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# A long wave approximation for capillary-gravity waves and an effect of the bottom

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## 1 Introduction

We are concerned with two-dimensional, irrotational flow of incompressible ideal fluid with free surface under the gravitational field. The domain occupied by the fluid is bounded below by a solid bottom and above by an atmosphere of constant pressure. The upper surface is free boundary and we take the influence of surface tension into account on the free surface. Our main interest is the motion of the free surface, which is called capillary-gravity wave. In the case without surface tension, it is called gravity wave or water wave.

Mathematically, the problem is formulated as a free boundary problem for incompressible Euler equation with the irrotational condition. After rewriting the equations in an appropriate non-dimensional form, we have two non-dimensional parameters $\delta$ and $\varepsilon$ the ratio of the water depth $h$ to the wave length $\lambda$ and the ratio of the amplitude of the wave $a$ to the water depth $h$, respectively. In this communication, we consider capillary-gravity waves characterized by the physical condition $\delta^2 = \varepsilon \ll 1$. In this long wave regime, Korteweg and de Vries [10] derived a very notable equation, which is nowadays called the KdV equation, from the equations for water waves. Here, we note that even in the formal level the bottom of the fluid is assumed to be flat in the derivation of the KdV equation.

Until now, there are several efforts to give a mathematically rigorous justification for the KdV equation as an approximate one to the full equations for water waves over a flat bottom. Kano and Nishida [9] gave the justification in a class of analytic functions. In order to guarantee the existence of solution for the full equation, they used an abstract Cauchy-Kowalevski theorem in a scaled Banach space, which is a modified version of those due to Ovsjannikov [14, 15] and Nirenberg [12], so that analyticity of the initial data is required. Based on the existence theorem due to Nalimov [11] and Yosihara [28], Craig [4] gave the justification in the framework of Sobolev spaces. In the long wave regime, the dynamics of the free surface is approximately translation of two wave packets without change of the shape, one moving to the right and the others to the left, for a short time interval $0 \le t \le O(1)$. The dynamics of each wave packets is very slow so that it is invisible for the short time interval. By introducing a fast time scale $\tau = \varepsilon t$, the dynamics can be visible and described by the KdV equation for a long time interval $0 \le t \le O(1/\varepsilon)$. One of the difficulties in the justification is to obtain a uniform estimate of the solution of the initial value problem for full water waves with respect to $\varepsilon$ for the long time interval. Craig established well the estimate under the restriction that the wave is almost one-directional. Recently, Schneider and Wayne tried to refine the Craig's result in [19] and to extend it to the capillary-gravity waves in [20]. However, their formulation of the problem is different from Craig's and ours and they treated the problem as regular perturbation, so that their results are weak.

Our main purpose is to analyze the long wave approximation of the capillary-gravity waves in the case that the bottom is not flat and to derive simple equations whose solution approximates that of original equations for a long time interval $0 \le t \le O(1/\varepsilon)$. If the amplitude of the bottom is comparable to that of the free surface, then the effect of the bottom can not be negligible and the approximate equations become coupled KdV like equations. Another purpose is to give a refined version of the claim in Schneider and Wayne [20]. Since the well-posedness of the initial value problem for fixed $\varepsilon > 0$ was established by Yosihara [29] and Iguchi [5], our main task is to obtain a priori estimate for the long time interval. To this end we follows basically the strategy due to Craig [4]. However, as explained in Iguchi, Tani, and Tanaka [6], we do not have to use the Lagrangian coordinates for the analysis to capillary-gravity waves, so that we will study the problem in the Eulerian coordinates. Owing to this choice of coordinates, an expression of the operator $K$, which is the Dirichlet-to-Dirichlet map for the Cauchy-Riemann equations, becomes simpler than Craig's.

**Notation.** For $s \in \mathbf{R}$, we denote by $H^s$ the Sobolev space of order $s$ on $\mathbf{R}$ equipped with the inner product $(u,v)_s = \frac{1}{2\pi}\int_{\mathbf{R}}(1+\xi^2)^s\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$, where $\hat{u}$ is the Fourier transform of $u$, that is, $\hat{u}(\xi) = \int_{\mathbf{R}} u(x)\mathrm{e}^{-ix\xi}dx$. We put $\|u\|_s = \sqrt{(u,u)_s}$, $(u,v) = (u,v)_0$, and $\|u\| = \|u\|_0$. For a non-negative integer $m$ and a real $\gamma$, we denote by $H^{m,\gamma}$ the weighted Sobolev space on $\mathbf{R}$ equipped with the norm $\|u\|_{m,\gamma} = \left(\sum_{l=0}^m \|\langle x\rangle^\gamma \left(\frac{d}{dx}\right)^l u\|\right)^{1/2}$, where $\langle x\rangle = (1+x^2)^{1/2}$. For $1 \le p \le \infty$, we denote by $|\cdot|_p$ the norm of the Lebesgue space $L^p = L^p(\mathbf{R})$. For a non-negative integer $m$, we denote by $W^{m,\infty}$ the Banach space of all functions $u = u(x)$ on $\mathbf{R}$ such that $\left(\frac{d}{dx}\right)^l u \in L^\infty$ for $0 \le l \le m$ with the norm $\|u\|_{W^{m,\infty}} = \max_{0\le l\le m}|\left(\frac{d}{dx}\right)^l u|_\infty$. For $0 < T < \infty$, a non-negative integer $j$, and a Banach space $X$, we denote by $C^j([0,T];X)$ the Banach space of all functions of $C^j$-class on the interval $[0,T]$ with the value in $X$. A pseudo-differential operator $P(D)$, $D=-i\frac{d}{dx}$, with a symbol $P(\xi)$ is defined by $P(D)u(x)=\frac{1}{2\pi}\int_{\mathbf{R}}P(\xi)\hat{u}(\xi)e^{ix\xi}d\xi$.

## 2 Formulation of the problem

We assume that the domain $\Omega(t)$ occupied by the fluid at time $t\geq 0$, the free surface $\Gamma(t)$, and the bottom $\Sigma$ are of the forms

$$\begin{aligned}
&\Omega(t)=\{(x,y)\in\mathbf{R}^2\,;\ b(x)<y<h+\eta(x,t)\},\\
&\Gamma(t)=\{(x,y)\in\mathbf{R}^2\,;\ y=h+\eta(x,t)\},\\
&\Sigma=\{(x,y)\in\mathbf{R}^2\,;\ y=b(x)\},
\end{aligned}$$

where $h$ is the mean depth of the fluid. In this paper $b$ is a given function, while $\eta$ is the unknown. The motion of the fluid is described by the velocity $v=(v_1,v_2)$ and the pressure $p$ satisfying the equations

$$\begin{cases}\rho\big(v_t+(v\cdot\nabla)v\big)+\nabla p=-\rho(0,g),\\ \nabla\cdot v=0,\quad \nabla^{\perp}\cdot v=0 \qquad \text{in}\quad \Omega(t),\quad t>0,\end{cases}\tag{1}$$

where $\rho$ is the constant density and $g$ is the gravitational constant. It is assumed that both $\rho$ and $g$ are positive constants. The dynamical and kinematical boundary conditions on the free surface are given by

$$\begin{cases}p=p_0-\sigma H,\\ (\partial_t+v\cdot\nabla)\big(y-\eta(x,t)\big)=0 \qquad \text{on}\quad \Gamma(t),\quad t>0,\end{cases}\tag{2}$$

where $p_0$ is the atmospheric pressure, $\sigma$ is the surface tension coefficient, and $H$ is the curvature of the free surface. It is assumed that $p_0$ is a constant and $\sigma$ is a positive constant. In our parametrization of the free surface the curvature $H$ at the point $\big(x,h+\eta(x,t)\big)$ is expressed as

$$H(x,t)=\Big(\big(1+\big(\eta_x(x,t)\big)^2\big)^{-1/2}\eta_x(x,t)\Big)_x.$$

The boundary condition on the bottom is given by

$$v\cdot N=0 \qquad \text{on}\quad \Sigma,\quad t>0,\tag{3}$$

where $N$ is the unit normal vector to $\Sigma$. Finally, we impose the initial conditions

$$\eta(x,0)=\eta_0(x),\quad v(x,y,0)=v_0(x,y).\tag{4}$$

It is assumed that the initial data satisfy the compatibility conditions, that is,

$$\begin{cases} \nabla \cdot v_0 = 0, \quad \nabla^{\perp} \cdot v_0 = 0 \quad \text{in} \quad \Omega(0), \\ v_0 \cdot N = 0 \quad \text{on} \quad \Sigma. \end{cases}$$

