10}{9}y^4 + x$$

has the quasi-Painlevé property, and, the quadratic version of this is the first Painlevé equation

$$\text{(I)} \qquad y'' = 6y^2 + x.$$

For (E), the corresponding version is

$$\text{(8)} \qquad y'' = 6y^2 + P(x).$$

In general, this equation does not always admit the quasi-Painlevé property. In fact, equation (8) with $P(x) = x^2$ possesses the solution of the form

$$y = \xi^{-2} - \frac{x_0^2}{10}\xi^2 - \frac{x_0}{3}\xi^3 + \left(c + \frac{1}{7}\log\xi\right)\xi^4 + \cdots, \qquad \xi = x - x_0,$$

which means that $x = x_0$ is a logarithmic branch point.

References

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