We proceed to rewrite the equations (1)–(4) in an appropriate non-dimensional form. Let $\lambda$ be the typical wave length and $a$ the typical amplitude of the free surface. We introduce two non-dimensional parameters $\delta$ and $\varepsilon$ by

$$\delta = \frac{h}{\lambda} \quad \text{and} \quad \varepsilon = \frac{a}{h},$$

respectively. We will consider asymptotic behavior of capillary-gravity waves when $\delta$ and $\varepsilon$ tend to zero keeping the relation

$$\delta^2 = \varepsilon.$$

We rescale the independent and dependent variables by

$$(5) \quad \begin{cases} x = \lambda\tilde{x}, \quad y = h\tilde{y}, \quad t = \dfrac{\lambda}{\sqrt{gh}}\tilde{t}, \\ v_1 = \dfrac{a}{h}\sqrt{gh}\,\tilde{v}_1, \quad v_2 = \dfrac{a}{\lambda}\sqrt{gh}\,\tilde{v}_2, \quad p = p_0 + \rho g h \tilde{p}, \quad \eta = a\tilde{\eta}, \quad b = a\tilde{b}. \end{cases}$$

These new variables are called Boussinesq ones. Here, we note that the function $b$ of the bottom is rescaled by $a$ the typical amplitude of the free surface. Putting these into (1)–(4) and dropping the tilde sign in the notation we obtain

$$(6) \quad \begin{cases} \varepsilon v_{1t} + \varepsilon^2(v_1 v_{1x} + v_2 v_{1y}) + p_x = 0, \\ \varepsilon^2 v_{2t} + \varepsilon^3(v_1 v_{2x} + v_2 v_{2y}) + p_y + 1 = 0, \\ v_{1x} + v_{2y} = 0, \quad v_{1y} - \varepsilon v_{2x} = 0 \quad \text{in} \quad \Omega^{\varepsilon}(t), \quad t > 0, \end{cases}$$

$$(7) \quad \begin{cases} p = -\varepsilon^2\mu\big((1+\varepsilon^3\eta_x^2)^{-1/2}\eta_x\big)_x, \\ \eta_t + \varepsilon v_1 \eta_x - v_2 = 0 \quad \text{on} \quad \Gamma^{\varepsilon}(t), \quad t > 0, \end{cases}$$

$$(8) \quad \varepsilon b' v_1 - v_2 = 0 \quad \text{on} \quad \Sigma^{\varepsilon}, \quad t > 0,$$

$$(9) \quad \eta(x,0) = \eta_0(x), \quad v(x,y,0) = v_0(x,y),$$

where

$$\begin{aligned} \Omega^{\varepsilon}(t) &= \big\{(x,y) \in \mathbf{R}^2\,;\, \varepsilon b(x) < y < 1 + \varepsilon\eta(x,t)\big\}, \\ \Gamma^{\varepsilon}(t) &= \big\{(x,y) \in \mathbf{R}^2\,;\, y = 1 + \varepsilon\eta(x,t)\big\}, \\ \Sigma^{\varepsilon} &= \big\{(x,y) \in \mathbf{R}^2\,;\, y = \varepsilon b(x)\big\}, \end{aligned}$$

and $\mu$ is a non-dimensional parameter called the Bond number and defined by

$$\mu = \frac{\sigma}{\rho g h^2}.$$

The function $b$ and the initial data $\eta_0$ and $v_0$ may depend on $\varepsilon$.

According to [5], we reformulate the initial value problem (6)–(9) as a problem on the free surface. Put

$$u(x,t) = v(x, 1+\varepsilon\eta(x,t), t),$$

which is the boundary value of the velocity on the free surface. Then, we see that the unknowns $\eta$ and $u = (u_1, u_2)$ are governed by the equations

$$\begin{cases} u_{1t} + \eta_x + \varepsilon u_1 u_{1x} + \varepsilon^2 \eta_x (u_{2t} + \varepsilon u_1 u_{2x}) = \varepsilon\mu\big((1+\varepsilon^3\eta_x^2)^{-1/2}\eta_x\big)_{xx}, \\ \eta_t + \varepsilon u_1 \eta_x - u_2 = 0, \\ u_2 = K(\eta, b, \varepsilon) u_1 \qquad \text{for} \quad t > 0, \end{cases} \tag{10}$$

$$\eta = \eta_0, \quad u_1 = u_0 \qquad \text{at} \quad t = 0. \tag{11}$$

This is the initial value problem that we are going to investigate in this communication. The Dirichlet-to-Dirichlet map $K = K(\eta, b, \varepsilon)$ for the Cauchy-Riemann equations appearing in (10) can be written explicitly in terms of integral operators as

$$K = -\varepsilon^{-1/2}\Big(\frac{1}{2} - B_2\Big)^{-1} B_1,$$

where

$$\begin{cases} B_1 = A_2 + (\varepsilon^{3/2} A_5 b' - A_6)\Big(\dfrac{1}{2} + A_3 + \varepsilon^{3/2} A_4 b'\Big)^{-1} A_7, \\ B_2 = A_1 - (\varepsilon^{3/2} A_5 b' - A_6)\Big(\dfrac{1}{2} + A_3 + \varepsilon^{3/2} A_4 b'\Big)^{-1} A_8. \end{cases}$$

Here, $A_1, \ldots, A_8$ are integral operators, which map real valued functions to real valued ones, and satisfy the relations

$$\begin{cases} (A_1 + iA_2) f(x) = \dfrac{i}{2}(i \operatorname{sgn} D) f(x) + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{3/2}\frac{\eta(y,t) - \eta(x,t)}{y - x}\Big)\frac{df}{dy}(y)dy, \\ (A_3 + iA_4) f(x) = \dfrac{i}{2}(i \operatorname{sgn} D) f(x) + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{3/2}\frac{b(y) - b(x)}{y - x}\Big)\frac{df}{dy}(y)dy, \\ (A_5 + iA_6) f(x) = \dfrac{1}{2}\mathrm{e}^{-\varepsilon^{1/2}|D|}\big(-1 + i(i \operatorname{sgn} D)\big) f(x) \\ \qquad + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{3/2}\frac{b(y) - \eta(x,t)}{y - x - i\varepsilon^{1/2}}\Big)\frac{df}{dy}(y)dy, \\ (A_7 + iA_8) f(x) = \dfrac{1}{2}\mathrm{e}^{-\varepsilon^{1/2}|D|}\big(1 + i(i \operatorname{sgn} D)\big) f(x) \\ \qquad + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{3/2}\frac{\eta(y,t) - b(x)}{y - x + i\varepsilon^{1/2}}\Big)\frac{df}{dy}(y)dy. \end{cases}$$

By using this expression we can expand the operator $K$ in terms of $(\eta, b)$ as

$$K = \sum_{k=0}^{n-1} K_k + \widetilde{K}_n,$$

where the operator $K_k$ is homogeneous of degree $k$ in $(\eta, b)$. Particularly, we have

$$\text{(12)} \qquad \begin{cases} K_0 = -\varepsilon^{-1/2} i \tanh(\varepsilon^{1/2} D), \\ K_1 = -\varepsilon\big(\eta + i\tanh(\varepsilon^{1/2}D)\eta\, i\tanh(\varepsilon^{1/2}D)\big)(iD) \\ \qquad\quad + \varepsilon \operatorname{sech}(\varepsilon^{1/2}D)(iD) b \operatorname{sech}(\varepsilon^{1/2}D). \end{cases}$$

**Remark 1.** Under suitable assumptions on $\eta$ and $b$, for each positive $\varepsilon$ the operator $K_1$ possesses a smoothing property and we do not need the expression of $K_1$ when we fix $\varepsilon$. However, in order to get uniform estimates of the solution for the initial value problem (10) and (11) with respect to $\varepsilon$ the above explicit formula for $K_1$ plays an important role.

For the remainder term $\widetilde{K}_n$, we have the following lemma.

**Lemma 1.** *Let $m$, $m_0$, and $n$ be positive integers satisfying $m, m_0 \geq 2$ and $n + m \geq m_0$. Put $m_1 = \max\{m, m_0 - 1\}$ and $m_2 = \max\{m, m_0\} + 1$. There exists constants $C > 0$ and $\delta_1 > 0$ such that for any $\eta \in H^{m_1}$, $b \in W^{m_2,\infty}$, and $\varepsilon \in (0, 1]$ satisfying $\varepsilon(\|\eta\|_{m_1} + \|b\|_{W^{m_2,\infty}}) \leq \delta_1$ we have*

$$\|\widetilde{K}_n f\|_m \leq C\varepsilon^{-(m-m_0+1)/2}\big(\varepsilon(\|\eta\|_{m_1} + \|b\|_{W^{m_2,\infty}})\big)^n \|f\|_{m_0}.$$

**Remark 2.** This estimate says that $\widetilde{K}_n$ has a smoothing property, which is very important to the existence theory for the initial value problem (10) and (11). But, if we use the smoothing property, then we lose a power of $\varepsilon$ and we shall face a difficulty when we try to get uniform estimates with respect to $\varepsilon$. However, taking $n$ sufficiently large, we gain a power of $\varepsilon$. For our problem, it is sufficient to expand the operator $K$ up to $n = 2$.

**Remark 3.** By virtue of Taylor's formula we have $\tanh x = x - \frac{1}{3}x^3 + O(x^5)$ and $\operatorname{sech} x = 1 + O(x^2)$ so that (12) implies

$$K_0 = -(iD) - \frac{\varepsilon}{3}(iD)^3 + O(\varepsilon^2), \quad K_1 = -\varepsilon\eta(iD) + \varepsilon(iD)b + O(\varepsilon^2).$$

Since $\widetilde{K}_2 = O(\varepsilon^2)$, we obtain

$$K = -(1 + \varepsilon\eta)(iD) + \varepsilon(iD)b - \frac{\varepsilon}{3}(iD)^3 + O(\varepsilon^2).$$

Here, we should note that the remainder term $O(\varepsilon^2)$ contains high order derivatives. This is one of the reason why we require much differentiability on the data.

## 3 Formal asymptotic analysis and main results

In this section we begin to study formally an asymptotic behavior of the solution $(\eta^\varepsilon, u^\varepsilon)$ to the initial value problem (10) and (11) when $\varepsilon$ tends to 0 and derive coupled KdV like equations, whose solution approximates $(\eta^\varepsilon, u^\varepsilon)$ in a suitable sense. Then, we state our main results.

It follows from the first equation in (10) that

$$u_{1t} + \eta_x + \varepsilon u_1 u_{1x} - \varepsilon\mu\eta_{xxx} = O(\varepsilon^2).$$

By the third and the fourth equations in (6) we have

$$v_{2y} = -v_{1x}, \quad v_{2yy} = -\varepsilon v_{2xx}, \quad v_{2yyy} = \varepsilon v_{1xxx}, \quad v_{2yyyy} = \varepsilon^2 v_{2xxxx}.$$

These relations and Taylor's formula imply that

$$\begin{aligned} v_2(x, y_0, t) &= v_2(x, y_1, t) + (y_1 - y_0)v_{1x}(x, y_1, t) \\ &\quad -\frac{\varepsilon}{2}(y_1 - y_0)^2 v_{2xx}(x, y_1, t) - \frac{\varepsilon}{6}(y_1 - y_0)^3 v_{1xxx}(x, y_1, t) \\ &\quad +\frac{\varepsilon^2}{6}(y_1 - y_0)^4 \int_0^1 v_{2xxxx}(x, sy_0 + (1-s)y_1, t)\, ds. \end{aligned}$$

Putting $y_1 = 1 + \varepsilon\eta(x,t)$ and $y_0 = \varepsilon b(x)$ in the above equation and using the relations

$$\frac{\partial^k u}{\partial x^k}(x,t) = \frac{\partial^k v}{\partial x^k}\big(x, 1 + \varepsilon\eta(x,t), t\big) + O(\varepsilon) \qquad \text{for} \qquad k = 1, 2, 3, \dots,$$

we obtain

$$\begin{aligned} u_2(x,t) &= \varepsilon b'(x) v_1\big(x, \varepsilon b(x), t\big) - \Big(1 + \varepsilon\big(\eta(x,t) - b(x)\big)\Big) v_{1x}\big(x, 1 + \varepsilon\eta(x,t), t\big) \\ &\quad +\frac{\varepsilon}{2} u_{2xx}(x,t) + \frac{\varepsilon}{6} u_{1xxx}(x,t) + O(\varepsilon^2), \end{aligned}$$

where we used (8) the boundary condition on the bottom. Similarly, we get

$$\begin{aligned} u_1(x,t) &= v_1\big(x, \varepsilon b(x), t\big) + \varepsilon \int_{\varepsilon b(x)}^{1+\varepsilon\eta(x,t)} v_{2x}(x, y, t) dy \\ &= v_1\big(x, \varepsilon b(x), t\big) + O(\varepsilon) \end{aligned}$$

and

$$\begin{aligned} u_{1x}(x,t) &= v_{1x}\big(x, 1 + \varepsilon\eta(x,t), t\big) + \varepsilon^2 \eta_x(x,t) v_{2x}\big(x, 1 + \varepsilon\eta(x,t), t\big) \\ &= v_{1x}\big(x, 1 + \varepsilon\eta(x,t), t\big) + O(\varepsilon^2). \end{aligned}$$

These three relations yield that

$$u_2 = -(1 + \varepsilon\eta)u_{1x} + \varepsilon(bu_1)_x + \frac{\varepsilon}{2}u_{2xx} + \frac{\varepsilon}{6}u_{1xxx} + O(\varepsilon^2).$$

Particularly, we have $u_2 = -u_{1x} + O(\varepsilon)$. Putting this into the right hand side of the above relation we obtain

$$u_2 = -(1+\varepsilon\eta)u_{1x} + \varepsilon(bu_1)_x - \frac{\varepsilon}{3}u_{1xxx} + O(\varepsilon^2), \tag{13}$$

which is exactly the same formula as that in Remark 3. This together with the second equation in (10) implies that

$$\eta_t + u_{1x} + \varepsilon\big((\eta - b)u_1\big)_x + \frac{\varepsilon}{3}u_{1xxx} = O(\varepsilon^2).$$

To summarize, we have derived the partial differential equations

$$\begin{cases} u_{1t} + \eta_x + \varepsilon u_1 u_{1x} - \varepsilon\mu\eta_{xxx} = O(\varepsilon^2), \\ \eta_t + u_{1x} + \varepsilon\big((\eta - b)u_1\big)_x + \dfrac{\varepsilon}{3}u_{1xxx} = O(\varepsilon^2), \end{cases} \tag{14}$$

which approximate the equations in (10) up to order $O(\varepsilon^2)$.

Now, let us consider the limiting case $\varepsilon = 0$. Then, the equations in (14) become

$$\begin{cases} u_{1t} + \eta_x = 0, \\ \eta_t + u_{1x} = 0. \end{cases}$$

Under the initial condition (11) this system can be easily solved and the solution has the form

$$\begin{pmatrix} u_1(x,t) \\ \eta(x,t) \end{pmatrix} = \begin{pmatrix} \alpha_1(x-t) - \alpha_2(x+t) \\ \alpha_1(x-t) + \alpha_2(x+t) \end{pmatrix},$$

where the functions $\alpha_1$ and $\alpha_2$ are determined from the initial data $\eta_0$ and $u_0$ by

$$\alpha_1(x) = \frac{1}{2}\big(\eta_0(x) + u_0(x)\big), \quad \alpha_2(x) = \frac{1}{2}\big(\eta_0(x) - u_0(x)\big).$$

For the case $0 < \varepsilon \ll 1$ we can show that under suitable assumptions on the data the initial value problem (10) and (11) has a unique solution $(\eta, u) = (\eta^\varepsilon, u^\varepsilon)$ on some time interval and that the solution satisfies

$$\begin{pmatrix} u_1^\varepsilon(x,t) \\ \eta^\varepsilon(x,t) \end{pmatrix} \simeq \begin{pmatrix} \alpha_1(x-t) - \alpha_2(x+t) \\ \alpha_1(x-t) + \alpha_2(x+t) \end{pmatrix} \tag{15}$$

in an appropriate sense. Therefore, the dynamics of the free surface is approximately as follows: the free surface divides into two wave packets, one moving to the right and the other to the left with the same speed 1 without changing their shapes. Here we should note that the approximation (15) is valid only for the time interval $0 \le t \le O(1)$. Roughly speaking, this means that the dynamics is only translation for such a time interval.

In order to study the dynamics for a long time interval $0 \le t \le O(1/\varepsilon)$ we have to take account of dynamics of the shapes of the two wave packets. Since the dynamics is very slow, it is convenient to use a fast time scale $\tau = \varepsilon t$ in order to make the dynamics to be visible. It is natural to expect that the shapes of the two wave packets shall change in this time scale $\tau$ so that the functions $\alpha_1(x)$ and $\alpha_2(x)$, which describe the shapes of the wave packets in moving coordinates, should be replaced by the functions $\alpha_1(x,\tau)$ and $\alpha_2(x,\tau)$. These considerations lead the ansatz

$$\begin{cases} u_1(x,t) = \alpha_1(x-t,\varepsilon t) - \alpha_2(x+t,\varepsilon t) + \varepsilon\big(\beta_1(x-t,\varepsilon t) - \beta_2(x+t,\varepsilon t)\big) + \varepsilon\bar{\phi}_1(x,t), \\ \eta(x,t) = \alpha_1(x-t,\varepsilon t) + \alpha_2(x+t,\varepsilon t) + \varepsilon\bar{\phi}_2(x,t). \end{cases}$$

Putting these into (14) we obtain

$$\begin{aligned} &(\alpha_{1\tau} + \alpha_1\alpha_{1x} - \mu\alpha_{1xxx}) - (\alpha_{2\tau} - \alpha_2\alpha_{2x} + \mu\alpha_{2xxx}) \\ &\quad - (\alpha_1\alpha_2)_x - \beta_{1x} - \beta_{2x} + \bar{\phi}_{1t} + \bar{\phi}_{2x} = O(\varepsilon) \end{aligned}$$

and

$$\begin{aligned} &\left(\alpha_{1\tau} + 2\alpha_1\alpha_{1x} + \frac{1}{3}\alpha_{1xxx}\right) + \left(\alpha_{2\tau} - 2\alpha_2\alpha_{2x} - \frac{1}{3}\alpha_{2xxx}\right) \\ &\quad - \big(b(\alpha_1 - \alpha_2)\big)_x + \beta_{1x} - \beta_{2x} + \bar{\phi}_{2t} + \bar{\phi}_{1x} = O(\varepsilon), \end{aligned}$$

which are equivalent to the equations

$$\begin{aligned} &2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} + \left(\frac{1}{3} - \mu\right)\alpha_{1xxx} - \left(2\beta_{2x} + \alpha_2\alpha_{2x} + \left(\frac{1}{3} + \mu\right)\alpha_{2xxx}\right) \\ &\quad - \big(\alpha_1\alpha_2 + b(\alpha_1 - \alpha_2)\big)_x + (\bar{\phi}_1 + \bar{\phi}_2)_t + (\bar{\phi}_1 + \bar{\phi}_2)_x = O(\varepsilon) \end{aligned}$$

and

$$\begin{aligned} &2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} - \left(\frac{1}{3} - \mu\right)\alpha_{2xxx} + \left(2\beta_{1x} + \alpha_1\alpha_{1x} + \left(\frac{1}{3} + \mu\right)\alpha_{1xxx}\right) \\ &\quad + \big(\alpha_1\alpha_2 - b(\alpha_1 - \alpha_2)\big)_x - (\bar{\phi}_1 - \bar{\phi}_2)_t + (\bar{\phi}_1 - \bar{\phi}_2)_x = O(\varepsilon). \end{aligned}$$

Here, we define the corrective terms $\beta = (\beta_1, \beta_2)$ and $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2)$ by

$$\begin{cases} \beta_1(x,\tau) = -\frac{1}{4}\alpha_1(x,\tau)^2 - \frac{1}{2}\left(\frac{1}{3} + \mu\right)\alpha_{1xx}(x,\tau), \\ \beta_2(x,\tau) = -\frac{1}{4}\alpha_2(x,\tau)^2 - \frac{1}{2}\left(\frac{1}{3} + \mu\right)\alpha_{2xx}(x,\tau) \end{cases} \tag{16}$$

and

$$\begin{cases} \bar{\phi}_1(x,t) + \bar{\phi}_2(x,t) \\ = b(x)\alpha_1(x-t,\varepsilon t) - \frac{1}{2}b(x)\alpha_2(x+t,\varepsilon t) + \frac{1}{2}\alpha_1(x-t,\varepsilon t)\alpha_2(x+t,\varepsilon t), \\ \bar{\phi}_1(x,t) - \bar{\phi}_2(x,t) \\ = \frac{1}{2}b(x)\alpha_1(x-t,\varepsilon t) - b(x)\alpha_2(x+t,\varepsilon t) - \frac{1}{2}\alpha_1(x-t,\varepsilon t)\alpha_2(x+t,\varepsilon t). \end{cases} \tag{17}$$

Then, the above equations become

$$\Big(2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} + \Big(\frac{1}{3} - \mu\Big)\alpha_{1xxx}\Big)(x - t, \varepsilon t)$$
$$- \big(b(x) + \alpha_2(x + t, \varepsilon t)\big)\alpha_{1x}(x - t, \varepsilon t) + \frac{1}{2}b'(x)\alpha_2(x + t, \varepsilon t) = O(\varepsilon)$$

and

$$\Big(2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} - \Big(\frac{1}{3} - \mu\Big)\alpha_{2xxx}\Big)(x + t, \varepsilon t)$$
$$+ \big(b(x) + \alpha_1(x - t, \varepsilon t)\big)\alpha_{2x}(x + t, \varepsilon t) - \frac{1}{2}b'(x)\alpha_1(x - t, \varepsilon t) = O(\varepsilon).$$

Neglecting the terms $O(\varepsilon)$ in the above equations we arrive at the following coupled KdV like equations

$$\begin{cases} 2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} + \Big(\frac{1}{3} - \mu\Big)\alpha_{1xxx} \\ -\big((T_{\tau/\varepsilon}b) + (T_{2\tau/\varepsilon}\alpha_2)\big)\alpha_{1x} + \frac{1}{2}(T_{\tau/\varepsilon}b')(T_{2\tau/\varepsilon}\alpha_2) = 0, \\ 2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} - \Big(\frac{1}{3} - \mu\Big)\alpha_{2xxx} \\ +\big((T_{-\tau/\varepsilon}b) + (T_{-2\tau/\varepsilon}\alpha_1)\big)\alpha_{2x} - \frac{1}{2}(T_{-\tau/\varepsilon}b')(T_{-2\tau/\varepsilon}\alpha_1) = 0, \end{cases} \tag{18}$$

where $T_\theta$ is the translation operator with respect to the spatial variable defined by $(T_\theta\alpha)(x, \tau) = \alpha(x + \theta, \tau)$. If the functions $\alpha_1$, $\alpha_2$, and $b$ decay at infinity, then we can expect that the coupling terms in the above equations converge to zero when $\varepsilon$ tends to zero and that the equations in (18) are reduced to the KdV equation (in the case $\mu = \frac{1}{3}$ they degenerate into the Burgers equation)

$$\begin{cases} 2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} + \Big(\frac{1}{3} - \mu\Big)\alpha_{1xxx} = 0, \\ 2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} - \Big(\frac{1}{3} - \mu\Big)\alpha_{2xxx} = 0. \end{cases} \tag{19}$$

It is natural to specify the initial conditions in the form

$$\alpha_1 = \frac{1}{2}(\eta_0 + u_0), \quad \alpha_2 = \frac{1}{2}(\eta_0 - u_0) \qquad \text{at} \quad \tau = 0. \tag{20}$$

Now, we are ready to give our main theorems.

**Theorem 1.** *Let $\mu$ and $M$ be positive constants and $m$ an integer such that $m \geq 4$. There exist positive constants $T$, $C$, and $\varepsilon_0$ such that the following holds. For any $\varepsilon \in (0, \varepsilon_0]$, $\eta_0, u_0 \in H^{m+11}$ and $b \in W^{m+9,\infty}$ satisfying*

$$\|(\eta_0, u_0)\|_{m+11} + \|b\|_{W^{m+9,\infty}} \leq M,$$

*the initial value problem* (10) *and* (11) *has a unique solution* $(\eta, u) = (\eta^\varepsilon, u^\varepsilon)$ *on the time interval* $[0, T/\varepsilon]$ *such that*

$$\begin{cases} \eta^\varepsilon \in C([0,T/\varepsilon]; H^{m+2}) \cap C^1([0,T/\varepsilon]; H^{m+1}), \\ u^\varepsilon \in C([0,T/\varepsilon]; H^{m+1}) \cap C^1([0,T/\varepsilon]; H^{m}). \end{cases} \tag{21}$$

*Moreover, the solution satisfies*

$$\sup_{0 \le t \le T/\varepsilon} \Big( \big\| \eta^\varepsilon(t) - \big( \alpha_1^\varepsilon(\cdot - t, \varepsilon t) + \alpha_2^\varepsilon(\cdot + t, \varepsilon t) \big) \big\|_{m+2} \\ + \big\| u_1^\varepsilon(t) - \big( \alpha_1^\varepsilon(\cdot - t, \varepsilon t) - \alpha_2^\varepsilon(\cdot + t, \varepsilon t) \big) \big\|_{m+1} \Big) \le C\varepsilon,$$

*where* $\alpha^\varepsilon = (\alpha_1^\varepsilon, \alpha_2^\varepsilon)$ *is a unique solution of the initial value problem for coupled KdV like equations* (18) *and* (20).

**Theorem 2.** *Let* $\mu$, $T$, *and* $M$ *be positive constants and* $m$ *an integer such that* $\mu \neq 1/3$ *and* $m \ge 4$. *There exist positive constants* $C$ *and* $\varepsilon_0$ *such that the following holds. For any* $\varepsilon \in (0, \varepsilon_0]$, $\eta_0, u_0 \in H^{m+11} \cap H^{m+3,2}$ *and* $b \in W^{m+9,\infty} \cap H^{m+2,2}$ *satisfying*

$$\|(\eta_0, u_0)\|_{m+11} + \|(\eta_0, u_0)\|_{m+3,2} + \|b\|_{W^{m+9,\infty}} + \|b\|_{m+2,2} \le M,$$

*the initial value problem* (10) *and* (11) *has a unique solution* $(\eta, u) = (\eta^\varepsilon, u^\varepsilon)$ *on the time interval* $[0, T/\varepsilon]$ *satisfying* (21) *and*

$$\sup_{0 \le t \le T/\varepsilon} \Big( \big\| \eta^\varepsilon(t) - \big( \alpha_1(\cdot - t, \varepsilon t) + \alpha_2(\cdot + t, \varepsilon t) \big) \big\|_{m+2} \\ + \big\| u_1^\varepsilon(t) - \big( \alpha_1(\cdot - t, \varepsilon t) - \alpha_2(\cdot + t, \varepsilon t) \big) \big\|_{m+1} \Big) \le C\varepsilon,$$

*where* $\alpha = (\alpha_1, \alpha_2)$ *is a unique solution of the initial value problem for the KdV equation* (19) *and* (20).

**Theorem 3.** *Let* $\mu$, $T$, *and* $M$ *be positive constants and* $m$ *an integer such that* $\mu \neq 1/3$ *and* $m \ge 4$. *There exist positive constants* $C$ *and* $\varepsilon_0$ *such that the following holds. For any* $\varepsilon \in (0, \varepsilon_0]$, $\eta_0, u_0 \in H^{m+11}$ *and* $b \in W^{m+9,\infty}$ *satisfying*

$$\|(\eta_0, u_0)\|_{m+11} + \varepsilon^{-1} \big( \|b\|_{W^{m+9,\infty}} + \|\eta_0 - u_0\|_{m+11} \big) \le M$$

*or*

$$\|(\eta_0, u_0)\|_{m+11} + \varepsilon^{-1} \big( \|b\|_{W^{m+9,\infty}} + \|\eta_0 + u_0\|_{m+11} \big) \le M,$$

*the initial value problem* (10) *and* (11) *has a unique solution* $(\eta, u) = (\eta^\varepsilon, u^\varepsilon)$ *on the time interval* $[0, T/\varepsilon]$ *satisfying* (21) *and*

$$\sup_{0 \le t \le T/\varepsilon} \Big( \|\eta^\varepsilon(t) - \alpha_1(\cdot - t, \varepsilon t)\|_{m+2} + \|u_1^\varepsilon(t) - \alpha_1(\cdot - t, \varepsilon t)\|_{m+1} \Big) \le C\varepsilon$$

*or*

$$\sup_{0\le t\le T/\varepsilon}\Big(\|\eta^\varepsilon(t)-\alpha_2(\cdot+t,\varepsilon t)\|_{m+2}+\|u_1^\varepsilon(t)+\alpha_2(\cdot+t,\varepsilon t)\|_{m+1}\Big)\le C\varepsilon,$$

*respectively, where* $\alpha=(\alpha_1,\alpha_2)$ *is a unique solution of the initial value problem for the KdV equation* (19) *and* (20).

**Remark 4.** Concerning the initial value problem (18) and (20), we merely know a local existence theorem in time of solution, so that in Theorem 1 the time $T$ may be small. On the contrary, the initial value problem for the KdV equation (19) and (20) has a global solution in time, so that in Theorems 2 and 3 we can take $T$ as an arbitrarily large constant.

**Remark 5.** Theorem 2 is a refined version of the claim in Schneider and Wayne [20], where they did not work in the Boussinesq variables but studied the equations in the case $\varepsilon=1$. Instead, they assumed that the initial data have the forms $\eta_0(x)=\varepsilon\Phi_1(\varepsilon^{1/2}x)$ and $u_0(x)=\varepsilon\Phi_2(\varepsilon^{1/2}x)$. Note that the solutions $(\eta,u)$ of (10) for general $\varepsilon>0$ are related to the solutions $(\tilde\eta,\tilde u)$ of (10) for $\varepsilon=1$ by the formulas $\tilde\eta(x,t)=\varepsilon\eta(\varepsilon^{1/2}x,\varepsilon^{1/2}t)$, $\tilde u_1(x,t)=\varepsilon u_1(\varepsilon^{1/2}x,\varepsilon^{1/2}t)$, and $\tilde u_2(x,t)=\varepsilon^{3/2}u_2(\varepsilon^{1/2}x,\varepsilon^{1/2}t)$ in the case $b=0$. Therefore, it follows from their estimates that the $L^\infty$-norms of the error terms are of order $O(\varepsilon^{1/6})$ in the Boussinesq variables, whereas we have $O(\varepsilon)$. Moreover, their estimates do not yield any uniform estimates for derivatives of the error terms in those variables.

**Remark 6.** The conditions $\|\eta_0+u_0\|_{m+11}\le M\varepsilon$ and $\|\eta_0-u_0\|_{m+11}\le M\varepsilon$ in Theorem 3 imply that there exists a positive constant $C_1$ depending only on $\mu$, $m$, $M$, and $T$ such that the solution $\alpha=(\alpha_1,\alpha_2)$ of (19) and (20) satisfy $\|\alpha_1(\tau)\|_{m+11}\le C_1\varepsilon$ and $\|\alpha_2(\tau)\|_{m+11}\le C_1\varepsilon$ for $0\le\tau\le T$ and $0<\varepsilon\le 1$, respectively. Therefore, the conditions in Theorem 3 assure that the wave is approximately one directional up to order $O(\varepsilon)$. The global existence theorem of the initial value problem for the KdV equation was established, for example, by Tsutsumi and Mukasa [25] and Bona and Smith [2] in Sobolev spaces of integer order and by Saut and Temam [18] and Bona and Scott [1] in Sobolev spaces of fractional order. See also Temam [24].

# 4 Reduction to a quasi-linear system

In this section we reduce the system (10) to a quasi-linear system of equations, which leads long time ($0\le t\le O(1/\varepsilon)$) existence of the solution. Throughout this and next sections we assume that $(\eta,u)$ is a solution of the system (10) and sufficiently smooth.

Let $\alpha=(\alpha_1,\alpha_2)$ be the solution of the initial value problem for coupled KdV like equations (18) and (20) and define $\beta=(\beta_1,\beta_2)$ and $\bar\phi=(\bar\phi_1,\bar\phi_2)$ by (16) and (17), respectively. We define an approximate solution $\phi = (\phi_1, \phi_2)$ by

$$\begin{cases} \phi_1(x,t) = \alpha_1(x-t,\varepsilon t) - \alpha_2(x+t,\varepsilon t) + \varepsilon\big(\beta_1(x-t,\varepsilon t) - \beta_2(x+t,\varepsilon t)\big) + \varepsilon\bar{\phi}_1(x,t), \\ \phi_2(x,t) = \alpha_1(x-t,\varepsilon t) + \alpha_2(x+t,\varepsilon t) + \varepsilon\bar{\phi}_2(x,t) \end{cases}$$

and remainder functions $\bar{\eta}$ and $\bar{u}_1$ by

$$\begin{cases} \eta(x,t) = \phi_2(x,t) + \varepsilon\bar{\eta}(x,t), \\ u_1(x,t) = \phi_1(x,t) + \varepsilon\bar{u}_1(x,t), \end{cases} \tag{22}$$

and put $\bar{\zeta} = \bar{\eta}_x$. Then, our task is to derive uniform estimates of these remainder functions $\bar{\eta}$ and $\bar{u}_1$ with respect to small $\varepsilon$ for long time interval $0 \le t \le O(1/\varepsilon)$. To this end, we derive quasi-linear equations for these remainder functions. The quasi-linear equations are of the forms

$$\begin{cases} \bar{u}_{1tt} + 2\varepsilon u_1\bar{u}_{1tx} + \varepsilon^2\big(u_1^2 + 3\mu(1+\varepsilon^3\zeta^2)^{-5/2}\zeta_x\big)\bar{u}_{1xx} \\ \qquad - \varepsilon\mu\big((1+\varepsilon^3\zeta^2)^{-3/2}K_0\bar{u}_{1xx}\big)_x + K_0\bar{u}_{1x} \\ \qquad - \varepsilon^2\mu L_1(\eta,b)\bar{u}_{1xxxx} + \varepsilon L_1(\eta,b)\bar{u}_{1xx} = \varepsilon h_1, \\ \bar{\zeta}_{tt} + 2\varepsilon u_1\bar{\zeta}_{tx} - \varepsilon\mu(1+\varepsilon^3\zeta^2)^{-3/2}K_0\bar{\zeta}_{xxx} + K_0\bar{\zeta}_x \\ \qquad - \varepsilon^2\mu L_1(\eta,b)\bar{\zeta}_{xxxx} + \varepsilon L_1(\eta,b)\bar{\zeta}_{xx} = \varepsilon h_2, \end{cases} \tag{23}$$

and

$$\begin{cases} \bar{u}_{1t} + \bar{\eta}_x = \varepsilon h_3, \\ \bar{\eta}_t + \bar{u}_{1x} = \varepsilon h_4, \end{cases} \tag{24}$$

where $L_1(\eta,b)$ is a linear operator defined by

$$L_1(\eta,b)f = -\big(\eta + i\tanh(\varepsilon^{1/2}D)\eta\, i\tanh(\varepsilon^{1/2}D)\big)f + \operatorname{sech}(\varepsilon^{1/2}D)b\operatorname{sech}(\varepsilon^{1/2}D)f.$$

For remainder terms $h_1, \dots, h_4$, we have the following lemma.

**Lemma 2.** *Let $M_1, M_2 > 0$, $m$ be an integer such that $m \ge 4$, and $b \in W^{m+9,\infty}$. There exist positive constants $\varepsilon_1$ and $C_1$ such that if $\|\alpha(\tau)\|_{m+11} \le M_1$ for $0 \le \tau \le T$ and the solution $(\eta, u)$ of (10) satisfies*

$$\begin{cases} \|\eta(t)\|_{m+2} + \|\eta_t(t)\|_{m+1} + \|u_1(t)\|_{m+1} + \|u_{1t}(t)\|_m \le M_2, \\ \|K_0u_1(t)\|_{m+1} + \|K_0u_{1t}(t)\|_m \le M_2 \end{cases}$$

*for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_1$, then we have*

$$\|h_1(t)\|_m^2 + \|h_2(t)\|_m^2 + \|h_3(t)\|^2 + \|h_4(t)\|^2 \le C_1\big(1 + \mathscr{E}(t)\big)$$

*for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_1$, where*

$$\mathscr{E}(t) = \|\bar{\eta}(t)\|_{m+2}^2 + \|\bar{\eta}_t(t)\|_{m+1}^2 + \|\bar{u}_1(t)\|_{m+1}^2 + \|\bar{u}_{1t}(t)\|_m^2.$$

In view of the quasi-linear equations in (23) we consider the linear equation

$$
\begin{aligned}
(25) \qquad & u_{tt} + \varepsilon p_1 u_{tx} + \varepsilon p_2 u_{xx} - \varepsilon a K_0 u_{xxx} + \varepsilon\gamma a_x K_0 u_{xx} \\
& + K_0 u_x + \varepsilon^2 L_1(q_1, b_1) u_{xxxx} + \varepsilon L_1(q_2, b_2) u_{xx} = F_1 + \varepsilon F_2,
\end{aligned}
$$

where $\varepsilon > 0$ is a parameter, $a$, $p_1$, $p_2$, $q_1$, $q_2$, $b_1$, $b_2$, $F_1$, and $F_2$ are given functions of $(x,t)$ and may depend on $\varepsilon$, and $\gamma$ is a real constant.

**Lemma 3.** *Let $M_3 > 0$, $r > 1$, and $m$ be an integer such that $m \geq 4$. There exist positive constant $\varepsilon_2$ and $C_2$ such that if*

$$
\begin{cases}
\varepsilon^{-1}\|a_x(t)\|_m + \|(p_1(t), p_2(t), q_1(t), q_2(t))\|_m + \|(b_1(t), b_2(t))\|_{W^{m,\infty}} \leq M_3, \\
\varepsilon^{-1}\|a_t(t)\|_3 + \|q_{1t}(t)\|_3 + \|q_{2t}(t)\|_1 + |(p_{2t}(t), b_{1t}(t), b_{2t}(t))|_\infty \leq M_3, \\
M_3^{-1} \leq a(x,t) \leq M_3 \quad \text{for} \quad (x,t) \in \mathbf{R} \times [0,T],
\end{cases}
$$

*and $u \in C^j([0,T]; H^{m+3-3j/2})$, $j = 0,1,2$, is a solution of (25), then we have*

$$
(26) \qquad E_m(t) \leq C_2\left(e^{C_2\varepsilon t} E_m(0) + \int_0^t e^{C_2\varepsilon(t-\tau)}\left((1+\tau)^r \|F_1(\tau)\|_m^2 + \varepsilon\|F_2(\tau)\|_m^2 d\tau\right)\right.
$$

*for $0 \leq t \leq T$ and $0 < \varepsilon \leq \varepsilon_2$, where*

$$
E_m(t) = \|u_t(t)\|_m^2 + \left\|\sqrt{\frac{D\tanh(\varepsilon^{1/2}D)}{\varepsilon^{1/2}}}u(t)\right\|_m^2 + \left\|\sqrt{\varepsilon^{1/2}D^3\tanh(\varepsilon^{1/2}D)}u(t)\right\|_m^2.
$$

**Remark 7.** By the inequality

$$
\frac{|x|}{1+\sqrt{|x|}} \leq \sqrt{x\tanh x} \leq |x| \quad \text{for} \quad x \in \mathbf{R},
$$

it holds that

$$
\begin{cases}
\xi^2 \leq 4\left(\varepsilon^{-1/2}\xi\tanh(\varepsilon^{1/2}\xi) + \varepsilon^{1/2}\xi^3\tanh(\varepsilon^{1/2}\xi)\right), \\
\varepsilon^{-1/2}\xi\tanh(\varepsilon^{1/2}\xi) \leq \xi^2, \quad \varepsilon^{1/2}\xi^3\tanh(\varepsilon^{1/2}\xi) \leq \varepsilon\xi^4 \quad \text{for} \quad \xi \in \mathbf{R},
\end{cases}
$$

which yields the following relations for the energy function $E_m(t)$

$$
\|u_t(t)\|_m^2 + 4^{-1}\|u_x(t)\|_m^2 \leq E_m(t) \leq \|u_t(t)\|_m^2 + \|u(t)\|_{m+2}^2.
$$

# 5 Outline of the proof

Since a local existence theorem in time of solution for the initial value problem (10) and (11) for fixed $\varepsilon > 0$ was already given in [5], it is sufficient to derive a priori estimates of the solution $(\eta^\varepsilon, u^\varepsilon)$ for long time interval $0 \leq t \leq O(1/\varepsilon)$.

First, we prove Theorem 1. By standard energy method and appropriate approximation argument of the system it is not difficult to show that under the assumption of Theorem 1 there exist constants $T, M_1 > 0$, which depend only on $\mu$, $m$, and $M$, such that the initial value problem (18) and (20) has a unique solution $\alpha = \alpha^\varepsilon \in C([0,T]; H^{m+11})$ satisfying

$$\|\alpha^\varepsilon(\tau)\|_{m+11} \le M_1 \quad \text{for} \quad 0 \le \tau \le T,\ \varepsilon > 0.$$

Then, there exists a constant $M_4 > 0$ such that the approximate solution $\phi = \phi^\varepsilon$ defined in section 4 satisfies

$$\|\phi^\varepsilon(t)\|_{m+9}^2 + \|\phi_t^\varepsilon(t)\|_{m+6}^2 \le M_4^2 \quad \text{for} \quad 0 \le t \le T/\varepsilon,\ 0 < \varepsilon \le 1.$$

Now, we assume that

$$\mathscr{E}(t) = \|\bar\eta^\varepsilon(t)\|_{m+2}^2 + \|\bar\eta_t^\varepsilon(t)\|_{m+1}^2 + \|\bar u_1^\varepsilon(t)\|_{m+1}^2 + \|\bar u_{1t}^\varepsilon(t)\|_m^2 \le N_1^2 \tag{27}$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$, where the constants $N_1$ and $\varepsilon_0$ will be determined later. Then, by (22) we have

$$\begin{cases} \|\eta^\varepsilon(t)\|_{m+2}^2 + \|\eta_t^\varepsilon(t)\|_{m+1}^2 + \|u_1^\varepsilon(t)\|_{m+1}^2 + \|u_{1t}^\varepsilon(t)\|_m^2 \le (2M_4)^2, \\ \|K_0 u_1^\varepsilon(t)\|_{m+1}^2 + \|K_0 u_{1t}^\varepsilon(t)\|_m^2 \le (2M_4)^2 \end{cases}$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_3$, if we take $\varepsilon_3 \in (0,1]$ so small that $\varepsilon_3 \le \varepsilon_0$ and $\varepsilon_3 N_1 \le M_4$. Thanks of these estimates and Lemma 2 we see that there exist constants $C_1 > 0$ independent of $N_1$ and $\varepsilon_1 \in (0, \varepsilon_3]$ such that

$$\|h_1(t)\|_m^2 + \|h_2(t)\|_m^2 + \|h_3(t)\|^2 + \|h_4(t)\|^2 \le C_1\big(1 + \mathscr{E}(t)\big)$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_1$. By (22) and (24) there exists a constants $C_3 > 0$ independent of $N_1$ such that

$$\|\bar\eta^\varepsilon(0)\|_{m+3}^2 + \|\bar u_1^\varepsilon(0)\|_{m+2}^2 + \|\bar\eta_t^\varepsilon(0)\|_{m+1}^2 + \|\bar u_{1t}^\varepsilon(0)\|_m^2 \le C_3$$

for $0 < \varepsilon \le \varepsilon_1$. Since $\bar\zeta$ and $\bar u$ satisfy (23), by Lemma 3 and Remark 7 it holds that there exist constant $C_2 > 0$ independent of $N_1$ and $\varepsilon_2 \in (0, \varepsilon_3]$ such that

$$\begin{aligned} &\big\|\big(\bar\zeta_t^\varepsilon(t), \bar\zeta_x^\varepsilon(t), \bar u_{1t}^\varepsilon(t), \bar u_{1x}^\varepsilon(t)\big)\big\|_m^2 \\ &\le C_2 e^{C_2\varepsilon t}\big(\big\|\big(\bar\zeta_t^\varepsilon(0), \bar u_{1t}^\varepsilon(0)\big)\big\|_m^2 + \big\|\big(\bar\zeta^\varepsilon(0), \bar u_1^\varepsilon(0)\big)\big\|_{m+2}^2\big) \\ &\quad + C_2\varepsilon \int_0^t e^{C_2\varepsilon(t-\tau)}\big(\|h_1(\tau)\|_m^2 + \|h_2(\tau)\|_m^2\big)d\tau \\ &\le C_2C_3e^{C_2\varepsilon t} + C_2C_1\varepsilon \int_0^t e^{C_2\varepsilon(t-\tau)}\big(1 + \mathscr{E}(\tau)\big)d\tau \end{aligned} \tag{28}$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_2$. Furthermore, $\bar\eta$ and $\bar u$ satisfy also (24) so that we have

$$(29)\quad \begin{aligned} \|(\bar\eta^\varepsilon(t), \bar u_1^\varepsilon(t))\|^2 &\le \mathrm{e}^{\varepsilon t}\|(\bar\eta^\varepsilon(0), \bar u_1^\varepsilon(0))\|^2 + \varepsilon \int_0^t \mathrm{e}^{\varepsilon(t-\tau)}\big(\|h_3(\tau)\|^2 + \|h_4(\tau)\|^2\big)d\tau \\ &\le C_3\mathrm{e}^{\varepsilon t} + C_1\varepsilon \int_0^t e^{\varepsilon(t-\tau)}\big(1+\mathscr{E}(\tau)\big)d\tau \end{aligned}$$

and that

$$(30)\quad \begin{aligned} \|\bar\eta_t^\varepsilon(t)\|^2 &\le 2\|\bar u_{1x}^\varepsilon(t)\|^2 + 2\varepsilon^2\|h_4(t)\|^2 \\ &\le 2\|\bar u_{1x}^\varepsilon(t)\|^2 + 2C_1\varepsilon\big(1+\mathscr{E}(t)\big) \end{aligned}$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_2$. Summarizing the above estimates we see that there exists a constant $C_4$ depending only on $\mu$, $m$, and $M$ such that

$$\mathscr{E}(t) \le C_4\mathrm{e}^{C_4\varepsilon t} + C_4\varepsilon \int_0^t \mathrm{e}^{C_4\varepsilon(t-\tau)}\big(1+\mathscr{E}(\tau)\big)d\tau$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$, by taking $\varepsilon_0 \in (0, \varepsilon_2]$ so small that $4C_1\varepsilon_0 \le 1$. This and Gronwall's inequality imply that

$$\mathscr{E}(t) \le (C_4+1)\mathrm{e}^{2C_4T} \quad \text{for} \quad 0 \le t \le T/\varepsilon,\ 0 < \varepsilon \le \varepsilon_0.$$

Therefore, by setting $N_1 = (C_4+1)^{1/2}\mathrm{e}^{C_4T}$ we see that (27) holds for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$. The proof of Theorem 1 is complete.

We proceed to prove Theorem 2. One of strategies for the proof is to compare the solution of (18) and (20) and that of (19) and (20). However, we do not know whether the solution of (18) and (20) exists globally in time or not, so that we can not take the time $T$ arbitrarily large if we use the solution. In order to take $T$ as an arbitrarily large time, we use the global existence theorem, for example, in [1, 2, 18, 24] and we should not use the solution of (18). Therefore, we have to modify the quasi-linearization given in section 4.

Let $\alpha = (\alpha^1, \alpha^2)$ be the solution of the initial value problem for the KdV equation (19) and (20) and define $\beta = (\beta_1, \beta_2)$ by (16) as before. We define an approximate solution $\phi = (\phi_1, \phi_2)$ by

$$\begin{cases} \phi_1(x,t) = \alpha_1(x-t, \varepsilon t) - \alpha_2(x+t, \varepsilon t) + \varepsilon\big(\beta_1(x-t, \varepsilon t) - \beta_2(x+t, \varepsilon t)\big), \\ \phi_2(x,t) = \alpha_1(x-t, \varepsilon t) + \alpha_2(x+t, \varepsilon t), \end{cases}$$

and remainder functions $\bar\eta$ and $\bar u_1$ by (22), and put $\bar\zeta = \bar\eta_x$. Then, in place of (23) and

(24) we obtain

$$
(31) \qquad \begin{cases} \bar{u}_{1tt} + 2\varepsilon u_1 \bar{u}_{1tx} + \varepsilon^2\big(u_1^2 + 3\mu(1+\varepsilon^3\zeta^2)^{-5/2}\zeta_x\big)\bar{u}_{1xx} \\ \qquad -\varepsilon\mu\big((1+\varepsilon^3\zeta^2)^{-3/2}K_0\bar{u}_{1xx}\big)_x + K_0\bar{u}_{1x} \\ \qquad -\varepsilon^2\mu L_1(\eta,b)\bar{u}_{1xxxx} + \varepsilon L_1(\eta,b)\bar{u}_{1xx} = \tilde{g}_1 + \varepsilon\tilde{h}_1, \\ \bar{\zeta}_{tt} + 2\varepsilon u_1\bar{\zeta}_{tx} - \varepsilon\mu(1+\varepsilon^3\zeta^2)^{-3/2}K_0\bar{\zeta}_{xxx} + K_0\bar{\zeta}_x \\ \qquad -\varepsilon^2\mu L_1(\eta,b)\bar{\zeta}_{xxxx} + \varepsilon L_1(\eta,b)\bar{\zeta}_{xx} = \tilde{g}_2 + \varepsilon\tilde{h}_2 \end{cases}
$$

and

$$
(32) \qquad \begin{cases} \bar{u}_{1t} + \bar{\eta}_x = \tilde{g}_3 + \varepsilon\tilde{h}_3, \\ \bar{\eta}_t + \bar{u}_{1x} = \tilde{g}_4 + \varepsilon\tilde{h}_4, \end{cases}
$$

respectively. Here, $\tilde{h}_1, \dots, \tilde{h}_4$ satisfy the same estimate in Lemma 2 as $h_1, \dots, h_4$. For $\tilde{g}_1, \dots, \tilde{g}_4$, we have the following lemma.

**Lemma 4.** *Let $m$ be a positive integer. There exists a positive constant $C$ such that*

$$
\begin{aligned} & \|\tilde{g}_1(t)\|_m + \|\tilde{g}_2(t)\|_m + \|\tilde{g}_3(t)\| + \|\tilde{g}_4(t)\| \\ & \quad \le \; C(1+t)^{-2}\big(\|\alpha(\varepsilon t)\|_{m+3,2} + \|b\|_{m+2,2}\big)\|\alpha(\varepsilon t)\|_{m+3,2} \end{aligned}
$$

*for $t \ge 0$ and $\varepsilon > 0$.*

Under the assumption of Theorem 2, there exists a constant $M_1 > 0$ such that the initial value problem for the KdV equation (19) and (20) has a unique solution $\alpha \in C([0,T]; H^{m+11} \cap H^{m+3,2})$ satisfying

$$
\|\alpha(\tau)\|_{m+11} + \|\alpha(\tau)\|_{m+3,2} \le M_1 \quad \text{for} \quad 0 \le \tau \le T,\ \varepsilon > 0,
$$

so that by Lemma 4 we have

$$
\|\tilde{g}_1(t)\|_m^2 + \|\tilde{g}_2(t)\|_m^2 + \|\tilde{g}_3(t)\|^2 + \|\tilde{g}_4(t)\|^2 \le C_1(1+t)^{-4}
$$

for $0 \le t \le T/\varepsilon$ and $\varepsilon > 0$. Now, we suppose (27) as before. In this case, in place of (28), (29), and (30) we obtain

$$
\begin{aligned} & \|\big(\bar{\zeta}_t^\varepsilon(t), \bar{\zeta}_x^\varepsilon(t), \bar{u}_{1t}^\varepsilon(t), \bar{u}_{1x}^\varepsilon(t)\big)\|_m^2 \\ & \quad \le \; C_2 e^{C_2\varepsilon t}\big(\|\big(\bar{\zeta}_t^\varepsilon(0), \bar{u}_{1t}^\varepsilon(0)\big)\|_m^2 + \|\big(\bar{\zeta}^\varepsilon(0), \bar{u}_1^\varepsilon(0)\big)\|_{m+2}^2\big) \\ & \qquad + C_2\int_0^t e^{C_2\varepsilon(t-\tau)}\big\{(1+\tau)^2\big(\|\tilde{g}_1(\tau)\|_m^2 + \|\tilde{g}_2(\tau)\|_m^2\big) + \varepsilon\big(\|\tilde{h}_1(\tau)\|_m^2 + \|\tilde{h}_2(\tau)\|_m^2\big)\big\}d\tau \\ & \quad \le \; C_2C_3e^{C_2\varepsilon t} + C_2C_1\int_0^t e^{C_2\varepsilon(t-\tau)}\big\{(1+\tau)^{-2} + \varepsilon\big(1+\mathscr{E}(\tau)\big)\big\}d\tau, \end{aligned}
$$

$$
\begin{aligned}
&\|(\bar{\eta}^\varepsilon(t), \bar{u}_1^\varepsilon(t))\|^2 \\
&\quad \leq \mathrm{e}^{1+\varepsilon t}\|(\bar{\eta}^\varepsilon(0), \bar{u}_1^\varepsilon(0))\|^2 \\
&\qquad + \int_0^t \mathrm{e}^{1+\varepsilon(t-\tau)}\big\{(1+\tau)^2\big(\|\tilde{g}_3(\tau)\|^2 + \|\tilde{g}_4(\tau)\|^2\big) + \varepsilon\big(\|\tilde{h}_3(\tau)\|^2 + \|\tilde{h}_4(\tau)\|^2\big)\big\}d\tau \\
&\quad \leq C_3\mathrm{e}^{1+\varepsilon t} + C_1\int_0^t \mathrm{e}^{1+\varepsilon(t-\tau)}\big\{(1+\tau)^{-2} + \varepsilon\big(1+\mathscr{E}(\tau)\big)\big\}d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\bar{\eta}_t^\varepsilon(t)\|^2 &\leq 3\|\bar{u}_{1x}^\varepsilon(t)\|^2 + 3\|\tilde{g}_4(t)\|^2 + 3\varepsilon^2\|\tilde{h}_4(t)\|^2 \\
&\leq 3\|\bar{u}_{1x}^\varepsilon(t)\|^2 + 3C_1\big\{(1+\tau)^{-2} + \varepsilon\big(1+\mathscr{E}(t)\big)\big\},
\end{aligned}
$$

respectively. Summarizing the above estimates we see that

$$
\begin{aligned}
\mathscr{E}(t) &\leq C_4\mathrm{e}^{C_4\varepsilon t} + C_4\int_0^t \mathrm{e}^{C_4\varepsilon(t-\tau)}\big\{(1+\tau)^{-2} + \varepsilon\big(1+\mathscr{E}(\tau)\big)\big\}d\tau \\
&\leq 2C_4\mathrm{e}^{C_4\varepsilon t} + \varepsilon C_4\int_0^t \mathrm{e}^{C_4\varepsilon(t-\tau)}\big(1+\mathscr{E}(\tau)\big)d\tau
\end{aligned}
$$

for $0 \leq t \leq T/\varepsilon$ and $0 < \varepsilon \leq \varepsilon_0$. This and Gronwall's inequality imply that

$$
\mathscr{E}(t) \leq (2C_4+1)\mathrm{e}^{2C_4T} \quad \text{for} \quad 0 \leq t \leq T/\varepsilon,\ 0 < \varepsilon \leq \varepsilon_0.
$$

Therefore, by setting $N_1 = (2C_4+1)^{1/2}\mathrm{e}^{C_4T}$ we see that (27) holds for $0 \leq t \leq T/\varepsilon$ and $0 < \varepsilon \leq \varepsilon_0$. The proof of Theorem 2 is complete.

It remains to prove Theorem 3. As explained in Remark 6, under the assumption of Theorem 3 the solution $\alpha = (\alpha_1, \alpha_2)$ of (19) and (20) satisfies $\|\alpha_1(\tau)\|_{m+11} \leq C\varepsilon$ or $\|\alpha_2(\tau)\|_{m+11} \leq C\varepsilon$, so that we have

$$
\|\tilde{g}_1(t)\|_m + \|\tilde{g}_2(t)\|_m + \|\tilde{g}_3(t)\| + \|\tilde{g}_4(t)\| \leq C\varepsilon
$$

for $0 \leq t \leq T/\varepsilon$ and $\varepsilon > 0$. Therefore, we can show Theorem 3 in the same way as the proof of Theorem 1.

The details will be published elsewhere.

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